A Unified Framework for Monetary Theory and Policy Analysis

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Introduction

We develop a framework that unifies micro and macro models of monetary exchange

Why?

Existing macro models are reduced-form models ...

Most existing micro models impose severe restrictions ...

Attempts to generalize micro models are very complicated ...

Outline for Today:

1. A brief review of the related literature
2. A basic version of our model
3. Policy implications and extensions
Micro Foundations of Money

Some things we ought not take for granted:

• the demand for money
  *money should not be a primitive in monetary theory*

• the auctioneer, multilateral trade, price-taking behavior
  *a model of money cannot be “too Walrasian”:
  it should build in frictions and strategic elements explicitly*

• commitment/enforcement, monitoring/memory
  *an essential role for money requires a double coincidence problem, imperfect commitment/enforcement, and imperfect memory/monitoring*
1st Generation Models

Assume $m \in \{0, 1\}$ and $q$ fixed

Implies BE

$$V_1 = b_1 + \alpha \sigma (1 - M) [u(q) + W_0]$$

$$+ [1 - \alpha \sigma (1 - M)] W_1$$

$$V_0 = b_0 + \alpha \sigma M [W_1 - c(q)]$$

$$+ (1 - \alpha \sigma M) W_0$$

Note: typically, $W_m = \beta V_m$

IR conditions:

$$u(q) + W_0 \geq W_1$$

$$W_1 - c(q) \geq W_0$$

Results:

Existence of equilibrium with valued money, welfare ...
2nd Generation Models

Keep $m \in \{0, 1\}$ but endogenize $q$

Add BS: choose $q$ to solve

$$\max_q \left[ u(q) + W_0 - T_1 \right]^\theta \left[ -c(q) + W_1 - T_0 \right]^{1-\theta}$$

IC conditions

$$u(q) + W_0 \geq W_1$$
$$W_1 - c(q) \geq W_0$$

Results:
As above (existence, welfare...), plus we can discuss price $p = 1/q$

Example:
Assumptions $\Rightarrow q < q^*$ but $q \rightarrow q^*$ as $\beta \rightarrow 1$.
This “looks like” standard monetary inefficiency (say, in CIA model)
1st and 2nd Generation Shortcomings

Extreme assumptions on inventories of money: $m \in \{0, 1\}$

- Restrictive upper bound on money holdings
  \[ \Rightarrow \text{models not useful to analyze policy experiments (e.g. changes in the money supply)} \]

- Indivisibility of money often drives results:
  Berentsen and Rocheteau (*JME* 2002)
  - No trade inefficiency (in 1st and 2nd generations)
  - Too much trade inefficiency (in 2nd generation)
3rd Generation Models

Let $m \in \mathcal{M} \subset R_+$ and let $F(\tilde{m})$ be the CDF of money holdings

$$V(m) = b(m) + (1 - 2\alpha \sigma)W(m)$$

$$+ \alpha \sigma \int \{ u[q(m, \tilde{m})] + W[m - d(m, \tilde{m})] \} \, dF(\tilde{m})$$

$$+ \alpha \sigma \int \{ W[m + d(\tilde{m}, m)] - c[q(\tilde{m}, m)] \} \, dF(\tilde{m})$$

where $q(m, \tilde{m})$ and $d(m, \tilde{m})$ solve

$$\max_{q,d} [u(q) + W(m - d) - T(m)]^\theta$$

$$\times [-c(q) + W(\tilde{m} + d) - T(\tilde{m})]^{1-\theta}$$

s.t.

$$u(q) + W(m - d) \geq W(m)$$

$$W(\tilde{m} + d) - c(q) \geq W(\tilde{m})$$

$$d \leq m$$

Typically: $W(m) = \beta V(m)$
3rd Generation Issues

- Model with $\mathcal{M} = \{0, 1, \ldots, \bar{m}\}$
  

- Model with $\mathcal{M} = \mathbb{R}_+$
  
  Analytic results (even existence) very difficult
  
  Some numerical examples: Molico (PhD thesis 1997)

One (big) complication: endogenous $F(m)$

$\rightarrow$ Shi (*Econometrica* 1997) provides a trick:

  - the $\infty$ family

$\rightarrow$ We provide a different trick:

  - competitive markets

Potential advantages of our approach:

* do not need “unpalatable” $\infty$ family
* can use standard and simple bargaining theory
* do not have to ignore incentive problems
* having some centralized trading can be desirable
Our Model

- Discrete time, infinite horizon, \([0, 1]\) continuum of agents

- 2 types of nonstorable, perfectly divisible goods: *general* and *special*

- *General* goods are consumed and produced by everyone
  \(U(Q)\) and \(C(Q)\) are utility of consumption and production
  \(U' > 0, U'' \leq 0\)

- *Special* goods (subject to double-coincidence problem)
  \(\Rightarrow 3\) types of meetings:
  - double-coincidence (with prob. \(\delta\))
  - single-coincidence (with prob. \(\sigma\))

  \(u(q)\) and \(c(q)\) are utility of consumption and production
  \(u(0) = c(0) = 0, u' > 0, c' > 0, u'' < 0, c'' \geq 0\), and
  \(u(\bar{q}) = c(\bar{q})\) for some \(\bar{q} > 0\).
  Let \(q^*\) be defined by \(u'(q^*) = c'(q^*)\)

- Key ingredients
  - two sub-periods (day and night)
  - special goods can only be produced during the day
  - general goods can only be produced at night
  - \(C(Q) = Q\)
Trading

Feasible trades:

- day: special for special goods and money for special goods
- night: general for general goods and money for general goods

Assume:

- day: decentralized trading
- night: centralized trading

Note:

Agents cannot commit to future actions; and no “memory”
⇒ role for money in decentralized trading
Value Function

\[ V(m) = \alpha \sigma \int \{ u[q(m, \tilde{m})] + W[m - d(m, \tilde{m})] \} dF(\tilde{m}) \]
\[ + \alpha \sigma \int \{ W[m + d(\tilde{m}, m)] - c[q(\tilde{m}, m)] \} dF(\tilde{m}) \]
\[ + \alpha \delta \int B(m, \tilde{m})dF(\tilde{m}) + (1 - 2\alpha \sigma - \alpha \delta)W(m) \]

\[ V(m) \] : value of entering the search market with \( m \) dollars
\[ W(m) \] : value of entering the centralized market with \( m \) dollars
\[ F(m) \] : CDF of money holdings (endogenous)
\[ M \] : total stock of money; \( \int m dF(m) = M \)

\[ q(m, \tilde{m}) \] : quantity of special good exchanged if the buyer has \( m \) and the seller \( \tilde{m} \) dollars
\[ d(m, \tilde{m}) \] : dollars exchanged if the buyer has \( m \) and the seller \( \tilde{m} \) dollars
\[ B(m, \tilde{m}) \] : expected net payoff from a barter trade if the buyer has \( m \) and the seller \( \tilde{m} \) dollars

Next: look at \( W(m) \); determine single-coincidence terms of trade \( q(m, \tilde{m}) \) and \( d(m, \tilde{m}) \); and value of barter \( B(m, \tilde{m}) \)
Centralized Market

\( \beta \): discount factor

\( \phi \): price of money in terms of general goods

In the centralized market agents solve:

\[
W(m) = \max_{X,Y,m'} U(X) - Y + \beta V(m')
\]

s.t. \( X + \phi m' = Y + \phi m \)

\[
\Leftrightarrow \quad W(m) = \max_{X,m'} U(X) - X + \phi m - \phi m' + \beta V(m')
\]

\[
\Leftrightarrow \quad W(m) = U(X^*) - X^* + \phi m + \max_{m'} \{\beta V(m') - \phi m'\}
\]

where \( U'(X^*) = 1 \)

**Observation:** \( m' \) is independent of \( m \)

**Corollary 1:** \( V(m) \) strictly concave \( \Rightarrow F(m) \) degenerate

**Corollary 2:** \( W(m) \) is affine, \( W(m) = W(0) + \phi m \)
Decentralized Market: Terms of Trade

Double-coincidence meetings

Symmetric Nash solution ⇒

(i) each agent produces $q^*$ for the other, and
(ii) no money changes hands

$⇒ B (m, \tilde{m}) = b + W (m); \quad \text{where} \quad b \equiv u (q^*) - c (q^*)$

Single-coincidence meetings

In general BS is

$$\max_{q,d} [u (q) + W (m - d) - T (m)]^\theta$$

$$\times [ -c (q) + W (\tilde{m} + d) - T (\tilde{m})]^{1-\theta}$$

subject to

$$u (q) + W (m - d) \geq W (m)$$

$$-c (q) + W (\tilde{m} + d) \geq W (\tilde{m})$$

$$d \leq m$$

$T (m)$: the threat point of an agent with $m$ units of money

Linear $W$ and $T (m) = W (m)$ ⇒ BS becomes:
\[
\max_{q,d} \left[u(q) - \phi d\right]^\theta \left[-c(q) + \phi d\right]^{1-\theta} \text{ s.t } d \leq m
\]

\[
\Rightarrow
\]

\[
q = \begin{cases} 
q(m) & m < m^* \\
q^* & m \geq m^*
\end{cases}
\]

\[
d = \begin{cases} 
m & m < m^* \\
m^* & m \geq m^*
\end{cases}
\]

where

\[
\phi m^* = \theta c(q^*) + (1 - \theta)u(q^*)
\]

and \(q(m)\) solves FOC:

\[
\phi m = \frac{\theta u'(q)c(q) + (1 - \theta)c'(q)u(q)}{\theta u'(q) + (1 - \theta)c'(q)} \equiv f(q)
\]
Observation

\( q(m) \) solves FOC:

\[
\phi_m = \frac{\theta u'(q)c(q) + (1 - \theta)c'(q)u(q)}{\theta u'(q) + (1 - \theta)c'(q)} = f(q)
\]

\[
q'(m) = \frac{\phi[\theta u' + (1 - \theta)c']^2}{u'c'[\theta u' + (1 - \theta)c'] + \theta(1 - \theta)u - c)(u'c'' - c'u'')}
\]

\[
\lim_{m \to m^*} q'(m) = \frac{1}{u'(q^*)} \left[ \frac{\phi}{1 + \theta(1 - \theta)u^* - c^*} \frac{c''(q^*) - u''(q^*)}{u'(q^*)^2} \right]
\]

\[
u'(q^*) q'(m^*) < \phi \text{ if } \theta < 1
\]

\[
u'(q^*) q'(m^*) = \phi \text{ if } \theta = 1
\]
Value Function

\[ V(m) = \alpha \delta [b + W(m)] + \alpha \sigma E_F \left\{ -c[q(\tilde{m})] + W[m + d(\tilde{m})] \right\} \\
+ \alpha \sigma \left\{ u[q(m)] + W[m - d(m)] \right\} \\
+ (1 - 2\alpha \sigma - \alpha \delta)W(m) \]

\[ = \alpha \delta b + \alpha \sigma E_F \left\{ -c[q(\tilde{m})] + \phi d(\tilde{m}) \right\} \\
+ \alpha \sigma \left\{ u[q(m)] - \phi d(m) \right\} + W(m) \]

Recall: \( W(m) = U(X^*) - X^* + \phi m + \max_{m'} \{ \beta V(m') - \phi m' \} \)

let: \( \kappa = \alpha \delta b + \alpha \sigma E_F \left\{ -c[q(\tilde{m})] + \phi d(\tilde{m}) \right\} + U(X^*) - X^* \)

and let: \( v(m) = \kappa + \alpha \sigma \left\{ u[q(m)] - \phi d(m) \right\} \)

Then the value function can be written as

\[ V(m) = v(m) + \phi m + \max_{m'} \{ \beta V(m') - \phi m' \} \]
**Value Function: Existence and Uniqueness**

\[ V(m) = \max_{m'} \{v(m) + \phi m - \phi m' + \beta V(m')\} \]

**Note:** The RHS defines a contraction on a complete metric space

\[ \mathcal{B}_\phi = \{f : \mathbb{R} \to \mathbb{R} | f(x) = g(x) + \phi x, \ g(x) \in \mathcal{B}\} \]

where:

\[ \mathcal{B} = \{g : \mathbb{R} \to \mathbb{R} | g \text{ is cont. and bounded in the sup norm}\} \]

Hence \( \exists! V(m) \) in \( \mathcal{B}_\phi \) solving BE

**Remark:** Our methods also work for \( V(m, \phi, F) \)
Value Function: Properties

\[ V(m) = v(m) + \phi m + \max_{m'} \{ \beta V(m') - \phi m' \} \]

where \( v(m) = \kappa + \alpha \sigma \{ u[q(m)] - \phi d(m) \} \)

If \( u \) and \( c \) are \( C^n \) then \( V \) is \( C^{n-1} \) a.e. and

\[ V'(m) = \begin{cases} \alpha \sigma \phi e(q) + (1 - \alpha \sigma) \phi & m < m^* \\ \phi & m \geq m^* \end{cases} \]

where \( e(q) = \frac{u'(q)q'(m)}{\phi} \) is the gain from having an additional unit of real balances when bargaining

**Lemma:** for \( \theta \in (0, 1) \)

(i) \( \lim_{m \to m^*} V'(m) < \phi \)

(ii) \( V''(m) < 0 \) for \( m < m^* \) if \( u' \) is log concave

**Note:** if \( \theta = 1 \) then:

(i) \( \lim_{m \to m^*} V'(m) = \phi \)

(ii) \( V''(m) < 0 \) for \( m < m^* \)
Solution to the Agent’s Problem in the Centralized Market

\[
\max_{m+1} \{ \beta V(m+1) - \phi m+1 \}
\]

Derivative is:

\[
\begin{align*}
\beta \phi_{+1} - \phi \\
\beta \{ \alpha \sigma u'[q(m+1)]q'(m+1) + (1 - \alpha \sigma) \phi_{+1} \} - \phi \\
\text{for } m+1 \geq m_{+1}^* \\
\text{for } m+1 < m_{+1}^*
\end{align*}
\]

where \( m_{+1}^* \equiv [\theta c(q^*) + (1 - \theta) u(q^*)] / \phi_{+1} \)

**Corollary.** In *any* equilibrium:

(i) \( \beta \phi_{+1} \leq \phi \), and

(ii) \( m_{+1} < m_{+1}^* \)

\[ \Rightarrow m_{+1} \text{ is characterized by the FOC} \]

\[ \beta V'(m_{+1}) - \phi m_{+1} = 0 \]

Recall: under mild conditions, \( V''(m) < 0 \) for \( m < m^* \)

Thus the FOC has a *unique* solution, which is *independent of* \( m \)

\[ \Rightarrow \]

\[ F(m) \text{ degenerate } \forall t > 0 \text{ in } \text{any equilibrium} \]
Equilibrium

**Definition.**
Given $M$ an equilibrium is a list $(V, q, d, \phi, F)$ satisfying:

1. the BE

$$V(m) = \max_{m'} \{v(m) + \phi m - \phi m' + \beta V(m')\}$$

2. the BS

$$q = \begin{cases} q(m) & m < m^* \\ q^* & m \geq m^* \end{cases} \quad d = \begin{cases} m & m < m^* \\ m^* & m \geq m^* \end{cases}$$

where $m^* = \frac{\theta c(q^*) + (1-\theta)u(q^*)}{\phi}$, and

$q(m)$ solves $\theta [ -c(q) + \phi m ] u'(q) = (1 - \theta) [ u(q) - \phi m ] c'(q)$

3. the FOC $\beta V'(m') = \phi$

4. $F$ degenerate at $m = M$
Analysis

Substitute $V'$ into FOC:

$$\phi_t = \beta [\alpha \sigma u'(q_{t+1})q'_{t+1}(m_{t+1}) + (1 - \alpha \sigma)\phi_{t+1}]$$

Use BS to eliminate $\phi$ and $q'$:

$$e_{\theta}(q_{t+1}) = 1 + \frac{f_{\theta}(q_t) - \beta f_{\theta}(q_{t+1})}{\alpha \sigma \beta f_{\theta}(q_{t+1})}$$

$$f_{\theta}(q) \equiv \frac{\theta cu' + (1 - \theta)uc'}{\theta u' + (1 - \theta)c'} \text{ and } e_{\theta}(q) \equiv \frac{\theta u' + (1 - \theta)c'}{u'c'[\theta u' + (1 - \theta)c'] + \theta(1 - \theta)(u - c)(u'c'' - c'u'')}$$

A monetary equilibrium is simply a sequence $\{q_t\}$ with $q_t \in (0, q^*]$ that solves this difference equation

Given $q_t$, the rest of the allocation is given by

$$d_t = M$$
$$\phi_t = f_{\theta}(q_t)/M$$
$$m_{t+1} = M \text{ with prob. 1}$$

General Results:

(i) Classical Neutrality

(ii) Inefficiency: $q_t < q^*$ for all $t$ in any equilibrium
Results (Steady State)

MSS solves

\[ e_\theta(q) = 1 + \frac{1 - \beta}{\alpha \sigma \beta} \]

A simple case. If \( \theta = 1 \), then equilibrium condition is

\[ \frac{u'(q)}{c'(q)} = 1 + \frac{1 - \beta}{\alpha \sigma \beta} \]

Results:

(i) If a MSS exists, it is unique
(ii) \( \exists \) MSS \( q^1 \) provided \( \frac{u'(0)}{c'(0)} > 1 + \frac{1 - \beta}{\alpha \sigma \beta} \)
(iii) \( q^1 < q^* \)
(iv) \( q^1 \to q^* \) as \( \beta \to 1 \); \( q^1 \to 0 \) as \( \alpha \sigma \beta \to 0 \)

General case. If \( 0 < \theta < 1 \), then \( \exists \) MSS \( q^\theta \) and \( q^\theta < q^1 \).
(Existence and uniqueness under mild conditions.)

Result: \( \theta < 1 \Rightarrow q^\theta \) is bounded away from \( q^* \) even as \( \beta \to 1 \)

Intuition. In general there are two distortions:

* “\( \beta \)-wedge” (standard in monetary models)
* “\( \theta \)-wedge” (hold up problem)
Monetary Policy

Suppose $M_{t+1} = (1 + \tau) M_t$

Form of injections: lump-sum transfers

Generalized steady state condition:

$$e_\theta(q) = 1 + \frac{1 + \tau - \beta}{\beta \alpha \sigma}$$

Results:

(i) Inflation reduces welfare
(ii) MSS exists iff $\tau \geq \beta - 1$
(iii) Friedman Rule is optimal
(iv) Friedeman Rule achieves $q^*$ iff $\theta = 1$

Intuition:
Monetary policy can correct the “$\beta$-wedge” but not the “$\theta$-wedge”
Welfare Cost of Inflation

Implication: Welfare costs of inflation can be much higher than predicted by standard reduced form models

Intuition: *Envelope Theorem*
Extensions and Applications

→ Dynamics

  nonstationary, cyclic, sunspot and chaotic equilibria

→ Real shocks

  can have $d < m$ with positive prob. (endogenous velocity)

→ Monetary shocks

  only persistent inflation affects $\phi$ (negatively)

→ Endogenous search or specialization

  also makes velocity $= \alpha \sigma$ endogenous

→ Heterogeneity (e.g. through “limited participation”)

  $F$ nondegenerate yet tractable
  inflation may increase welfare (through redistribution)

→ Empirical implementation

  (e.g. use the model to quantify welfare cost of inflation)