Multiple Equilibria in a Simple Asset Pricing Model with Foresight

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Abstract

Motivated by recent Federal Reserve’s unconventional monetary policies, this paper explicitly models the central bank’s large-scale asset purchases (LSAPs) along with its foresight brought by forward guidance by characterizing the central bank’s actions as a liquidity trader who sends signals to the public period in advance. Assuming negative-exponential utility with Gaussian uncertainty (CARA-Normal framework), we show that there are exactly two stationary equilibria (one stable and one unstable) in the single period model, with the effects of introducing foresight makes the stable equilibrium more stable and the unstable equilibrium more unstable. Extending the single period model to the N-period dynamic model will possibly yield $2^{\max(N-q,1)}$ equilibria, both in the case of expected terminal wealth maximization and intertemporal expected utility maximization.

Keywords: LSAPs, Foresight, Multiple Equilibria, CARA-Normal, Multi-period Optimization

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HAN: Multiple Equilibria in a Simple Asset Pricing Model with Foresight
1 INTRODUCTION

Following the aftermath of the financial crisis 2007-2008 induced by the bursting of the housing bubble, the United States entered a new round of recessions. During the last five years, with Federal Reserves’ main policy instrument: the targeted Federal Funds Rate stuck at zero, the U.S. central bank is facing an unprecedented challenge in rescuing the real economy. With nominal interest rate constrained at the zero lower bound (ZLB) thus no longer an available toolkit at the time being, central bankers have been pushed to explore new norms of monetary policy like “Large Scale Asset Purchasing” (LSAPs)\(^1\) and “Forward Guidance”. While LSAPs mainly changes the size and composition of the central bank’s balance sheet, the general public usually refers to forward guidance as being used by the central bank to influence, with their own foresights, market expectations of future levels of interest rates. The Fed started its first round of LSAPs, also known as QE1, in late November 2008 with $600 billion purchases in mortgage-backed securities (MBS). At almost the same time, on December 16, 2008, the FOMC statement explicitly mentioned that the economic conditions “are likely to warrant exceptionally low levels of the federal funds rate for some time”. Similar forward guidance statements can be found in each FOMC’s statement since then. The Fed consequently conducts two later rounds of QE, purchasing mainly long-term treasury bonds and mortgage-backed securities. With three rounds of QE, the Fed’s balance sheet quadrupled, reaching to an all-time record 4 trillion dollars.

While these unconventional monetary policies (LSAPs and forward guidance) have become popular on social media and thus publicly well-known, the effects of them are still much debated in the academic literature. No common consensus has been made on the effects of LSAPs, or forward guidance, or the combination of these two. Eggertsson and Woodford (2003) shows that in a baseline New Keynesian model the LSAPs in the sense of injecting reserves in exchange for longer term securities is a neutral operation, thus will have no effect on the real economy. They, on the other hand, provide theoretical evidence on forward guidance and argues its effectiveness. Chen, Crdia, and Ferrero (2012) introduces financial market segmentation in their DSGE model and concludes the effects of LSAPs is small. Others estimate the effects of LSAPs from an empirical perspective and reach various conclusions. Hamilton and Wu (2012) uses a discrete time version of Vayanos and Vila (2009)’s preferred-habitat exponential affine term structure model to develop measures of how the maturity structure of debt held by the public might affect the pricing of level, slope, and curvature term structure risk. They provide evidence that LSAPs are effective at bringing down term premiums and thus reducing longer-term rates. Other empirical work also claim LSAPs are bringing down term premiums, but the effects varies a lot, ranging from several basis points to several hundred basis points. \(^2\)

\(^{1}\)Or equivalently, Quantitative Easing (QE).
\(^{2}\)See http://www.federalreserve.gov/newsevents/speech/bernanke20130301a.htm where former Federal Reserve Chairman Bernanke discusses the past and future of monetary policy under the talk title “Long-Term Interest Rate”; Footnote 10 of the
While the literature has provided some insights on these unconventional monetary policies and their effectiveness, what seems missing is that effects of LSAPs and forward guidance are closely entangled and cannot be separated. In particular, while people generally interpret forward guidance as policy guidance on future interest rates, forward guidance provides additional information on Fed’s other actions, especially on LSAPs. Rather than considering them as independent policies, LSAPs and forward guidance are twins born together and the anticipated foresight on shocks brought by forward guidance cannot be ignored when analysing the effects of LSAPs. One prominent example is when then Fed Chairman Bernanke hinted to taper the Fed’s QE policies from $85 billion to $65 billion a month contingent upon continued positive economic data on June 19, 2013, the stock markets dropped approximately 4.3% over the three trading days following Bernanke’s announcement. On the other hand, market reactions to the “real” tapering of QE3 starting on December 2013 was much smaller since the action was well anticipated. Another recent example is when the current Fed Chair Yellen said interest-rate increases could begin in the first half of 2015, around six months after it winds down its bond-buying program on March 19, 2014, the Dow-Jones index fell more than 100 points with the 10-year Treasury bond yield spike from 2.68% to 2.77%.

This paper tries to provide insight on the joint effects of LSAPs and forward guidance, especially the foresight effects of forward guidance on LSAPs. Obviously, as people usually recognized, forward guidance have many other effects, especially its direct influence on people’s expectations of future interest rates. In this sense, this paper is far from being complete. The small step we are making here is try to bring the ignored foresight effect on LSAPs to the attention. With this goal, we focus our attention on the asset pricing implications and consider a partial equilibrium model. By following Singleton (1987), we explicitly models the central bank’s LSAPs and forward guidance by characterizing its actions as a liquidity trader who sends demand signals to the public $q$ period in advance. This stylized way of modelling foresight will yield the equilibrium assets prices consisting of two parts: a fundamental part that is the discounted present value of future payoffs and a non-fundamental part that is driven by anticipated news shocks. Under the constant relative risk aversion-Gaussian framework, we find there are two rational equilibria(one stable and one unstable) consistent with Walker and Whiteman (2007), with the effects of foresight makes the stable equilibrium more stable and the unstable equilibrium more unstable. Extending the single period model to a $N$-period model will possibly yield $2^{\max(N-q-1)}$ equilibria, both in the case of expected terminal wealth maximization and intertemporal expected utility maximization.

The paper is organized in six sections. Section 2 lays out the benchmark single-period asset pricing model and derives the equilibrium condition. Section 3 solves the benchmark model and Section 4 discusses the multiplicity of equilibria and their properties. Section 4 extends the single period model to a multi-period

3 Source: http://en.wikipedia.org/wiki/Quantitative_easing
model and solves the problem by backward induction and argues the multiplicity problem exaggerates. Section 6 concludes.

2 The Single Period Model

In this section, we build a simple single period asset pricing model to describe the Federal Reserve’s recent large scale asset purchases (LSAP) and forward guidance actions. The structure of our model follows Singleton (1987), which is motivated mainly by the market microstructure of U.S. bond market. There are three agents in the benchmark model: the U.S. Treasury Department, the Federal Reserve, and a representative speculative trader. The implicit assumptions underlying the representative speculative trader setup comes from the fact that the U.S. government bond market is a deep liquid market and we assume there is a continuum of homogeneous traders of measure 1 who are price takers. There are two assets in the market, one is risk-free, which earns constant interest rate during each period and one is risky. The risky asset we are considering here corresponds to a generalized long-term U.S. treasury bond. For simplicity and tractability, we model the risky asset as a perpetuity that will never expire, which is subject to price changes and stochastic dividend payments. In the single period model, we call the speculative trader myopic. As in Singleton (1987), we assume the myopic trader is speculative who chooses his holding of the asset to maximize his expected utility while the Fed is a “liquidity” trader who participates in the market primarily for non-speculative reasons and the Treasury is the supplier of the asset.

2.1 The Treasury

The treasury issues the long term treasuries. For simplicity, we assume there is a fixed supply of the securities provided by the Treasury. This is equivalent to say that during each period, the Treasury issues the same amount of new securities as those matured ones to keep the supply stable. Since our only long term risky asset is a perpetuity that will never expire, the treasury’s role is very passive in the sense that it only need to guarantee a constant amount of perpetuities circulated in the market. Rather than modelling the behavior of the Treasury as \( Z_t^T = Z^T \), we transform the equation into levels so the risky asset is in zero net supply \(^4\)

\[ z_t^T = 0, \]

\(^4\)Singleton (1987) models the net supply of non-speculative traders (i.e., the U.S. Treasury, the Fed, some type of financial intermediaries, etc.) as \( z_t = A(L)\tilde{e}_t + B(L)\tilde{\eta}_t + \zeta p_t \) where \( A(L)\tilde{e}_t \) stands for the news shock that are noisy to the public and \( B(L)\tilde{\eta}_t \) are news shock that are in the information set of the public, \( \zeta > 1 \) to reflect the fact that the liquidity trader, even though non-speculative, is tend to supply more securities to the market when the price of the security is high. In our case here, this would imply the Treasury would like to issue more national debt when the borrowing cost is low. We do not consider the price factor here to make our analysis simple and sharp. Effects of including prices can be easily incorporated to the current framework and the main conclusion of the paper, i.e., the multiplicity of equilibria, remains.
where the superscript $T$ denotes the Treasury. Other agents’ (the Fed, the speculative trader) behaviors will also be characterized in levels.

2.2 The Fed

We model the Federal Reserve as a liquidity trader who demands the long term risky asset in levels as

$$z_F^t = \eta_{t-q}, \quad (2.2)$$

and the transaction is accomplished by paying out risk-free assets (cash or reserves) which in theory the central bank can create out of thin air. If $z_F^t < 0$, then the Fed is short selling the security and is a net supplier at time $t$. We assume $z_F^t$ is determined by a signal released by the Fed at time period $t - q$, where $q = 0, 1, 2, ...$ can be interpreted as the “foresight horizon”. Iterate (2.2) $q$-period forward, we get:

$$z_{t+q}^F = \eta_t$$

which means the Fed’s asset buying action $q$-period ahead is determined by the news $\eta_t$ released today at time $t$. Trivially if $q = 0$, then there is no foresight at all. Further assume:

$$\eta_t \sim N(0, \sigma^2_\eta)$$

We now turn to the behavior of the speculative trader.

2.3 The Speculative Trader

In the single period model, the myopic speculative trader allocates his wealth between the risk-free asset and the risky asset. The single risky security comes with price $p_t$ and stochastic dividend payment $d_t$ at time $t$. The dividend stream $\{d_t\}$ is assumed to be normally distributed and given by:

$$d_t = \bar{d} + D(L)\varepsilon^d_t, \quad \varepsilon^d_t \sim N(0, \sigma^2_d) \quad (2.3)$$

where $D(L)$ is (possibly) an infinite-order square summable polynomial in the lag operator $L$ (i.e., $\sum_{j=0}^{\infty} D_j^2 < \infty$). The Wold Decomposition Theorem implies that we have assumed $d_t$ is a covariance-stationary process. Simply setting $\bar{d} = 0$ will transform (2.3) in levels. We don’t model shocks $\varepsilon^d_t$ but they can be interpreted as fundamental shocks to the real economy that influences the coupon payment. Purchases of the security is financed by borrowing at a constant rate $r$. The wealth of the trader evolves according to:

$$w_{t+1} = z_t(p_{t+1} + d_{t+1}) - (1 + r)(z_t p_t - w_t) \quad (2.4)$$

where $z_t$ denotes the holding of the security at time $t$. Note that we can also depict (2.4) by arguing that the trader’s portfolio consists of $z_t$ risky asset and $w_t - z_t p_t$ risk-free asset. The reason why we write in the
form of (2.4) simply reflects the fact that traders usually finance their long-term bond holding by short-term borrowing so that \( r \) can be interpreted as some type of real federal funds rate in reality.

The trader chooses \( z_t \) to maximize the expected value of a constant absolute risk aversion (CARA) utility function

\[
-\mathbb{E}_t \exp(-\gamma w_{t+1})
\]

where \( \mathbb{E}_t \) denotes the expectation of the trader condition on his information set \( \Omega_t \) (which we will specify explicitly in the next section) at time \( t \) and \( \gamma \) is the absolute risk aversion parameter.

Given the assumptions that the underlying uncertainties \( \{\eta_t, \varepsilon^d_t\} \) facing the investors are normally distributed, the distribution of the equilibrium price of the risky asset at time \( t+1 \), conditioning on \( \Omega_t \) is normal, with mean \( \mathbb{E}_t(p_{t+1}) \) and variance \( \text{var}_t(p_{t+1}) \), which in turn guarantees the normally of \( w_{t+1}\mid \Omega_t \). Thus, (2.5) can be calculated from the conditional moment generating function for the normal random variable \(-\gamma w_{t+1}\). That is,

\[
-\mathbb{E}_t \exp(-\gamma w_{t+1}) = -\exp\left\{-\gamma \mathbb{E}_t(w_{t+1}) + \frac{1}{2} \gamma^2 \text{var}_t(w_{t+1}) \right\}
\]

where \( \text{var}_t \) denotes conditional variance. Notice that \( \text{var}_t(w_{t+1}) = z_t^2 \text{var}_t(p_{t+1} + d_{t+1}) \equiv z_t^2 \delta \), where \( \delta \) is a constant.\(^5\) The trader’s demand for the security follows from the first-order necessary conditions for maximization and is given by

\[
z_t^{ST} = \frac{1}{\gamma \delta} \left( \mathbb{E}_t p_{t+1} - \alpha p_t + \mathbb{E}_t d_{t+1} \right)
\]

where \( \alpha \equiv 1 + r > 1 \) and the superscript \( ST \) stands for the speculative trader.

### 2.4 Market Clearing

The market clearing condition requires that the supply equals to the demand at each time period \( t \). That is,

\[
z_t^T = z_t^F + z_t^{ST}
\]

Substituting (2.1), (2.2), (2.6) into the above equation gives the equilibrium price to be

\[
p_t = \frac{1}{\alpha} \left[ \mathbb{E}_t(p_{t+1} + d_{t+1}) + \gamma \delta \eta_{t-\eta} \right]
\]

As emphasized in Walker (2007), the equilibrium price of the asset at time \( t \) depend upon the market-wide average expectation of the asset price at \( t + 1 \), while the trader’s forecast of \( p_{t+1} \) will depend upon the market-wide forecast of \( p_{t+2} \), and so on, \textit{ad infinitum}: a phenomenon Singleton (1987) describes as an “infinite regress problem”.

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\(^5\)This is due to the covariance stationarity of the price and the dividend processes; More explicitly, because \( \delta = \nu_t(p_{t+1} + d_{t+1}) = \text{var}_t(p_{t+1}) + \text{var}_t(d_{t+1}) + 2\text{cov}_t(p_{t+1}, d_{t+1}) \), where each term on the right hand side is a constant, as we will see.


3 INFORMATION AND SOLUTION TECHNIQUES

In this section we first specify the speculative trader’s information set and then work out a perfect information case of the above model.

For each realization of the random variable \( q \), suppose that the speculative trader observes past prices, the dividend stream, the supply history of the Treasury, and Fed’s all past and current asset buying actions signals. Then the information set of the trader is given by:

\[
\Omega_t = \{ p_{t-j}, d_{t-j}, z^T_{t-j}, \eta_{t-j-1}, \eta_t : j \geq 0 \}.
\]

We also assume the underlying distributions of \( \eta_t, \epsilon^d_t \) along with the form of \( D(L) \) are known to the investors.\(^6\)

3.1 SOLUTION TECHNIQUE In the absence of sunspots, the equilibrium price will be driven by the underlying shocks \( \{ \epsilon^d_t, \eta_t \} \), which are assumed to be orthogonal at all leads and lags. Two equivalent approaches can be applied to solve the equilibrium price \( p_t \) of the risky asset. One is guess and verify, which we call the backward approach; and one is the forward approach. Walker and Whiteman (2007) proposes an equilibrium price of the form

\[
p_t = F(L)\epsilon^d_t + G(L)\eta_t \tag{3.1}
\]

where \( F(L) \) and \( G(L) \) are (possibly) infinite-order square summable polynomials in the lag operator \( L \) with coefficients to be determined in the equilibrium and solves the problem backward. Here, we will solve the equilibrium price by the forward approach. The appendix (A.1) demonstrates that the backward solution is exactly the same as the forward solution we are deriving here.

**Proposition 1.** There are exactly two equilibria in the single-period model.

**Proof.** Recall the equilibrium price is given by (2.7), where \( \delta = \text{var}_t(p_{t+1} + d_{t+1}) \). By assuming the covariance stationarity of \( p_t + d_t \), \( \delta \) is a constant. Define \( |\alpha| = |1 + r| > 1 \), and let \( L \) denote the usual lag operator, we could rewrite (2.7) as:

\[
(1 - \alpha^{-1}L^{-1})p_t = \frac{1}{\alpha} (\mathbb{E}_t(d_{t+1}) + \gamma \delta \eta_{t-q}). \tag{3.2}
\]

\(^6\)Clearly this is a case of perfect information, a scenario of asymmetric information and disparately informed traders will be left to future work.
Thus, iterating forward will yield

\[ p_t = \frac{\alpha^{-1}(E_t(d_{t+1}) + \gamma \delta \eta_{-q})}{1 - \alpha^{-1}L^{-1}} \]

\[ = E_t \sum_{j=0}^{\infty} \frac{1}{\alpha^{j+1}} [d_{t+1+j} + \gamma \delta \eta_{-q+j}] \]

\[ = E_t \sum_{j=0}^{\infty} \frac{1}{\alpha^{j+1}} D(L)\varepsilon_{t+j+1}^d + \frac{\gamma \delta}{\alpha} (\eta_{-q} + \frac{1}{\alpha} \eta_{-q+1} + \ldots + \frac{1}{\alpha^q} \eta_t). \]

(3.3)

As showed in the appendix (A.1), the discounted present value of future dividend payoffs is an instance of Hansen and Sargent (1980) prediction formula and characterizes the cross-equation restrictions typical of rational expectations models,

\[ E_t \sum_{j=0}^{\infty} \frac{1}{\alpha^{j+1}} D(L)\varepsilon_{t+j+1}^d = \sum_{j=0}^{\infty} \frac{1}{\alpha^{j+1}} (\frac{D(L)}{L^{j+1}}) \varepsilon_t^d = \frac{D(\alpha^{-1}) - D(L)}{1 - \alpha L} \varepsilon_t^d = F(L)\varepsilon_t^d, \]

(3.4)

where we define \( F(z) := \frac{D(\alpha^{-1}) - D(z)}{1 - \alpha z} \). We still need to pin down \( \delta = \text{var}_t(p_{t+1} + d_{t+1}) \) in (3.3). Iterating \( p_t \) one-period forward, we get

\[ p_{t+1} = F(L)\varepsilon_{t+1}^d + \frac{\gamma \delta}{\alpha} (\eta_{-q+1} + \frac{1}{\alpha} \eta_{-q+2} + \ldots + \frac{1}{\alpha^{q-1}} \eta_t + \frac{1}{\alpha^q} \eta_{t+1}). \]

Thus

\[ E_t p_{t+1} = L^{-1}[F(L) - F(0)]\varepsilon_t^d + \frac{\gamma \delta}{\alpha} (\eta_{-q+1} + \frac{1}{\alpha} \eta_{-q+2} + \ldots + \frac{1}{\alpha^{q-1}} \eta_t), \]

and

\[ \text{var}_t(p_{t+1}) = \text{var}_t(p_{t+1} + d_{t+1}) = \text{var}_t(p_{t+1}) + \text{var}_t(d_{t+1}) + 2\text{cov}_t(p_{t+1}, d_{t+1}) \]

Since \( \delta = \text{var}_t(p_{t+1} + d_{t+1}) = \text{var}_t(p_{t+1}) + \text{var}_t(d_{t+1}) + 2\text{cov}_t(p_{t+1}, d_{t+1}) \) and \( d_t = D(L)\varepsilon_t^d \), by applying the Wiener-Kolmogorov formula, we have \( \text{var}_t(d_{t+1}) = D(0)^2 \sigma_d^2 \). By assuming \( \varepsilon_t^d \) and \( \eta_t \) are orthogonal at all leads and lags and observing the form of \( p_{t+1} \) and \( d_{t+1} \), we will have

\[ \text{cov}_t(p_{t+1}, d_{t+1}) = \text{cov}_t(F(0)\varepsilon_t^d + \frac{\gamma \delta}{\alpha^{q+1}} \eta_{t+1})(D(0)\varepsilon_{t+1}^d) \]

\[ = F(0)D(0)\sigma_d^2. \]
Plugging those results into $\delta$, we get a quadratic equation that $\delta$ has to satisfy

$$\frac{\gamma^2 \sigma^2}{\alpha^{2q+2}} \delta^2 - \delta + (F(0) + D(0))^2 \sigma_d^2 = 0,$$

which implies

$$\delta = \frac{1 \pm \sqrt{1 - 4 \frac{\gamma^2 \sigma^2}{\alpha^{2q+2}} (F(0) + D(0))^2 \sigma_d^2}}{2 \frac{\gamma^2 \sigma^2}{\alpha^{2q+2}}}.$$

Thus, two stationary equilibria have been found to the single period model. In summary, the equilibrium price is given by:

$$p_t = F(L)\varepsilon_t^d + \frac{\gamma \delta}{\alpha} (\eta_t - q + \frac{1}{\alpha} \eta_{t-q+1} + \ldots + \frac{1}{\alpha^q} \eta_t);$$  \hspace{1cm} (3.5)

where $\delta$ is the solution to the quadratic equation:

$$\frac{\gamma^2 \sigma^2}{\alpha^{2q+2}} \delta^2 - \delta + (D(\alpha^{-1}))^2 \sigma_d^2 = 0.$$  \hspace{1cm} (3.6)

and is given by:

$$\delta_{1,2} = \frac{1 \pm \sqrt{1 - 4 \frac{\gamma^2 \sigma^2}{\alpha^{2q+2}} D(\alpha^{-1})^2 \sigma_d^2}}{2 \frac{\gamma^2 \sigma^2}{\alpha^{2q+2}}} = \frac{\alpha^{2q+2} \pm \alpha^{q+1} \sqrt{\alpha^{2q+2} - 4 \gamma^2 \sigma^2 D(\alpha^{-1})^2 \sigma_d^2}}{2 \gamma^2 \sigma^2};$$

and

$$F(z) = \frac{D(\alpha^{-1}) - D(z)}{1 - \alpha z}$$

is a square-summable polynomial in the lag operator $L$.

4 Effects of Foresight

It is noteworthy that the multiplicity of equilibria is not caused by the introduction of foresight; rather, it is entirely due to the nonlinearity in the presence of the conditional variance term $\delta = \text{var}(p_{t+1} + d_{t+1})$. In this section, we will discuss the effects of foresight and the differences between the two equilibria.

4.0.1 Multiplicity of Equilibria   As emphasized above, there are always exactly two equilibria outcome in this model, either in the case $q = 0$ or $q > 0$. Let’s define one equilibrium price to be a high-volatility equilibrium price $p_t^{HV}$ (which corresponds to $\text{var}(p_{t+1} + d_{t+1}) = \delta_1$) and the other one a low-volatility equilibrium price $p_t^{LV}$ (which corresponds to $\text{var}(p_{t+1} + d_{t+1}) = \delta_2$). This is because by (3.5), we can explicitly
calculate the unconditional variances of equilibria as:

\[
\begin{align*}
\text{var}(p_t) &= \left( \sum_{i=0}^{\infty} F_i^2 \sigma_d^2 \right) \frac{\gamma^2 \delta^2 \sigma_n^2}{\alpha^2} \frac{1 - \alpha^{-2q+2}}{1 - \alpha^{-2}} \\
&= \left( \sum_{i=0}^{\infty} F_i^2 \sigma_d^2 \right) \frac{\gamma^2 \delta^2 \sigma_n^2}{\alpha^{2q+2}} - 1 \\
&= \left( \sum_{i=0}^{\infty} F_i^2 \sigma_d^2 \right) \frac{\delta_1 \alpha^2 - 1}{\alpha^2}.
\end{align*}
\]

where we use the restriction of the quadratic equation (3.6) in the last equality. Since \( \delta_1 \geq \delta_2 \), \( \text{var}(p_{HV}^t) \geq \text{var}(p_{LV}^t) \). Also we claim the same conclusion holds for the conditional variance, that is, \( \text{var}(p_{HV}^t) \geq \text{var}(p_{LV}^t) \). This is due to:

\[
\begin{align*}
\text{var}(p_{t+1}) &= \mathbb{E}_t (p_{t+1} - \mathbb{E}_t(p_{t+1}))^2 \\
&= F(0)^2 \sigma_d^2 + \frac{\gamma^2 \delta^2 \sigma_n^2}{\alpha^{2q+2}} \\
&= \delta_1 \alpha^2 + (D(\alpha^{-1}))^2 \sigma_d^2.
\end{align*}
\]

So the naming of the two equilibrium prices is legitimate.

4.0.2 Effects of q

It is obvious from (3.5) that the random variable \( q \) plays two roles in determining \( p_t \): it influences the conditional variance \( \delta := \text{var}(p_{t+1} + d_{t+1}) \) and it adds moving average component \( \eta_{t-q} + \frac{1}{\alpha} \eta_{t-q+1} + \ldots + \frac{1}{\alpha^q} \eta_t \) into \( p_t \). This MA component shows that agents discount recent news heavier than old news by a factor of \( \frac{1}{\alpha} \). This seemingly ‘reversed’ way of discounting is due to the fact that recent news \( \eta_t \) contains information about Fed’s action at distant future \( t + q \), thus should be discounted heavier, while old news \( \eta_{t-q} \) are informative about contemporaneous Fed actions, hence should be discounted less.

This type of discounting referring to news along with how foresight and optimizing behavior create equilibria with moving-average representations have been emphasized in Leeper, Walker, and Yang (2013).

\( q \) will also influence the conditional and unconditional variances of \( p_{HV}^t, p_{LV}^t \) differently. Simple differentiation will show:

\[
\frac{\partial \text{var}(p_{t+1}^{HV})}{\partial q} = \frac{\partial \delta_1}{\partial q} > 0,
\]

\[
\frac{\partial \text{var}(p_{t+1}^{LV})}{\partial q} = \frac{\partial \delta_2}{\partial q} > 0,
\]

So both the unconditional and conditional variances of the high-volatility price increase along with the increase of \( q \). On the other hand,

\[
\frac{\partial \text{var}(p_{t+1}^{LV})}{\partial q} = \frac{\partial \delta_2}{\partial q} < 0,
\]
\[
\frac{\partial \text{var}(p_{t+1}^{LV})}{\partial q} = \frac{\partial \delta_2}{\partial q} \left( \frac{\alpha^2q^2 - 1}{\alpha^2 - 1} \right) + \left[ \delta_2 + (D(\alpha^{-1}))^2 \sigma_d^2 \right] \frac{2\alpha^{2q+2}\ln(\alpha)}{\alpha^2 - 1} > 0 \text{ when } q \text{ is large.}
\]

The above calculation shows that \( \text{var}_t(p_{t+1}^{HV}) - \text{var}_t(p_{t+1}^{LV}) \) is an increasing function with respect to \( q \), a result which we will use below.

4.0.3 Likelihood of multiplicity when \( q = 0 \) & \( q > 0 \) We argue when \( q > 0 \), multiplicity of equilibria will become less likely in a perfect information environment as we considered here. This is because in this set-up there is no mechanism for prices to shift from one equilibrium to another equilibrium. Suppose agents have no idea on which equilibrium path the prices is on, then as \( q \) increases, the gap between \( \text{var}_t(p_{t+1}^{HV}) \) and \( \text{var}_t(p_{t+1}^{LV}) \) widens so that it is easier for agents to determine which equilibrium the price actually is on. Suppose we calculate theoretically

\[
\text{var}_t(p_{t+1}^{HV}) - \text{var}_t(p_{t+1}^{LV}) = x > 0,
\]

as the way we did above. Then an observation of \( \text{var}_t(p_{t+1}) < x \) in the reality will effectively eliminate \( p_{t+1}^{HV} \). The above discussion shows that \( q > 0 \) always ‘partially’ eliminates the high-volatility price equilibrium, a point where we could argue to be one of the benefits of the Federal Reserves’ “Forward Guidance”.

But when information is imperfect, our conjecture is that multiplicity will rise more frequently due to the uncertainty of \( q \). For example, there might be asymmetric information between the traders and the Fed on the random variable \( q \): the public coincidentally misunderstand the Fed’s signal sending actions; or heterogeneous agents hold different private information on \( q \) and those disparately informed traders makes higher-order belief dynamics more prone to multiple equilibria.

4.0.4 Stability Properties of Equilibria in the Time Domain Following the discussion in Walker and Whiteman (2007), let \( \delta_t = \text{var}_t(p_{t+1}) \), then

\[
\delta_t = (D(\alpha^{-1}))^2 \sigma_d^2 + \frac{\gamma^2 \sigma_n^2}{\alpha^{2q+2}} \delta_{t+1}^2.
\]

Stationarity implies \( \delta_t = \delta_{t+1} = \delta \) and the above equation just degenerates to the quadratic equation (3.6). We now can check the stability of the high-volatility and the low-volatility solutions with or without foresight. Notice that

\[
\frac{d\delta_t}{d\delta_{t+1}} = 2 \frac{\gamma^2 \sigma_n^2}{\alpha^{2q+2}} \delta_{1.2}.
\]

An equilibrium is stable if \( |d\delta_t/d\delta_{t+1}| < 1 \). Then it is clear that the high-volatility equilibrium is not stable and can be called a “knife-edge” equilibrium. As Walker and Whiteman (2007) argued, \( p_{t}^{HV} \) can only be sustained if the agent believes the price will remain in the high-volatility regime indefinitely.
It is worthwhile to see how $q$ influences the stability of the low-volatility solution. Clearly,

$$\left| \frac{d\delta_{LV}}{dt_{t+1}} \right| = |1 - \sqrt{1 - 4\gamma^2\sigma^2_\eta\alpha^{-(2q+2)}D(\alpha-1)^2\sigma^2_\nu}|.$$ 

Given $\alpha = 1 + r > 1$, as $q$ increases, $\left| \frac{d\delta_{LV}}{dt_{t+1}} \right|$ decreases. And as $q \to \infty$, $\left| \frac{d\delta_{LV}}{dt_{t+1}} \right|$ approaches to 0 from above. This shows that foresight $q$ increases the stability of the low-volatility solution.

On the other hand, the existences of $q > 0$ exacerbate the unstability of the high-volatility solution. This is because

$$\left| \frac{d\delta_{HV}}{dt_{t+1}} \right| = |1 + \sqrt{1 - 4\gamma^2\sigma^2_\eta\alpha^{-(2q+2)}D(\alpha-1)^2\sigma^2_\nu}|.$$ 

As $q \to \infty$, $\left| \frac{d\delta_{HV}}{dt_{t+1}} \right|$ approaches to 2 from below.

5 The Multi-Period Model

It is natural to extend the single-period model into a multi-period model. However, to consider a finite- or an infinite-horizon problem is another modelling choice. We choose to close our model in finite-horizon and there are mainly two reasons. First, we want to argue that the static model setup (i.e, the constant foresight horizon $q$) has to change in the future. More specifically, the LSAPs probably will not continue forever so the whole model will break down. The second reason is solely due to the tractability of the finite-horizon model under which we can get analytical solutions. We will consider two cases, the expected terminal wealth maximization and the intertemporal expected utility maximization.

5.1 Expected Terminal Wealth Maximization

Suppose now the speculative traders want to maximize his expected terminal CARA utility $N$-period into the future at time $t$, then naturally, we can rephrase the problem as\(^7\):

$$\max - E_t\exp(-\gamma w_{t+N})$$

s.t: $w_{t+s} = z_{t+s-1}(p_{t+s} + d_{t+s}) - (1 + r)(z_{t+s-1}p_{t+s-1} - w_{t+s-1}) \quad \forall s = 1, 2, ..., N$

By the above formulation it is easy to see that we assumed these traders do not consume anything until $w_{t+N}$ has been realized and then they consume everything. Notice that now we cannot use the normal moment generating function directly to $w_{t+N}$ because given all available information at time $t$, $w_{t+N}$ is not necessarily normal\(^8\); rather, we will solve this multi-period problem by backward induction by following Stapleton and Subrahmanyam (1978), in which Stapleton and Subrahmanyam (1978) provides a list of assumptions under which the backward induction method can be applied to a multi-period asset pricing model. One key element

\(^7\)We can call $N$ the termination horizon with respect to current time index $t$.

\(^8\)In fact it will not be normal as we will see below in the most general case.
in their work is they have to verify the conclusions of Fama (1970) regarding the conditions under which the consumers’ multi-period optimization problem can be reduced to a series of single-period problems. All assumptions are satisfied under our setup. What’s different between Stapleton and Subrahmanyam (1978) classic model and our model is that they only have speculative traders in the market so that market clearing can be determined simultaneously inside the market among those speculative traders while in our set-up, we have another type of trader: the liquidity trader, i.e, the Fed, so the market clearing condition has to be satisfied between the two types of traders. Another difference is that we introduce foresight in our model. It turns out that these differences will twist the backward induction procedure and complicate the algebra to some extent.

To start with, let’s assume the current time index is $t$, the termination time is $t+N$, thus the termination horizon is given by $N$. During time period $t+N-1$, the speculative trader’s problem will be the static myopic problem as before.

5.1.1 Prices at $t+N-1$ Given the speculative trader’s wealth level $w_{t+N-1}$, the speculative trader solves

$$V(w_{t+N-1}) = \max_{z_{t+N-1}} \left[ -\mathbb{E}_{t+N-1} \exp(-\gamma w_{t+N}) \right]$$

s.t: $w_{t+N} = z_{t+N-1}(p_{t+N} + d_{t+N}) - (1+r)(z_{t+N-1}|t+N-1|p_{t+N-1}|t+N-1|w_{t+N-1})$

where we use $V(.)$ to denote the value of wealth$^9$. The choice variable is $z_{t+N-1}|t+N-1|$, which is state dependent, conditioning on all information available at time $t+N-1$ (more precisely, after the realizations of $d_{t+N-1}$ and $\eta_{t+N-1}$). Our target, the asset price $p_{t+N-1}|t+N-1|$, is defined in the same fashion. This is exactly the same static myopic problem we have dealt with so far and $p_{t+N-1}|t+N-1|$ is given by

$$p_{t+N-1}|t+N-1| = F(L|\epsilon_{t+N-1} + \frac{\gamma \delta}{\alpha}(\eta_{t+N-1} - q + \frac{1}{\alpha} \eta_{t+N-1} - q + 1 + \frac{1}{(\alpha-1)^2} \eta_{t+N-1}) (5.1)$$

where

$$F(L) = \sum_{j=0}^{\infty} \frac{1}{\alpha^{j+1}} \frac{D(\alpha^{-1})-D(L)_{j}}{1-nL} \quad \text{with} \quad F(0) = D(\alpha^{-1}) - D(0) \quad (5.2)$$

$$\delta = \frac{1 \pm \sqrt{1 - 4 \frac{\gamma^2 \sigma^2}{\alpha^2 \sigma^2} (D(\alpha^{-1}))^2 \sigma^2}}{2 \alpha \sigma^2} \quad (5.3)$$

$^9$More rigorously, we should use $V(.)$ with time index as subscript.
where as before $\alpha = 1 + r$. Given the equilibrium price, we can easily back out the value function $V(w_{t+N-1})$ by combining $p_{t+N-1|t+N-1}$ with the speculative trader’s budget constraint, that is,

$$V(w_{t+N-1}) = -\exp[-\gamma(\gamma \delta \eta_{t+N-1}^2 + (1 + r)w_{t+N-1}) + \frac{1}{2} \gamma^2 \delta \eta_{t+N-1}^2]$$

$$= -\exp(-\gamma(1 + r)w_{t+N-1} - \frac{1}{2} \gamma^2 \delta \eta_{t+N-1}^2).$$

The detailed derivation is given in appendix (A.2).

5.1.2 Prices at $t + N - 2$ At period $t + N - 2$, we have the speculative trader’s problem

$$V(w_{t+N-2}) = \max \mathbb{E}_{t+N-2} V(w_{t+N-1}) = \max -\mathbb{E}_{t+N-2} \exp(-\gamma(1 + r)w_{t+N-1} - \frac{1}{2} \gamma^2 \delta \eta_{t+N-1}^2)$$

s.t: $w_{t+N-1} = z_{t+N-2|t+N-2}(p_{t+N-1} + d_{t+N-1}) - (1 + r)(z_{t+N-2|t+N-2} + p_{t+N-2|t+N-2} - w_{t+N-2})$

Now we have to be careful about the term $\eta_{t+N-1-q}$ in the above value function. If $q \geq 1$, then $\eta_{t+N-1-q}$ is known to agents at $t + N - 2$, so we could simply take the term $\frac{1}{2} \gamma^2 \delta \eta_{t+N-1-q}^2$ out of the expectation and apply the same procedure as in the time period $t + N - 1$. In fact, we will have the same problem as before with the only exception that we have to adjust the risk aversion parameter from $\gamma$ to $\gamma(1 + r)$.

This motivates the following two cases: $N \leq q + 1$ and $N > q + 1$.

5.1.3 Cases when $N \leq q + 1$ When $N \leq q + 1$, then at time $t$, the speculative traders will know every Fed’s future actions they want to know before they finally quit the market. This corresponds to we can take every $\eta_{t+N-i-q}, i = 1, 2, ... , N$ out of the expectation sign when doing backward induction during each step. To see this more clearly, at $t + N - 2$, the equilibrium price is given by

$$p_{t+N-2|t+N-2} = F(L)\epsilon_{t+N-2}^d + \frac{(\gamma \alpha) \delta_2}{\alpha}(\eta_{t+N-2-q} + \frac{1}{\alpha} \eta_{t+N-2-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N-2})$$

where

$$\delta_2 = 1 \pm \frac{1}{2} \sqrt{1 - 4 \frac{(\gamma \alpha)^2 \sigma_2^2}{\alpha^2 \alpha^2 + \alpha^2}}(D(\alpha^{-1}))^2 \sigma_2^2$$

Compared to $p_{t+N-1|t+N-1}$, it is easy to see we change nothing except substituting the risk aversion parameter from $\gamma$ to $\gamma \alpha$. For consistency, let’s denote $\delta_1 = \delta$. And now

$$V(w_{t+N-2}) = -\exp[-\gamma \alpha^2 w_{t+N-2} - \frac{1}{2} \gamma^2 \delta_1 \eta_{t+N-1-q}^2 - \frac{1}{2} (\gamma \alpha)^2 \delta_2 \eta_{t+N-2-q}^2]$$
To generalize,

\[ p_t = p_{t|t} = F(L)\delta t^d + \frac{\gamma \alpha^{N-1}}{\alpha} \delta N \left( \eta_t + \frac{1}{\alpha} \eta_{t-q+1} + \ldots + \frac{1}{\alpha^q} \eta_t \right) \]  

(5.4)

where

\[ \delta_N = \frac{1 \pm \sqrt{1 - 4 \frac{\gamma \alpha^{N-1} \gamma^2 \sigma^2_2}{\alpha^2 \sigma^2_2 - \delta^2 \sigma^2_2}}}{2 \frac{\gamma \alpha^{N-1} \gamma^2 \sigma^2_2}{\alpha^2 \sigma^2_2 - \delta^2 \sigma^2_2}} \]  

(5.5)

To compare with the myopic solution:

\[ p_t^{myopic} = F(L)\delta t^d + \frac{\gamma \alpha^{N-1}}{\alpha} \delta_1 \left( \eta_t + \frac{1}{\alpha} \eta_{t-q+1} + \ldots + \frac{1}{\alpha^q} \eta_t \right) \]  

where

\[ \delta_1 = \frac{1 \pm \sqrt{1 - 4 \frac{(\gamma^2 \sigma^2_2)^2}{\alpha^2 \sigma^2_2 - \delta^2 \sigma^2_2} (\alpha(\alpha-1)) (\gamma^2 \sigma^2_2)^2}}{2 \frac{(\gamma^2 \sigma^2_2)^2}{\alpha^2 \sigma^2_2 - \delta^2 \sigma^2_2}} \]  

(5.6)

Clearly we can see that under the assumption \( N \leq q + 1 \), the multi-period equilibrium price is analogous to the myopic equilibrium price, with adjusted risk aversion \( \gamma \alpha^{N-1} \). The value function for the initial wealth \( w_t \) is given by

\[ V(w_t) = -\exp\left[ -\gamma \alpha^N w_t - \frac{1}{2} \sum_{i=1}^{N} (\gamma \alpha^{i-1})^2 \delta_i \eta_{t+N-i-q}^2 \right] \]  

(5.7)

By looking at \( V(w_t) \), we can see that the assumption \( N \leq q + 1 \) guarantees that all \( \eta_{t+N-(i+1)-q} \)'s are known to the traders at time \( t \) for \( i = 0, 1, \ldots, N - 1 \).

Things will have to change when \( N > q + 1 \), i.e., when the termination horizon is greater than the anticipation horizon plus 1. At this time, when the speculative trader makes his decision at time \( t \), he has to put the future uncertainty of the Fed’s action into his consideration, and this will influence the equilibrium price in a new way. To illustrate it, let’s consider an extreme case when \( q = 0 \) and \( N > 1 \), the problem at \( t + N - 1 \) is still the same, but at time period \( t + N - 2 \), when the speculative trader

\[ \max -E_{t+N-2} \exp \left( -\gamma (1 + r) w_{t+N-1} - \frac{1}{2} \gamma^2 \delta \eta_{t+N-1}^2 \right) \]  

the moment generation function for the normal variable cannot be applied to the above expression directly. This is because \( -\gamma (1 + r) w_{t+N-1} - \frac{1}{2} \gamma^2 \delta \eta_{t+N-1}^2 \) is not normally distributed once \( \eta_{t+N-1} \) becomes a random variable conditional on information at \( t + N - 2 \). In fact, since \( w_{t+N-1} \) will depend on \( p_{t+N-1} \) and \( p_{t+N-1} \) will be determined partially by \( \eta_{t+N-1} \) and \( w_{t+N-1} \) and \( \eta_{t+N-1} \) must be correlated. Furthermore, given the fact \( \eta_t \) is normal, \( \eta_t^2 \) follows \( \chi \)-square. This leads to the analysis to the case when \( N > q + 1 \) below.
5.1.4 Cases when $N > q + 1$  

Recall that at time period $t + N - 2$, $\eta_{t+N-1-q}$ appears in the speculative trader’s utility function. If $q \geq 1$, then we can take $\eta_{t+N-1-q}$ out of $E_{t+N-2}$ and apply the same procedure as in the myopic case with some adjusted risk aversion. Similarly, at time period $t + N - 3$, we need $q \geq 2$ to take $\eta_{t+N-2-q}$ out of $E_{t+N-3}$ and the backward induction continues. Given the assumption $N > q + 1$, we have to pause and adjust our backward induction once we reach the time period $t + N - q - 2$. Notice that given $N > q + 1, t + N - q - 2 \geq t$.

At period $t + N - q - 2$, the speculative trader’s problem is

$$V(w_{t+N-q-2}) = \max -E_{t+N-q-2}\exp(-\gamma\alpha^{q+1}w_{t+N-q-1} - 1/2 \sum_{i=1}^{q+1} (\gamma\alpha^{i-1})^2\delta_i\eta_{t+N-i-q}$$

subject to:

$$w_{t+N-q-1} = z_{t+N-q-2|t+N-q-2}(p_{t+N-q-1} + d_{t+N-q-1}) - \alpha(z_{t+N-q-2|t+N-q-2}p_{t+N-q-2|t+N-q-2} - w_{t+N-q-2})$$

To avoid long subscripts, let’s denote $s = t + N - q - 2$ and rewrite the above problem as:

$$V(w_s) = \max -AE_s\exp(Bw_{s+1} + C\eta_{s+1}^2)$$

$$s.t. w_{s+1} = z_{s|s}(p_{s+1} + d_{s+1}) - \alpha(z_{s|s}p_{s|s} - w_s)$$

where $A = \exp(-1/2 \sum_{i=2}^{q+1} (\gamma\alpha^{i-1})^2\delta_i\eta_{t+N-i-q}^2), B = -\gamma\alpha^{q+1}, C = -1/2\gamma^2\delta_1$ are all constants in (5.8). We have the following proposition for the trader’s problem at $t + N - q - 2$.

**Proposition 2.** At time period $s = t + N - q - 2$, the speculative trader solves

$$V(w_s) = \max -AE_s\exp(Bw_{s+1} + C\eta_{s+1}^2),$$

$$s.t. w_{s+1} = z_{s}(p_{s+1} + d_{s+1}) - \alpha(z_{s}p_{s} - w_s);$$

There are exactly two equilibria to this problem. The derived indirect utility $V(w_s)$ is of the form

$$V(w_s) = -A'\exp(B'w_s + C'[\eta_s^2]),$$

where $A'$, $B'$ and $C'$ are constants conditional on $F_{s-1}$. In consequence, theoretically there could have $2^{N-\eta}$ rational equilibria in the $N$-period dynamic model.

**Proof.** See appendix (A.3).
5.2 Intertemporal Expected Utility Maximization  It turns out the solution method of intertemporal expected utility case will be very similar to the terminal wealth case thus we put it into the appendix. See appendix (A.4). What’s more important, the conclusion of Proposition 2 still holds. Two main differences exist between the two cases. First, instead of only one choice variable, the investment decision, as in the terminal wealth case, speculative traders now have two choice variables, both the investment decision and the contemporaneous consumption. This second choice variable adds another layer of optimization inside each period and complicates the algebra; Second, the opportunities of having consumption decreases the trader’s adjusted risk aversion in a systematic way: As the termination horizon $N$ approaching to $\infty$, the representative trader acts as if he is risk-neutral. This behaviour is in sharp contrast with the terminal wealth maximization case, under which the trader becomes more and more risk averse as $N$ increasing since he only have one single measure, the terminal wealth, to evaluate his success. See appendix (A.4) for more discussion on the characteristics of traders’ risk aversions.

6 Conclusions

Under a constant relative risk aversion-Gaussian framework, we analyze the asset pricing implications of the central bank’s LSAPs combined with foresight shocks and show that multiple equilibria exist. The multiplicity of equilibrium solely comes from the presence of the conditional variance in the equilibrium condition and is independent of any information assumptions and parameter values. Extending the benchmark single-period model to a $N$-period dynamic model will possibly introduce another layer of uncertainty(depending on the relationship between the terminal horizon $N$ and the foresight horizon $q$), which will exacerbate the multiple equilibria phenomenon. Increasing foresight horizon $q$ alleviates the multiplicity problem, either by pinning down only 2 equilibria in the case of $N \leq q + 1$, or by decreasing the number of equilibria to $2^{N-q}$ in the case of $N > q + 1$.

References


A Appendix

A.1 Backward Solution to the Single Period Model  Following Walker and Whiteman (2007), we assume that every investor believes the equilibrium price to be given by

\[ p_t = F(L)\varepsilon_t^d + G(L)\eta_t \]  \hspace{1cm} (A.1)

where \( F(L) \) and \( G(L) \) are (possibly) infinite-order square summable polynomials in the lag operator \( L \) with coefficients to be determined in the equilibrium. The expectations in (2.7) are given by the Wiener-Kolmogolov optimal prediction formulas.

\[
E_t(p_{t+1}) = L^{-1}[F(L) - F(0)]\varepsilon_t^d + L^{-1}[G(L) - G(0)]\eta_t
\]

\[
E_t(d_{t+1}) = L^{-1}[D(L) - D(0)]\varepsilon_t^d
\]
These implies that

\[ \text{var}_t(p_{t+1}) = \mathbb{E}_t\{[p_{t+1} - \mathbb{E}_t(p_{t+1})]^2\} \]
\[ = \mathbb{E}_t\{[F(L)\varepsilon_{t+1}^d + G(L)\eta_{t+1} - L^{-1}[F(L) - F(0)]\varepsilon_t^d - L^{-1}[G(L) - G(0)]\eta_t]^2\} \]
\[ = \mathbb{E}_t\{[F(0)\varepsilon_{t+1}^d + G(0)\eta_{t+1}]^2\} \]
\[ = F(0)^2\sigma_d^2 + G(0)^2\sigma_\eta^2 \]

Similar calculations yield \( \text{var}_t(p_{t+1}) = D(0)^2\sigma_d^2 \) and \( \text{cor}_t(p_{t+1}, d_{t+1}) = F(0)D(0)\sigma_d^2 \). Then, the conditional variance term \( \delta \) is given by

\[ \delta = \text{var}_t(p_{t+1}) + \text{var}_t(d_{t+1}) + 2\text{cov}_t(p_{t+1}, d_{t+1}) \]
\[ = G(0)^2\sigma_\eta^2 + (F(0) + D(0))^2\sigma_d^2 \]

Substituting \( \eta_{t-q} \) by \( L^q\eta_t \) and expanding (2.7) by using the above results gives

\[ \{\alpha F(L) - L^{-1}[F(L) - F(0)] - L^{-1}[D(L) - D(0)]\} \varepsilon_t^d = \]
\[ \{-\alpha G(L) + L^{-1}[G(L) - G(0)] + \gamma(G(0)^2\sigma_\eta^2 + (F(0) + D(0))^2\sigma_d^2)L^q\} \eta_t \]

Since the above expression must hold for all realizations of the two orthogonal processes \( \{\varepsilon_t^d, \eta_t\} \), the coefficients in front of \( \varepsilon_t^d \) and \( \eta_t \) have to be 0. Rather than solving \( F(L), G(L) \) as an infinite sequential problem, we apply the Riesz-Fisher Theorem and solve an equivalent functional fixed-point problem by examining the corresponding power series equalities. Imposing the coefficient in front of \( \varepsilon_t^d \) equals to 0 gives

\[ F(z) = \frac{F(0) + D(0) - D(z)}{1 - \alpha z} \quad \text{(A.2)} \]

As in Walker and Whiteman (2007), the hole \( \alpha^{-1} \), which lies in the open unit circle given \( |\alpha| > 1 \), has to be removed to guarantee the square-summability of \( F(L) \) in the time domain. This pins down the free parameter \( F(0) \) by applying

\[ \lim_{z \to \alpha^{-1}} F(0) + D(0) - D(z) = 0 \]

which implies \( F(0) = D(\alpha^{-1}) - D(0) \). Thus \( F(z) \) is uniquely given by

\[ F^*(z) = \frac{D(\alpha^{-1}) - D(z)}{1 - \alpha z} \quad \text{(A.3)} \]
Repeat the same procedure to the coefficient in front of \( \eta_t \), we have\(^{10} \)

\[
G(z) = \frac{G(0) - \gamma (G(0)^2 \sigma_n^2 + (F(0) + D(0))^2 \sigma_d^2) z^{q+1}}{1 - \alpha z}
\]  

(A.4)

Again, \( G(0) \) must be chosen to remove the singularity at \( z = \alpha^{-1} \), thus

\[
[\gamma \sigma_n^2 \alpha^{-(q+1)}]G(0)^2 - G(0) + \gamma D(\alpha^{-1}) \sigma_d^2 \alpha^{-(q+1)} = 0
\]

(A.5)

The quadratic formula gives the root as

\[
G(0)_{1,2} = \frac{1 \pm \sqrt{1 - 4\gamma^2 \sigma_n^2 \sigma_d^2 D(\alpha^{-1})^2 \alpha^{-2(q+1)}}}{2\gamma \sigma_n^2 \alpha^{-(q+1)}}
\]

Thus

\[
G^*(z) = \frac{G(0)_{1,2} - \gamma (G(0)^2 \sigma_n^2 + (D(\alpha^{-1})^2 \sigma_d^2) z^{q+1}}{1 - \alpha z}
\]

(A.6)

Combining equations (A.3), (A.6) gives the equilibrium price as \( p_t = F^*(L)\varepsilon_t^q + G^*(L)\eta_t \). A straightforward comparison will show that the backward and the forward solutions are identical.

**A.2 DERIVATION OF \( V(w_{t+N-1}) \) IN THE TERMINAL WEALTH MAXIMIZATION CASE**

Given \( p_{t+N-1|t+N-1} \) determined by (5.1) and \( z_{t+N-1|t+N-1} = -z_{t+N-1}^F = -\eta_{t+N-1-q} \) implied by the market clearing condition, we now can write \( V(w_{t+N-1}) \) explicitly. Notice that, by the moment generating function

\[
V(w_{t+N-1}) = -\exp[-\gamma \tilde{E}_{t+N-1}(w_{t+N}) + \frac{1}{2} \gamma^2 \text{var}_{t+N-1}(w_{t+N})]
\]

(A.7)

where

\[
w_{t+N} = -\eta_{t+N-1-q}(p_{t+N} + d_{t+N}) - (1 + r)(-\eta_{t+N-1-q} p_{t+N-1|t+N-1} - w_{t+N-1})
\]

\[
= -\eta_{t+N-1-q}[p_{t+N} + d_{t+N} - (1 + r)p_{t+N-1|t+N-1}] + (1 + r)w_{t+N-1}
\]

so \((1 + r)w_{t+N-1} \) can be interpreted as the end of period certain value of initial wealth \( w_{t+N-1} \) and

\[-\eta_{t+N-1-q}[p_{t+N} + d_{t+N} - (1 + r)p_{t+N-1|t+N-1}] \]

as the extra amount due to the opportunity to invest in the asset market.

Naturally, we will have

\[
\tilde{E}_{t+N-1}w_{t+N} = -\eta_{t+N-1-q}(\tilde{E}_{t+N-1}p_{t+N} + \tilde{E}_{t+N-1}d_{t+N} - (1 + r)p_{t+N-1|t+N-1}) + (1 + r)w_{t+N-1}
\]

(A.8)

\(^{10}\)Notice that the sign in front of the second term in the numerator is negative while in Walker and Whiteman (2007) it is positive, this is because we assume \( \tilde{z}_t^F \) is Fed’s net demand, not net supply here.
Now given the fact that \( d_{t+N} = D(L)e_{t+N}^d \), we have \( \mathbb{E}_t d_{t+N} = L^{-1}[D(L) - D(0)]e_{t+N-1}^d \). And

\[
p_{t+N} = F(L)e_{t+N}^d + \frac{\gamma \delta}{\alpha} (\eta_{t+N-q} + \frac{1}{\alpha} \eta_{t+N-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N})
\]

we have

\[
\mathbb{E}_{t+N-1} p_{t+N} = L^{-1}[F(L) - F(0)]e_{t+N-1}^d + \frac{\gamma \delta}{\alpha} (\eta_{t+N-q} + \frac{1}{\alpha} \eta_{t+N-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N-1})
\]

Combine with (5.1), we get

\[
\mathbb{E}_{t+N-1} p_{t+N} + \mathbb{E}_{t+N-1} d_{t+N} - (1 + r)p_{t+N-1} = L^{-1}[F(L) - F(0)]e_{t+N-1}^d + \frac{\gamma \delta}{\alpha} (\eta_{t+N-q} + \frac{1}{\alpha} \eta_{t+N-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N-1})
\]

\[
+ L^{-1}[D(L) - D(0)]e_{t+N-1}^d - \alpha[F(L)e_{t+N-1}^d + \frac{\gamma \delta}{\alpha} (\eta_{t+N-q} + \frac{1}{\alpha} \eta_{t+N-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N-1})]
\]

\[
= L^{-1}[F(L) - F(0) + D(L) - D(0) - \alpha F(L)]e_{t+N-1}^d - \gamma \delta \eta_{t+N-1-q}
\]

Using the result we get in (5.2), then

\[
F(L) - F(0) + D(L) - D(0) - \alpha F(L) L
\]

\[
= \sum_{j=0}^{\infty} \frac{1}{\alpha^j} (\frac{D(L)}{\alpha^j})_+ - D(\alpha^{-1}) + D(0) + D(L) - D(0) - \alpha L \sum_{j=0}^{\infty} \frac{1}{\alpha^j} (\frac{D(L)}{\alpha^j})_+
\]

\[
= \sum_{j=1}^{\infty} \frac{1}{\alpha^j} (\frac{D(L)}{\alpha^j})_+ - D(\alpha^{-1}) + D(L) - \sum_{j=0}^{\infty} \frac{1}{\alpha^j} (\frac{D(L)}{\alpha^j})_+ + \sum_{j=0}^{\infty} \frac{1}{\alpha^j} D_j
\]

\[
= -D(L) - D(\alpha^{-1}) + D(L) + D(\alpha^{-1}) = 0
\]

where we use the fact that \( \alpha L \sum_{j=0}^{\infty} \frac{1}{\alpha^j} (\frac{D(L)}{\alpha^j})_+ = \sum_{j=0}^{\infty} \frac{1}{\alpha^j} [(\frac{D(L)}{\alpha^j})_+ - D_j] \) in the fourth equality and \( D(\alpha^{-1}) = \sum_{j=0}^{\infty} \frac{1}{\alpha^j} D_j \) in the last equality. So it turns out

\[
\mathbb{E}_{t+N-1} p_{t+N} + \mathbb{E}_{t+N-1} d_{t+N} - (1 + r)p_{t+N-1} = -\gamma \delta \eta_{t+N-1-q}
\]

(A.9)

Plugging (A.9) into (A.8) yields

\[
\mathbb{E}_{t+N-1} w_{t+N} = \gamma \delta \eta_{t+N-1-q}^2 + (1 + r)w_{t+N-1}
\]

(A.10)

At the same time, since we defined \( \delta = \text{var}_t(p_{t+1} + d_{t+1}), \forall t \)

\[
\text{var}_{t+N-1}(w_{t+1}) = \eta_{t+N-1-q}^2 \text{var}_{t+N-1}(p_{t+1} + d_{t+N}) = \eta_{t+N-1-q}^2 \delta
\]

(A.11)
Plugging the results of (A.10),(A.11) into (A.7), finally we get

\[ V(w_{t+N-1}) = -\exp[-\gamma(\gamma \delta \eta_{t+N-1-q}^2 + (1+r)w_{t+N-1}) + \frac{1}{2} \gamma^2 \delta \eta_{t+N-1-q}^2] \]
\[ = -\exp(-\gamma(1+r)w_{t+N-1} - \frac{1}{2} \gamma^2 \delta \eta_{t+N-1-q}^2) \]  
(A.12)

A.3 Proof of Proposition 2 in the Terminal Wealth Maximization Case

Plugging the budget constraint, (5.8) is equivalent to:

\[ V(w_t) = \max z_{z_t} - A\exp(-\alpha B(z_{|s}p_{|s} - w_s))\mathbb{E}_s \exp(B z_{|s}(p_{s+1} + d_{s+1}) + C \eta_{s+1}^2) \]  
(A.13)

To proceed, we guess the equilibrium price is of the form

\[ p_s = F(L)\epsilon_s^d + G(L)\eta_s \]  
(A.14)

Then

\[ V(w_t) = \max z_{z_t} - A\exp(-\alpha B(z_{|s}p_{|s} - w_s))\mathbb{E}_s \exp(B z_{|s}(F(L)\epsilon_s^d + G(L)\eta_s + D(L)\epsilon_s^d + C \eta_{s+1}^2) \]
\[ = \max z_{z_t} - A\exp(-\alpha B(z_{|s}(F(L)\epsilon_s^d + G(L)\eta_s) - w_s))\exp(B z_{|s}(G(L) - G_0)\eta_{s+1}) \]
\[ \mathbb{E}_s \exp(B z_{|s}(F(L) + D(L))\epsilon_s^d)\mathbb{E}_s \exp(C \eta_{s+1}^2 + B z_{|s}G_0\eta_{s+1}) \]
\[ = \max z_{z_t} - A\exp(-\alpha B(z_{|s}(F(L)\epsilon_s^d + G(L)\eta_s) - w_s))\exp(B z_{|s}(G(L) - G_0)\eta_{s+1}) \]
\[ \mathbb{E}_s \exp(B z_{|s}(F(L) + D(L))\epsilon_s^d)\mathbb{E}_s \exp(C \eta_{s+1}^2 + B z_{|s}G_0\eta_{s+1}) \]  
(A.15)

where we use the assumption that \( \{\epsilon_t^d, \eta_t\} \) are orthogonal at all leads and lags in the third equality and the fact that \( (G(L) - G_0)\eta_{s+1} = L^{-1}(G(L) - G_0)\eta_s = \left(\frac{G(L)}{L}\right)_+ \eta_s \) is known to traders at time \( s \) in the fourth equality.

Apply the normal moment generating function to \( \mathbb{E}_s \exp(B z_{|s}(F(L) + D(L))\epsilon_s^d) \) in (A.15), we get

\[ \mathbb{E}_s \exp(B z_{|s}(F(L) + D(L))\epsilon_s^d) = \exp(B z_{|s}\mathbb{E}_s(F(L) + D(L))\epsilon_s^d + \frac{1}{2} B^2 z_{|s}^2 \text{var}(F(L) + D(L))\epsilon_s^d) \]
\[ = \exp(B z_{|s}L^{-1}[F(L) + D(L) - F(0) - D(0)]\epsilon_s^d)\exp\left(\frac{1}{2} B^2 z_{|s}^2(F(0) + D(0))^2\sigma^2 \right) \]  
(A.16)

On the other hand, by completing the squares of \( C \eta_{s+1}^2 + B z_{|s}G_0\eta_{s+1} \), we get

\[ \mathbb{E}_s \exp(C \eta_{s+1}^2 + B z_{|s}G_0\eta_{s+1}) = \exp(-\frac{B^2 z_{|s}^2 G_0^2}{2C})\mathbb{E}_s \exp(C(\eta_{s+1} + \frac{B z_{|s}G_0}{2C})^2) \]
Notice that \( \eta + \frac{Bz_s G_0}{2C} \sim N\left(\frac{Bz_s G_0}{2C}, \sigma_\eta^2\right) \), thus \( \left(\frac{\eta + \frac{Bz_s G_0}{2C}}{\sigma_\eta}\right)^2 \) follows a noncentral \( \chi^2 \)-square distribution with degree of freedom 1 and another parameter which related to the mean and characterises the noncentral \( \chi^2 \)-square distribution: \( \left(\frac{Bz_s G_0}{2\sigma_\eta}\right)^2 \). Apply the moment generating function for noncentral \( \chi^2 \)-square distribution\(^\text{11}\), we get

\[
\mathbb{E}_s \exp(C \eta_{s+1}^2 + Bz_s G_0 \eta_{s+1}) = \exp\left(-\frac{B^2 z_s^2 G_0^2}{4C}\right) \mathbb{E}_s \exp(C(\eta_{s+1} + \frac{Bz_s G_0}{2C})^2)
\]

\[
= \exp\left(-\frac{B^2 z_s^2 G_0^2}{4C}\right) \mathbb{E}_s \exp(C \eta_{s+1}^2(\frac{Bz_s G_0}{\sigma_\eta})^2)
\]

\[
= \exp\left(-\frac{B^2 z_s^2 G_0^2}{4C}\right)(1 - 2C \eta_{s+1}^2 - \frac{1}{2} \exp\left(\frac{B^2 z_s^2 G_0^2}{2C \eta_{s+1}^2}\right))
\]

\[
= (1 - 2C \eta_{s+1}^2) - \frac{1}{2} \exp\left(-\frac{B^2 z_s^2 G_0^2}{2C \eta_{s+1}^2}\right)
\]

(A.17)

Plugging (A.16),(A.17) into (A.15) and maximize \( V(w_s) \) with respect to the choice variable \( z_s | s \), finally we can get

\[
-\alpha(F(L) \epsilon_s^d + G(L) \eta_s) + L^{-1}(G(L) - G_0) \eta_s + L^{-1}[F(L) + D(L) - F(0) - D(0)] \epsilon_s^d
\]

\[
+ B(F(0) + D(0))^2 \sigma_\eta^2 z_s | s + \frac{B^2 \sigma_{\epsilon_s}^2 G_0^2}{1 - 2C \eta_{s+1}^2} z_s | s = 0
\]

Imposing the market clearing condition \( z_s | s = -\eta_s - q = -L^q \eta_s \), and setting the coefficients in front of \( \epsilon_s^d \) equal to 0, we will get the following result

\[
F(L) = \frac{D(\alpha^{-1}) - D(L)}{1 - \alpha L}
\]

i.e, \( \epsilon_s^d \) plays exactly the same role as in the myopic case and in the multi-period \( N \leq q + 1 \) case. Setting the coefficients of \( \eta_s \) equals to 0, we will get

\[
G(L) = \frac{G_0 + B(D(\alpha^{-1}) \sigma_\eta^2 + \frac{\sigma_\eta^2 G_0^2}{1 - 2C \sigma_\eta^2}) L^q + 1}{1 - \alpha L}
\]

Plugging \( B = -\gamma \alpha^q + 1 \) and \( C = -\frac{1}{2} \gamma^2 \delta_\eta \) into the above expression, we have

\[
G(L) = \frac{G_0 - \gamma \alpha^q (D(\alpha^{-1}) \sigma_\eta^2 + \frac{\sigma_\eta^2 G_0^2}{1 + \gamma \delta_\eta \sigma_\eta}) L^q + 1}{1 - \alpha L}
\]

(A.18)

It will be helpful to recall the corresponding \( G(L) \) we get in the myopic case:

\[
G(L)^\text{myopic} = \frac{G_0 - \gamma (D(\alpha^{-1}) \sigma_\eta^2 + \sigma_\eta^2 G_0^2) L^q + 1}{1 - \alpha L}
\]

\(^\text{11}\)If \( X \sim \chi^2(r, \lambda) \), where \( r \) is the degree of freedom and \( \lambda \) related to the mean of \( X \), then \( M_X(t) = \mathbb{E}^t X = (1 - 2t)^{-r/2} \exp\left(\frac{\lambda}{1 - 2t}\right)\).
We should keep in mind that \( G(L) \) is nothing complicated but a polynomial of degree \( q \) since \( 1 - \alpha L \) is a common factor between the numerator and the denominator! If we write out \( G(L) \) explicitly, it will be exactly the same as its counterpart in the forward solution without guessing the solution form.

Thus again, during period \( s = t + N - q - 2 \), we have to adjust the risk aversion parameter from \( \gamma \) to \( \gamma^{q+1} \), which is consistent along with the cases where \( N \leq q + 1 \) we considered above. On the other hand, we notice that \( \delta_1 \), which defined as \( \text{var}_t(p_{t+1} + d_{t+1}) \), appears in the expression of \( G(L) \) in (A.18). There are exactly two values for \( \delta_1 \). For convenience, we rewrite \( \delta_1 \) here:

\[
\delta_1 = 1 \pm \sqrt{1 - 4 \frac{\gamma^2 \sigma_\eta^2 (D(\alpha^{-1}))^2 \sigma_d^2}{2 \alpha^2 \sigma_d^2}}
\]

Also the form of \( G(L) \) in (A.18) reveals that we have to pin down the free parameter \( G_0 \). Again, there are two possible values of \( G_0 \) given each \( \delta_1 \), due to the quadratic equation it has to satisfy:

\[
\frac{\gamma \sigma_\eta^2}{1 + \gamma^2 \delta_1 \sigma_d^2} G_0^2 - G_0 + \gamma (D(\alpha^{-1}))^2 \sigma_d^2 = 0
\]

Due to the multiplicity of \( \delta_1 \), theoretically we could have 4 \( G_0 \)’s that are solutions to the above equation. Thus, we could have 4 different equilibrium prices that are consistent with agents’ rational expectations! Notice that we just solve the problem at \( t + N - q - 2 \), and we still need to conduct the backward induction until we solve for \( p_{t|t} = p_t \), given now \( V(w_{t+N-q-2}) \) could have 4 different functional forms depending on which equilibrium price it lies on, my conjecture is that we could have 8 possible equilibrium prices during the period \( t + N - q - 3 \). To generalize, at time period \( t \), we could have \( 2^{N-q} \) rational expectation equilibrium \( p_t \)’s.

If we write out \( V(w_s) \) explicitly in (5.8), it will turn out that it is still in the same form of \( V(w_{s+1}) = -A \exp(Bw_{s+1} + c \eta^2_{s+1}) \) so that our backward induction can continue and the number of solutions will explode by the factor of 2.

A.4 Dynamic Multi-Period Optimization with Intertemporal Consumption

Now let’s bring consumption into the picture. Recall that so far, we express all our variables in levels, i.e, we express them in terms of deviations from steady state. For simplicity, we will inherit this convention and abstract from constant terms when outlining the model setup in the following. Basically we consider a Lucas endowment economy where two types of traders of total measure 1 live in: The speculative trader chooses between a durable risky asset and a risk-free asset to maximize his present value of discounted utility while the liquidity trader influences the total supply of the risky asset.
1. The speculative trader:

\[
\max - E_0 \sum_{t=0}^N \beta^t \exp(-\gamma c_t) \\
\text{s.t.} \quad c_0 + p_0 z_0 + m_0 \leq w_0 \\
\quad c_t + p_t z_t + m_t \leq z_{t-1}(p_t + d_t) + m_{t-1}(1 + r), \quad t \geq 1 \\
C_N \leq z_{N-1}(p_N + d_N) + m_{N-1}(1 + r)
\]

where \( w_0 \) is the given initial wealth level and \( N \) could be infinity (If \( N = \infty \), then naturally we should omit the last inequality in the speculative trader’s problem ). During each period \( t \), the speculative trader has to make his consumption decision \( c_t \), his risky asset holding decision \( z_t \), and his borrowing-lending decision \( m_t \). We assume the dividend process follows:

\[d_t = D(L)c_t^d, \quad c_t^d \sim N(0, \sigma_d^2)\]

2. The liquidity trader demands the risky asset:

\[z^F_t = \eta_t - q, \quad \eta_t \sim N(0, \sigma^2_\eta)\]

where \( q \) denotes the foresight horizon.

3. The risky asset market clears:

\[(1 - \theta) z_t + \theta z^F_t = 0, \quad \forall t\]

i.e, we give measure \( \theta \) to the liquidity trader and \( 1 - \theta \) to the speculative trader.

We will first consider the case when \( N \) is finite. Following the procedure described in “DynamicModel”, we again will use backward induction to solve the problem. The only difference here is that besides the risky asset holding decision \( z_t \) and the borrowing-lending decision \( m_t \), the speculative trader also need to choose consumption \( c_t \) optimally during each time period.

**A.4.1 Prices at \( t + N - 1 \)** At time period \( t + N - 1 \), the trader's wealth \( w_{t+N-1} \) is a state variable. For now, let’s simply assume \( c_{t+N-1} \) is given; or, equivalently, let’ assume the trader just make his consumption decision \( c_{t+N-1} \). Then with his allocatable wealth \( w_{t+N-1} - c_{t+N-1} \), the trader wants to solve:

\[V(w_{t+N-1} - c_{t+N-1}) = \max E_{t+N-1} - \exp(-\gamma c_{t+N})\]
subject to:

\[ c_{t+N} = w_{t+N} = z_{t+N-1|t+N-1}(p_{t+N} + d_{t+N}) + (1 + r)(w_{t+N-1} - c_{t+N-1} - p_{t+N-1|t+N-1}z_{t+N-1|t+N-1}) \]

This is exactly the static myopic problem with initial wealth level \( w_{t+N-1} - c_{t+N-1} \). By the results we derived in “DynamicModel”, we know the equilibrium price is given by:

\[ p_{t+N-1|t+N-1} = F(L)s^d_{t+N-1} + \frac{\gamma \delta}{\alpha} \left( \frac{\theta}{1 - \theta} \right) (\eta_{t+N-1-q} + \frac{1}{\alpha} \eta_{t+N-1-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N-1}) \quad (A.19) \]

where

\[ F(L) = \sum_{j=0}^{\infty} \frac{1}{\alpha^{j+1}} \left( \frac{D(L)}{D(L)^j} \right)_+ = \frac{D(\alpha^{-1}) - D(0)}{1 - \alpha} \quad \text{with} \quad F(0) = D(\alpha^{-1}) - D(0) \quad (A.20) \]

\[ \delta = \frac{1 \pm \sqrt{1 - 4 \frac{\gamma^2}{\sigma^2} (\frac{\theta}{1 - \theta})^2 (D(\alpha^{-1}))^2 \delta^2}}{2 \frac{\gamma^2}{\sigma^2} (\frac{\theta}{1 - \theta})^2} \quad (A.21) \]

We should also keep in mind that we defined \( \alpha = 1 + r \).

At the same time, we will have:

\[ V(w_{t+N-1} - c_{t+N-1}) = -\exp \left( -\gamma (1 + r)(w_{t+N-1} - c_{t+N-1}) - \frac{1}{2} \gamma^2 \left( \frac{\theta}{1 - \theta} \right)^2 \delta \eta_{t+N-1-q}^2 \right) \quad (A.22) \]

In the above expression, \( w_{t+N-1} - c_{t+N-1} \) is the amount available for investment after consumption at time \( t + N - 1 \). We still have to use the principle of optimality in dynamic programming to determine \( c_{t+N-1} \).

Notice that the Bellman equation is given by:

\[ V(w_{t+N-1}) = \max_{c_{t+N-1}} \{ u(c_{t+N-1}) + \beta V(w_{t+N-1} - c_{t+N-1}) \} \]

Plugging the result we get in equation (A.22) along with \( u(c) = -\exp(-\gamma c) \) and applying the first order conditions yield:

\[ c_{t+N-1} = \frac{\gamma \alpha w_{t+N-1} + \frac{1}{2} \gamma^2 \left( \frac{\theta}{1 - \theta} \right)^2 \delta \eta_{t+N-1-q}^2 - \log(\beta \alpha)}{\gamma (1 + \alpha)} \quad (A.23) \]

which then determines:

\[ V(w_{t+N-1}) = -\left( \frac{\beta \alpha}{1 + \alpha} \right)^{\frac{1}{\alpha}} \frac{1}{\alpha} \exp \left( -\frac{\gamma \alpha}{1 + \alpha} w_{t+N-1} - \frac{1}{1 + \alpha} \frac{1}{2} \gamma^2 \left( \frac{\theta}{1 - \theta} \right)^2 \delta \eta_{t+N-1-q}^2 \right) \quad (A.24) \]
A.4.2 Cases when $N \leq q + 1$ Under this scenario we know we can take all $\eta$'s out of expectation sign when applying backward induction. In particular, at period $t + N - 2$, we first will have:

$$V(w_{t+N-2} - c_{t+N-2}) = \max E_{t+N-2} V(w_{t+N-1})$$

subject to:

$$w_{t+N-1} = z_{t+N-2|t+N-2}(p_{t+N-1} + d_{t+N-1}) + (1 + r)(w_{t+N-2} - c_{t+N-2} - p_{t+N-2|t+N-2} z_{t+N-2|t+N-2})$$

By looking at (A.24) we realize that after adjusting the risk aversion parameter from $\gamma$ to $\frac{\gamma_0}{1 + \alpha}$ this is essentially the same myopic problem. In consequence, the equilibrium price is given by:

$$p_{t+N-2|t+N-2} = F(L)c_{t+N-2}^d + \frac{\gamma_2 \delta_2}{\alpha} \left( \frac{\theta}{1 - \theta} \right) (\eta_{t+N-2-q} + \frac{1}{\alpha} \eta_{t+N-2-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N-2})$$

where we defined

$$\gamma_2 = \frac{\gamma_0}{1 + \alpha}, \delta_2 = \frac{1}{2} \sqrt{1 - 4 \frac{\gamma_0^2}{\alpha^2} \frac{\theta}{1 - \theta}^2 (D(\alpha - 1))^2 \text{sech}^2(\theta)}$$

We could also write out $V(w_{t+N-2} - c_{t+N-2})$ explicitly:

$$V(w_{t+N-2} - c_{t+N-2}) = -(\beta \alpha)^{\frac{1}{1 + \alpha}} \frac{1 + \alpha}{\alpha} \exp(-\frac{\gamma_2}{1 + \alpha} (w_{t+N-2} - c_{t+N-2})) \exp(-\frac{1}{2} \frac{1 + \alpha}{1 + \alpha} \gamma_1^2 \delta_1 (\frac{\theta}{1 - \theta})^2 \eta_{t+N-1-q}^2 - \frac{1}{2} \gamma_2^2 \delta_2 (\frac{\theta}{1 - \theta})^2 \eta_{t+N-2-q}^2) \quad (A.25)$$

where we rename $\gamma$ and $\delta$ to $\gamma_1$, $\delta_1$ for consistency. Repeat the same procedure as we did in the $t + N - 1$ case to get $c_{t+N-2}$:

$$c_{t+N-2} = \frac{\gamma_2}{1 + \alpha} w_{t+N-2} + \frac{1}{2} \gamma_2^2 \delta_2 (\frac{\theta}{1 - \theta})^2 \eta_{t+N-2-q}^2 + \frac{1}{2} \frac{1 + \alpha}{1 + \alpha} \gamma_1^2 \delta_1 (\frac{\theta}{1 - \theta})^2 \eta_{t+N-1-q}^2 - (1 + \frac{1}{1 + \alpha}) \log(\beta \alpha)$$

which then determines

$$V(w_{t+N-2}) = -\exp(-\gamma c_{t+N-2}) + \beta V(w_{t+N-2} - c_{t+N-2})$$

$$= -C_2 \exp(-\frac{\gamma \alpha^2}{1 + \alpha} w_{t+N-2}) \exp(-\frac{1}{2} \frac{1 + \alpha}{1 + \alpha} \gamma_1^2 \delta_1 (\frac{\theta}{1 - \theta})^2 \eta_{t+N-1-q}^2 - \frac{1}{2} \frac{1 + \alpha}{1 + \alpha} \gamma_2^2 \delta_2 (\frac{\theta}{1 - \theta})^2 \eta_{t+N-2-q}^2)$$

where $C_2$ is a constant. Now it is clear that to determine $p_{t+N-3|t+N-3}$, we only need to consider the myopic problem again, with adjusted risk aversion $\frac{\gamma_0}{1 + \alpha + \alpha^2}$. 

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To generalize, we will have:

\[ p_{t+N-i|t+N-i} = F(L)e_t^d + \frac{\gamma_i \delta_i}{\alpha} \left( \frac{\theta}{1-\theta} \right) (\eta_{t+N-i-q} + \frac{1}{\alpha} \eta_{t+N-i-q+1} + \ldots + \frac{1}{\alpha^q} \eta_{t+N-i}) \]

with

\[ \gamma_i = \frac{\gamma \alpha^{i-1}}{1 + \alpha + \alpha^2 + \ldots + \alpha^{i-1}} \]

\[ \delta_i = \frac{1 \pm \sqrt{1 - 4 \frac{\gamma_i^2 \sigma^2_i}{\alpha^2 + \sigma^2_i} \left( \frac{\theta}{1-\theta} \right)^2 (D(\alpha-1))^2 \sigma^2_d}}{2 \frac{\gamma_i^2 \sigma^2_i}{\alpha^2 + \sigma^2_i} \left( \frac{\theta}{1-\theta} \right)^2} \]

for \( i = 1, \ldots, N \). The corresponding value functions are given by:

\[ V(w_{t+N-i}) = -C_i \exp \left( -\frac{\gamma \alpha^i}{1 + \alpha + \ldots + \alpha^i} w_{t+N-i} - \frac{1}{2} \theta \left( \frac{\theta}{1-\theta} \right)^2 \sum_{j=1}^{i} \frac{1 + \alpha + \ldots + \alpha^{j-1}}{1 + \alpha + \ldots + \alpha^i} \gamma_j^2 \delta_j \eta_{t+N-j-q} \right) \]

where \( C_i \) are constants. In particular,

\[ p_t = F(L)e_t^d + \frac{\gamma N \delta N}{\alpha} \theta (\eta_{t-q} + \frac{1}{\alpha} \eta_{t-q+1} + \ldots + \frac{1}{\alpha^q} \eta_t) \]

\[ V(w_t) = -C_N \exp \left( -\frac{\gamma \alpha^N}{1 + \alpha + \ldots + \alpha^N} w_t - \frac{1}{2} \theta \left( \frac{\theta}{1-\theta} \right)^2 \sum_{j=1}^{N} \frac{1 + \alpha + \ldots + \alpha^{j-1}}{1 + \alpha + \ldots + \alpha^N} \gamma_j^2 \delta_j \eta_{t+N-j-q} \right) \]

Now let’s compare the above solutions with the case where we only maximize the terminal utility as described in “DynamicModel”. We use superscript \( t.u. \) to denote terminal utility. Then:

\[ p^{t.u.}_t = F(L)e_t^d + \frac{\gamma^{t.u.} \delta^{t.u.}}{\alpha} \theta (\eta_{t-q} + \frac{1}{\alpha} \eta_{t-q+1} + \ldots + \frac{1}{\alpha^q} \eta_t) \]

where

\[ \gamma^{t.u.} = \gamma \alpha^{i-1}, \]

\[ \delta^{t.u.} = \frac{1 \pm \sqrt{1 - 4 \frac{\gamma^{t.u.}_i \sigma^2_i}{\alpha^2 + \sigma^2_i} \left( \frac{\theta}{1-\theta} \right)^2 (D(\alpha-1))^2 \sigma^2_d}}{2 \frac{\gamma^{t.u.}_i \sigma^2_i}{\alpha^2 + \sigma^2_i} \left( \frac{\theta}{1-\theta} \right)^2} \]

for \( i = 1, \ldots, N \). We have already shown that those adjusted risk aversions play crucial roles in both scenarios. Now it’s clear that all differences in asset pricing also come from those adjusted risk aversion. In particular, in the case \( N \leq q + 1 \),

\[ \gamma^{cons}_i = \frac{\gamma \alpha^{i-1}}{1 + \alpha + \alpha^2 + \ldots + \alpha^{i-1}} \leq \gamma^{t.u.}_i = \gamma \alpha^{i-1}, \quad i = 1, \ldots, N \]

where we use superscript \( cons \) to denote the case where agents care about intertemporal substitution.
In other words, when the trader’s goal is to maximize discounted present value of utility that comes from each period’s consumption, rather than only maximizing the terminal consumption utility, the trader behaves ‘as if’ he is less risk averse in the former case if we observe his behaviour period by period, although the absolute risk aversion parameter has been kept constant in both cases! An intuitive explanation for this finding could be: Suppose an investment bank evaluates its traders’ performances according to two criteria: either by a history of his profit making abilities, or his terminal profit making ability. Then the trader will act more aggressively under the first criteria as he boldly looking for investment opportunities throughout his career to increase his performance grades, rather than overly worrying that one step of miss calculation could cause disastrous results to his terminal wealth utility, thus will ruin his whole career. To see this more clearly, notice that, as $N \to \infty$ under the case $N \leq q + 1$: $\gamma^{cons}_N \to \gamma (1 - \frac{1}{\alpha})$ while $\gamma^{t.u.}_N \to \infty$ ! Given the fact $\alpha \approx 1, \gamma^{cons}_N \approx 0$, i.e, traders look like risk-neutral when the termination horizon is far and he cares about intertemporal consumption. The $\gamma^{t.u.}_N \to \infty$ result may seem weird at first glance; although it doesn’t have to be. In fact, given $N \to \infty$, it is natural for the trader to put all of his wealth on the risk-free asset and enjoy the constant growth rate $1 + r$ into very far future and then consume a very large amount of terminal consumption; thus, display infinite large risk aversion.

Another fact worth discussing is that while $\gamma^{t.u.}_i$ is increasing with respect to $i$, $\gamma^{cons}_i$ is decreasing with respect to $i$, where $i$ denotes number of periods traders will remain active in the market.

Finally we will compare two $V(w_t)$’s. For convenience,

\[
V(w^{cons}_t) = -C_N \exp\left(-\frac{\gamma \alpha^N}{1 + \alpha + \ldots + \alpha^N} w_t - \frac{1}{2} \frac{\theta}{1 - \theta} \sum_{j=1}^{N} \frac{1 + \alpha + \ldots + \alpha^j}{1 + \alpha + \ldots + \alpha^N} \gamma^2_j \delta^2_j \eta_{t+N-j-q}\right)
\]

\[
V(w^{t.u.}_t) = -\exp\left(-\gamma \alpha^N w_t - \frac{1}{2} \frac{\theta}{1 - \theta} \sum_{j=1}^{N} \gamma^2_j \delta^2_j \eta_{t+N-j-q}\right)
\]

Since I do not calculate $C_N$ explicitly, I cannot compare two value functions quantitatively. But it is interesting to see that after abstracting from effects of different $\gamma_j$’s, $\delta_j$’s, $\eta_{t+N-j-q}$ itself plays different roles in determining $V(w_t)$’s. In particular, while all future investment opportunities $\eta_{t+N-j-q}$ are equally important in the terminal utility case, traders give weight $1 + \alpha + \ldots + \alpha^j$ to $\eta_{t+N-j-q}$ once there is an intertemporal substitution effect when they make their consumption decision at each period. Since the weights increases with $j$, traders put more importance to latest investment opportunities as they influence recent consumption decisions more heavily.

Recall that as in “DynamicModel”, so far we only consider cases $N \leq q + 1$ under which $\eta_{t+N-j-q}, j = 1, \ldots, N$ are all known to traders at time $t$. Now we turn to the case $N > q + 1$ below.
A.4.3 Cases when $N > q + 1$  Again, for the same reason as we described in “DynamicModel”, we have to adjust our backward induction procedure once we reach time period $t + N - q - 2$. The speculative trader’s problem then is given by:

$$
V(w_{t+N-q-2} - c_{t+N-q-2}) = \max E_{t+N-q-2}V(w_{t+N-q-1})
$$

$$
= \max E_{t+N-q-2} - C_{q+1}\exp(-\frac{\gamma \alpha^{q+1}}{1 + \alpha + \ldots + \alpha^{q+1}} w_{t+N-q-1})
$$

$$
\exp(-\frac{1}{2}(\frac{\theta}{1-\theta})^2 \sum_{j=1}^{q+1} 1 + \alpha + \ldots + \alpha^{j-1} \frac{\gamma_j \delta_j \eta_j^2}{1 + \alpha + \ldots + \alpha^{q+1}})
$$

subject to

$$
w_{t+N-q-1} = z_{t+N-q-2|t+N-q-2}(p_{t+N-q-1} + d_{t+N-q-1}) +
$$

$$
\alpha(w_{t+N-q-2} - c_{t+N-q-2} - z_{t+N-q-2|t+N-q-2}p_{t+N-q-2|t+N-q-2})
$$

To avoid long subscripts, let’s denote $s = t + N - q - 2$ and rewrite the above problem as:

$$
V(w_s - c_s) = \max -AE_s \exp(Bw_{s+1} + C\eta_{s+1}^2)
$$

s.t: $w_{s+1} = z_{s|s}(p_{s+1} + d_{s+1}) + \alpha(w_s - c_s - z_{s|s}p_{s|s})$

where

$$
A = C_{q+1}\exp(-\frac{1}{2}(\frac{\theta}{1-\theta})^2 \sum_{j=2}^{q+1} 1 + \alpha + \ldots + \alpha^{j-1} \frac{\gamma_j \delta_j \eta_j^2}{1 + \alpha + \ldots + \alpha^{q+1}})
$$

$$
B = -\frac{\gamma_1 \alpha^{q+1}}{1 + \alpha + \ldots + \alpha^{q+1}}
$$

$$
C = -\frac{1}{2}(\frac{\theta}{1-\theta})^2 \frac{\gamma_1 \delta_1}{1 + \alpha + \ldots + \alpha^{q+1}}
$$

are all constants once we condition on information at $t + N - q - 2$. By looking at (A.26) we realize we basically get the same problem as in the terminal utility maximization problem with different constants $A, B$ and $C$ as the only difference. Guess the equilibrium price

$$
p_s = F(L)d_s + G(L)\eta_s
$$

and solve the problem (A.26) by method of undetermined coefficients, we will arrive\textsuperscript{12}:

$$
F(L) = \frac{D(\alpha^{-1}) - D(L)}{1 - \alpha L}
$$

\textsuperscript{12}Notice that in “DynamicModel”, I do not consider weights on demand, thus the term $\frac{\theta}{1-\theta}$ doesn’t appear in that document.
\[ G(L) = \frac{G_0 + \frac{\theta}{1-\theta} B(D(\alpha^{-1})^2 \sigma_d^2 + \frac{\sigma^2 G_0^2}{1-2C \sigma_n^2}) L \alpha^{q+1}}{1 - \alpha L} \]

Plugging \( B, C \) into the above expression and remove the pole at \( \alpha^{-1} \) yield a quadratic equation \( G_0 \) must satisfy:

\[ \frac{\gamma \sigma_n^2}{1 + (\frac{\theta}{1-\theta})^2 \gamma^2 \delta_1 \sigma_n^2} G_0^2 - G_0 + \frac{\theta}{1 - \theta} \gamma ((D(\alpha^{-1}))^2 \sigma_d^2 = 0 \]  
\[ \frac{\gamma \sigma_n^2}{1 + (\frac{\theta}{1-\theta})^2 \gamma^2 \delta_1 \sigma_n^2} G_0^2 - G_0 + \frac{\theta}{1 - \theta} \gamma ((D(\alpha^{-1}))^2 \sigma_d^2 = 0 \]  

(A.28)

Thus, we still will confront the same multiplicity problem as we found in the terminal utility maximization problem. In fact, I claim the multiplicity problem will exacerbate if we consider intertemporal consumption. To see this more clearly, recall that the corresponding quadratic equation related to \( G_0 \) we get in the terminal utility maximization problem is given by:

\[ \frac{\gamma \sigma_n^2}{1 + (\frac{\theta}{1-\theta})^2 \gamma^2 \delta_1 \sigma_n^2} G_0^2 - G_0 + \frac{\theta}{1 - \theta} \gamma ((D(\alpha^{-1}))^2 \sigma_d^2 = 0 \]  

(A.29)

where

\[ \delta_1 = 1 \pm \sqrt{1 - \frac{4 \gamma^2 \sigma_n^2 ((\frac{\theta}{1-\theta})^2 (D(\alpha^{-1}))^2 \sigma_d^2}{2 \gamma \sigma_n^2 (\frac{\theta}{1-\theta})^2}} \]

could take 2 values. (Here, we implicitly assume that \( 1 - \frac{4 \gamma^2 \sigma_n^2 ((\frac{\theta}{1-\theta})^2 (D(\alpha^{-1}))^2 \sigma_d^2}{2 \gamma \sigma_n^2 (\frac{\theta}{1-\theta})^2} \geq 0 \) is always satisfied so that \( \delta_1 \) is well defined.) Then, for each possible \( \delta_1 \) fixed, we claim: If there are two \( G_0 \)'s are solutions of (A.29), then (A.28) must also have two \( G_0 \) solutions. This can be easily verified since:

\[ 1 - \frac{4 \gamma^2 \sigma_n^2}{1 + (\frac{\theta}{1-\theta})^2 \gamma^2 \delta_1 \sigma_n^2} \theta \gamma ((D(\alpha^{-1}))^2 \sigma_d^2 \geq 0 \]  

(A.30)

implies

\[ 1 - \frac{4 \gamma^2 \sigma_n^2}{1 + (\frac{\theta}{1-\theta})^2 \gamma^2 \delta_1 \sigma_n^2} \theta \gamma ((D(\alpha^{-1}))^2 \sigma_d^2 \geq 0 \]  

(A.31)

In other words, considering the fact that we have 2 \( \delta_1 \)'s, if we have 4 equilibrium prices in the terminal utility maximization case, then we must have 4 equilibrium prices in the corresponding intertemporal consumption case. So, the other way around, we could have a set of calibrations under which we have 2 equilibrium prices in the terminal utility maximization case while we will have 4 equilibrium prices in the intertemporal consumption case. The backward induction can be continued until we reach current time \( t \), which could yields \( 2^{N-\theta} \) equilibria as we showed in the terminal wealth case.