Leadership and peer effects

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Abstract

Consider an organization composed by a Leader and by a finite group of agents (or followers). The Leader has some ideal organization composition or vector of preferred actions one for each agent and can invest in costly socialization trying to instill this “corporate culture” in all the agents of the organization. Each agent has as well her ideal action. When an agent makes a decision each period her behavior is driven by two competing motives: she wants her behavior to agree with her personal ideal action and she wants also her behavior to be as close as possible to the average behavior of her peers. Ideal actions or preferences evolve over time. There are two sources of preference (and therefore, action) change. On the one hand, there exists a costly corporate socialization effort exerted by the Leader trying to transform the ideal action of each agent into his own ideal action. On the other hand, each agent’s ideal action changes in the direction of actual behavior (self-persuasion or cognitive dissonance). We are interested in the long-run outcomes of this situation and in particular in the ability of the Leader to fully instil the corporate culture in the members of the organization.

1 Introduction

A crucial problem faced by any large organization is to build a high performance team culture in order to encompass the experiences, knowledge, and language shared

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by those belonging to it. This typically involves aligning agents incentives or preferences in an optimal way. But usually some mix of homogeneity and diversity in the preferences and behaviour of the members might be optimal for the goals of management; although many corporations are interested and are benefitted by creating a corporate culture, they also prefer that different subgroups of the organization display different preferences and behaviour as profit-oriented or mission-oriented (Prendergast, 2007).

In this context, a problem faced by any leadership is the presence in closed groups of strong forces that lead to preference and behaviour homogeneity. These are psychological forces such as informational conformity or peer effects (Fischer and Huddart, 2008) and consistency or cognitive dissonance (Kuran and Sandholm, 2008). According to the former, individuals prefer to conform with the average behaviour of their peers which in turn depends on the average preferences of the group. According to the latter, individuals tend to update their preferences and beliefs to their current behaviour that it is indeed adjusted strategically to the influence of the peer. Consequently, both conformism and consistency generate a strong tendency towards homogeneity.

This paper is about charismatic leadership and its capacity and limits in influencing large and closed organizations where peer effects are operating. Leadership matters both at the company and at the country or society level as recent research has shown (Jones and Olken, 2005, Bertrand and Schoar, 2003, Bloom and Van Reenen, 2007). So far economic theory has focused mostly on a concept of leadership based on power that has the ability to influence others’ behavior by rewards and punishment rather than leadership based on prestige or charisma that has the ability to induce others to follow absent the power to compel. Accordingly, leadership has been formalized using the principal agent model (Hermalin, 1998). Notwithstanding, the Leader may find more suitable to make agents behave as he desires by socializing the agents to the “right” preferences, instead of providing them the “right” monetary incentives (for instance, the army, a religious organization or a revolutionary party). Namely, we know from agency theory that often, money cannot be used to align incentives (see Prendergast, 2007). In this situation, the leader can make the agents conform to his views through indoctrination or socialization.

Some very recent work focuses in the role of leadership in an organization as a coordination device or a motivator facing a coordination problem in a changing environment (Bolton et al., 2012, Van den Steen, 2010c). Therefore, this literature shares the idea of Kreps (1990) of corporate culture as an equilibrium selection mechanism. It explains the role of the leader as someone that helps to solve the
coordination problem by influencing the equilibrium behaviour and beliefs of the followers. Our approach is different. To the best of our knowledge, our model is the first attempt to analyse the role of the leader as a shaper of preferences. We adhere to Weber’s idea of charismatic leadership which is distinct from formal authority. It is the ability to induce others to follow absent the power to provide formal contractual incentives.

A leader is someone with followers, who follow voluntarily. The critical question then becomes how does a leader induce others to behave or to follow? He can influence the preferences of the followers because he has got some given charisma towards them or because of purposeful and costly socialization efforts which obviously, might include leading with his own example. Nevertheless, this influence competes with other well-known forces that drive the change and evolution of preferences such as the existence of endogenous social norms, that is, social norms of behavior established by your peers, the presence of intransigent agents (agents that are not influenced by charisma or socialization) or the inertia of the agents’ idiosyncratic preferences, among others.

The key features of our model are the following: Consider an organization composed by a Leader (or a Principal) and by a finite group of followers (or agents). At a given period all agents choose actions simultaneously. An agent has an intrinsic ideal action (or preference), but when she makes a decision her behavior is driven by two competing motives: she wants her behavior to agree with her personal ideal action and she also wants her behavior to be as close as possible to the average behavior of her peers. The relative relevance of each contribution depends on the innate conformity level of each agent. On the other side, the corporate culture of the organisation tailored by the Leader is some ideal composition or vector of preferred actions, one for each agent. We assume these leader’s preferred actions to be time independent and exogenous. The leader is also a charismatic leader, which means that he has a non-negative influence in the change of the ideal action of each agent towards the leader’s target concerning this agent. Therefore, we can think that the Leader’s charisma is modelled as a vector of non negative numbers, each one associated to each agent. We might think of a situation as this one where the leader has a fixed and exogenously given charisma but also there are scenarios where the leader can invest in costly socialization trying to instil his “corporate culture” in all the agents of the organization. In this context the charisma vector has to be interpreted as a vector of socialization or indoctrination efforts targeted to the different members of the organization.

Ideal actions or preferences evolve over time. From an initial condition of ideal
action, there are two possible sources for this evolution. On the one hand, there exists a costly socialization effort exerted by the Leader trying to transform the ideal action of each agent into his preferred action. Alternatively, the Leader has a given charisma and each agent’s ideal action changes in the direction of the leader’s target for this agent. On the other hand, each agent’s ideal action changes in the direction of actual behavior (self-persuasion or cognitive dissonance). These are, in essence, two psychological sources of change, conformism or to follow the leader and self-consistency. Each source has associated a positive parameter leading the different levels of conformism and consistency, respectively.

We are interested in the long-run outcomes of the above situation and, in particular, in the ability of the Leader to fully instil the organizational culture in the members of the organization. The main findings of our work are the following. We first analyze as a benchmark the case in which there is no leader in the group. We show that this organization tends towards complete homogeneity. Every agent has as ideal action in the steady state, and plays in equilibrium, a weighted average of the ideal actions in the initial condition of the group. This result is robust even if individuals have different conformity weights (the conformity weight measures the intensity of the social, endogenous, norm). A awkward result is that if there are non-conformist members in the group, i.e. individuals with a conformity weight of zero, the long run common ideal action of the rest of the organization is exclusively determined by this type of individuals.

We turn next to the case in which there is a leader with a given charisma and a vector of target ideal actions, one for each member of the organization. Now, in contrast with the situation with no leader, the initial condition of the organization does not affect the final steady state provided the leader has positive levels of charisma towards all the followers. We show that if the leader has the same target ideal action for all the organization, then in the unique steady state of the dynamics, he will succeed in instilling this ideal action. But, if the leader has different targets for different agents, then there is not a steady state where he never success in the sense that he never instills the ideal action for each agent. We define a measure of the success of the Leader as the average distance between the steady state preferences of the members of the organization and the targets of the leader and, we analyze its determinants. The higher is his charisma and the more concentrated it is, the higher is his success. But the higher are the peer effects, that is, higher levels of conformism and consistency, the lower is his success. Remarkably and showing the limits of charismatic leadership, the higher is the purposed diversity of the leader for the organization, the smaller is his success. The organization ends up always
less heterogeneous than what the leader desires. The presence of a subset of intransigent agents, i.e. non-conformist and inconsistent agents with whom the leader has no charisma at all, reduces the success of the leader. The final corporate culture of the rest of the followers depends not only on the average goals of the leader but also on the initial ideal actions of those intransigent agents.

We show similar results in a situation where the Leader can choose the levels of costly socialization efforts trying toindoctrinate the agents. The cost of socialization also depends on the “cultural” distance between the leader’s socialization target for each agent and the current ideal action of the latter. (We solve the dynamic programming problem by the leader faced with positive peer effects.) We characterize the optimal socialization function and the steady state solution of the organization (the resulting corporate culture). The success of a socializer leader decreases in the levels of conformism and consistency and in the costs of socialization, moreover it increases in the discount factor. Similarly to charismatic leadership, he only fully succeeds if he pursues a completely homogeneous organization. But again, the higher is the purposed diversity of the leader for the organization, the smaller is his success. Moreover, we characterise the socialization effort in the steady state. If the leader pursues a completely homogeneous organization and once the steady state is reached, the peer effects maintain the homogeneous organizational culture. Hence, the leader does not need to invest any additional amount on socialization. But if the leader desires some degree of diversity, he has to invest a constant and time independent amount on socialization effort in each period, the same for each agent, in order to maintain the corporate culture. This amount is independent of the level of desired diversity of the leader. Paradoxically, it is decreasing in the levels of consistency and of conformism of the followers. The intuition is that high peer effects collaborate in maintaining the steady state corporate culture. Our model allows us to compare charismatic leadership with socialization and obtain clear predictions on the relative advantages of each kind of leadership. Charismatic leadership is more likely to perform better than socialization in terms of its level of success in groups with strong peer effects, that is, with high levels of conformism and consistency.

Related Literature.

This paper is related to several strands of the literature. First, it continues a large economics literature on conformity and endogenous social norms (see, for instance, Bernheim, 1994, Kandel and Lazear, 1992, Akerlof, 1997, Kandori, 2003, Fischer and Huddart, 2008, Huck, Kbler and Weibull, 2012). This literature analyzes in a static scenario the interplay between social norms and monetary incentives, for example, the condition under which monetary incentives crowd out social norms or
intrinsic motivation. The more recent literature on peer effects and social networks (such as Cabrales, Calvó-Armengol and Zenou, 2011) displays the dynamic feature of a changing network. But nevertheless, there is no purposeful investment on socialization by a leader.

Second, our model is related to the few existing works analyzing corporate culture in a dynamic setting. See, Kandori, 2003 and Rob and Zemsky, 2002. These are “effort” models and their main goal is again to analyze the interplay between norms and monetary incentives. But none of them includes the possibility that firm could invest in socialization.

More closely related to our work is a paper from the cultural transmission literature of Cordes, Richerson, McElreath and Strimling, 2008. These authors analyze how an entrepreneur can shape human behavior within a firm. They model it as biased transmission of cultural contents via a social learning process. Besides there are some important technical differences between our and their model. For instance, they work with two behavioral traits while we work with a continuous trait or preference. A major difference is that these authors take the entrepreneur’s charisma as a given parameter. In other words, they do not analyze the socialization activities of the entrepreneur.

Finally, our model ties with the recent literature from economic theory on leadership. As it was mentioned previously, economic theory has only recently devoted attention to roles of a leader distinct from the role as a motivator. Van den Steen (2010c) and Bolton et al. (2012) analyse the role of a leader as a coordinator. Some personal characteristic of the leader might affect the organizational culture. Traits of the leader such as his overconfidence or his resoluteness might influence and determine the endogenous equilibrium beliefs of the members of the organization. Therefore, this literature shares the idea of Kreps (1990) of corporate culture as an equilibrium selection mechanism. And it explains the role of the leader as someone that helps to solve the coordination problem. But to the best of our knowledge there is no formal analysis of the role of a leader as a shaper of preferences. Our paper is a first step in this direction.

2 The model

Consider an organization composed by a Leader L and by a finite group of followers or agents of fixed size \( N \). The leader has some ideal organization composition or vector of preferred actions \( S \) targeted one for each agent \( i \). We assume these leader’s preferred actions to be time independent and also exogenous. Each agent
$i = 1, \ldots, N$ holds an ideal action $S_i^{(t)} \in X = [0, 1]$ and at discrete times $t = 0, 1, 2, \ldots$ she chooses, simultaneously with all other agents, an action $x_i^{(t)} \in X$. When an agent makes a decision on $x_i^{(t)}$ her behavior is driven by two competing motives: she wants her behavior to agree with her personal ideal action $S_i^{(t)}$ and she wants also her behavior to be as close as possible to the average behavior of her peers. The leader does not participate directly in the choice of the actions $x_i^{(t)}$ by the agents but, as we will explain later on, he intervenes by influencing the evolution of the agents’ ideal actions $S_i^{(t)}$.

2.1 The stage game

Consider a $N$-player game under complete information where the instantaneous payoff function of agent $i = 1, \ldots, N$ at period $t$ is:

$$u_i^{(t)}(x_1^{(t)}, \ldots, x_N^{(t)}) = -\omega_i(x_i^{(t)} - \langle x^{(t)} \rangle)^2 - (x_i^{(t)} - S_i^{(t)})^2. \quad (1)$$

Here $x^{(t)} \equiv (x_1^{(t)}, \ldots, x_N^{(t)})$ is the vector of actions chosen by all the agents and $\langle x^{(t)} \rangle$ is the average action at period $t$. The first term of this instantaneous payoff function implies that individuals are conformists and incur in costs associated with choosing actions that diverge from endogenously determined social norms modeled as the average action of the peer. The parameter $\omega_i \geq 0$ describes the taste for conformity\(^2\) and measures the intensity of the social (endogenous) norm for agent $i$. The second term establishes individuals derive utility from choosing actions close to their intrinsic “bliss point” $S_i^{(t)}$.

Research in sociology and psychology indicate that individuals want to minimize the social distance between themselves and the social norms established by their peers. And also that there are costs (that might be psychological or material) associated with the distance between your actual behavior and your ideal behavior (see specially, the notion of “identity” of Akerlof and Kranton, 2005).

We will assume $N$ large enough that each agent has a negligible effect on the group average action $\langle x^{(t)} \rangle$, so that they view $\langle x^{(t)} \rangle$ as exogenously given. Therefore, the salient features of the norm are that it is endogenously determined and it is

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\(^1\)Hereafter for each function $f$ that takes values $f_j$ for agent $i$ we define the average value $\langle f \rangle$ as:

$$\langle f \rangle = \frac{1}{N} \sum_{j=1}^{N} f_j.$$  

\(^2\)This is the standard way to model conformity and social norms in economics (see references above).
not affected by the action of any powerful agent. Note that we do not impose any particular structure neither on actions \(x_i^{(t)}\) nor on preferred actions \(S_i^{(t)}\). The formulation of the agent’s payoff function posits that her payoff in each period depends on the actions only through (i) the distance between the agent’s action and her ideal and (ii) the distance between the agent’s action and the average action of the organization. Therefore, we run the analysis in this general setting without assuming for the time being, for instance an “effort” model, as it is usual in agency and economic theory.

We will assume that in each period there is immediate behavioral adjustment that maintains equilibrium play. Therefore, at period \(t\) agents play the Nash equilibrium \(\hat{x}_i^{(t)}\) of the simultaneous game in which each one has the above utility function. Next lemma states the unique pure Nash equilibrium of the above game.

**Lemma 1** There exists a unique pure Nash equilibrium \((\hat{x}_1^{(t)}, \ldots, \hat{x}_N^{(t)})\) of the stage game given by:

\[
\hat{x}_i^{(t)} = \frac{S_i^{(t)} + \tilde{\omega}_i \langle \langle S^{(t)} \rangle \rangle}{1 + \tilde{\omega}_i},
\]

where we define \(\tilde{\omega}_i \equiv \omega_i \left(1 - \frac{1}{N}\right)\) and the weighted average \(\langle \langle S^{(t)} \rangle \rangle \equiv \frac{\langle S^{(t)} \rangle}{1 + \tilde{\omega}_i}\).

See in Appendix 1 the proof of Lemma 1.

The Nash equilibrium action of each player in each period is a convex combination of her ideal action \(S_i^{(t)}\) and a weighted average \(\langle \langle S^{(t)} \rangle \rangle\) of the ideal actions of all members in the organization. The relative relevance of each contribution depends on the level of conformism of the agent. We will denote as “non-conformist” agents, those agents with \(\omega_i = 0\). A non-conformist agent does not care at all about the social norm and obviously her Nash equilibrium action in each period is \(\hat{x}_i^{(t)} = S_i^{(t)}\).

### 2.2 Evolution of preferences

Agents’ ideal actions or preferences \(S_i^{(t)}\) evolve over time. There are two sources of preference (and therefore, action) change. On the one hand, each agent’s ideal action changes in the direction of actual behavior acting as a self-persuasion or cognitive dissonance force. On the other hand, each agent’s ideal action changes in the direction of the Leader’s target for this agent. These are in essence two psychological sources for change: self-consistency and follow the leader, respectively.

We study three scenarios. The first one considers that there is no leader in the organization. Thus, at stage \(t\), every agent \(i\) plays her Nash equilibrium action and
updates her ideal action $S_i^{(t)}$ reducing the dissonance between her current action and her ideal. On the other two scenarios we analyze two different kinds of leadership: a charismatic leader or a socializer leader. In the former case, the principal has a fixed, time-independent, charisma $(c_1, \ldots, c_N)$. Each $c_i \geq 0$ is the influence of the leader in the change of the ideal action of agent $i$ towards the leader’s target $\bar{S}_i$ concerning this agent. Finally, for the socializer leader, an optimal control problem arises by considering that the leader chooses a temporal sequence of costly vectors of socialization or indoctrination efforts $(c_1^{(t)}, \ldots, c_N^{(t)})$ targeted to the different members of the organization.

3 Preference evolution in the absence of a leader

In this subsection we study the case where there is no leader in the organization. Hence, we assume that only cognitive dissonance matters for the evolution of preferences. When there is a discrepancy between the ideal action and the current equilibrium behavior, the ideal action of each member $i$ of the organization $S_i^{(t)}$ gradually evolves over time in the direction of her actual Nash equilibrium behavior according to the following difference equation:

$$S_i^{(t+1)} = \gamma_i \bar{x}_i^{(t)} + (1 - \gamma_i) S_i^{(t)},$$

where $\gamma_i \in [0, 1)$. The parameter $\gamma_i$ measures the level of consistency of agent $i$.

This source for preference change has a robust psychological foundation and has been widely used not only in this discipline but in economics as well (see for instance, Akerlof, 1982 and more recently, Kuran and Sandholm, 2008, Nordblom and Zamac, 2012). Let us observe that the foregoing dynamics is two-speed dynamics: gradual changes in preferences are accompanied by immediate behavioral adjustment in each period’s equilibrium play.

The next proposition states the steady state when all members of the organization show strictly positive levels of consistency and conformity.

**Proposition 2** Assume that $\omega_i > 0$ and $\gamma_i > 0$, $\forall i$, then in the absence of a leader, the organization tends to complete homogeneity determined by the distribution of the initial ideal actions. The steady state is common for all agents and equals to

$$S_i^{st} = \frac{\langle S_i^0 \rangle}{\langle 1/\gamma_i \rangle}.$$  
(4)

Moreover, the organization is fully homogeneous not only in preferences or ideal actions but also in behavior i.e., $\bar{x}_i^{st} = S^{st}, \forall i$. 

9
Proof of Proposition 2 First, by substituting in (3) the Nash equilibrium expression (2), we obtain:

\[ S_i^{(t+1)} = S_i^{(t)} + \frac{\gamma_i \tilde{\omega}_i}{1 + \tilde{\omega}_i} \left( \langle S^{(t)} \rangle - S_i^{(t)} \right) = S_i^{(t)} + \frac{\gamma_i \tilde{\omega}_i}{1 + \tilde{\omega}_i} \left( \langle \frac{S^{(t)}}{1 + \tilde{\omega}} \rangle - S_i^{(t)} \right). \]  

(5)

Dividing both terms of the equation by \( \gamma_i \tilde{\omega}_i \) and taking averages we get the following relation \( \langle \frac{S_i^{(t+1)}}{\gamma \tilde{\omega}} \rangle = \langle \frac{S^{(t)}}{\gamma \tilde{\omega}} \rangle \). This can be interpreted as a “conservation law”, a function of the preferences \( S_i^{(t)} \) that remains constant with time \( t \). Therefore, \( \langle \frac{S_i^{(t)}}{\gamma \tilde{\omega}} \rangle = \langle \frac{S_0}{\gamma \tilde{\omega}} \rangle \), dictated only by the initial condition. A consequence is that the steady state satisfies the same relation:

\[ \langle \frac{S_{st}}{\gamma \tilde{\omega}} \rangle = \langle \frac{S_0}{\gamma \tilde{\omega}} \rangle. \]  

(6)

Furthermore, in the steady state \( S_{st} = S_i^{(t+1)} = S_i^{(t)} \) and, as \( \gamma_i \tilde{\omega}_i > 0, \forall i \), then it follows from (5) that \( S_{st} = \langle S^{st} \rangle \), a constant value independent on the agent index \( i \). This constant value can be factorized from the average in the left hand side of (6), leading to the result (4). Finally, note also that there is convergence to the same Nash equilibrium action i.e. \( \hat{x}_{st}^i = S_{st}^i, \forall i \).

Under the combination of no leader in the organization with consistency or cognitive dissonance as the unique driving force of the evolution of preferences, Proposition 2 states that if all members of the organization display positive levels of conformity and consistency, the organization will tend to complete homogeneity. Moreover, the homogeneous steady state ideal action is a weighted average of the distribution of the initial preferences in the organization. In fact, this average is kept constant along all the trajectory of the organization. Note also that the initial preferences \( S_0^i \) of those agents with relatively low levels of conformity and/or consistency have a higher weight in the common steady state ideal action of the organization.

Notice that both conformism or peer effects and consistency, the tendency to adjust preferences into actual behavior, operate in the same direction, contributing to obtain an homogeneous organization. Conformism implies to move your behavior towards the average action of the group. But this average equilibrium action coincides with a weighted average of the ideal actions of the members of the group. Nevertheless, an agent only changes her ideal action when she displays positive levels of consistency, moving it towards her equilibrium action. But this latter behavior
depends again on the weighted average of the ideal actions of the members of the organization.

Thus, the combination of peer effects and consistency explains the two basic features in the long run of an organization without a leader and where all members display positive levels of conformity and consistency: homogeneity and dependence on the initial distribution of ideal actions or preferences. Previous literature on organizations has shown the existence of a strong tendency towards homogeneity (Carrillo and Gromb, 1999, Carrillo and Gromb, 2007) and also the relevance of hysteresis, that is, the impact of initial historical conditions. Our first result provides microfoundations for this finding, but also indicates that there might be additional causes for heterogeneity.

Consider, for instance, that there are \( N \) agents in the organization which are non-conformists \((\omega_i = 0)\) or never update their ideal action \((\gamma_i = 0)\). As now \( \gamma_i \omega_i = 0 \) for all these agents, then by equation (5) we get \( S_i^{(t+1)} = S_i^{(t)} \) and consequently \( S_i^{\text{st}} = S_i^0 \). In the case of a non-conformist agent \((\omega_i = 0)\), by (2), her equilibrium action coincides always with her ideal action and therefore she never suffers cognitive dissonance. In the case of an inconsistent agent with positive levels of conformism \((\gamma_i = 0, \omega_i > 0)\), although she will play an equilibrium Nash action different to her ideal, her ideal action remains invariable along time.

For the rest \( N - N_0 \) members of the organization, we get again from (5) that in the steady state \( S_i^{\text{st}} = \langle S_i^{\text{st}} \rangle \), or

\[
S_i^{\text{st}} = \frac{S_i^{\text{st}}}{1 + \omega_i} = \frac{\sum_{i \in N} \frac{S_i^{\text{st}}}{1 + \omega_i}}{\sum_{i \in N} \frac{1}{1 + \omega_i}} = \frac{\sum_{i \in N - N_0} \frac{S_i^{\text{st}}}{1 + \omega_i} + \sum_{i \in N_0} \frac{S_i^0}{1 + \omega_i}}{\sum_{i \in N} \frac{1}{1 + \omega_i}}.
\]

Solving for \( S_i^{\text{st}} \) and, after using \( \sum_{i \in N} \frac{1}{1 + \omega_i} = \sum_{i \in N - N_0} \frac{1}{1 + \omega_i} + \sum_{i \in N_0} \frac{1}{1 + \omega_i} \), we obtain the following expression:

\[
S_i^{\text{st}} = \frac{\sum_{i \in N_0} \frac{S_i^0}{1 + \omega_i}}{\sum_{i \in N_0} \frac{1}{1 + \omega_i}} \equiv \langle S_i^0 \rangle_{N_0}.
\]

Therefore, the subset of agents in the organization which are either non-conformist or inconsistent fully determines the steady state common ideal action of the rest of the organization. This steady state preference is a weighted average of the initial ideal actions of the subset of former agents. Note again that the impact of non-conformist agents in the long run is higher than that of inconsistent though conformist individuals. For instance, if the whole subset \( N_0 \) consists of non-conformists
\( (\omega_i = 0) \), the rest of the organization converges to the average initial ideal action of non-conformists: \( S^{st} = (1/N_0) \sum_{i \in N_0} S^0_i \). We state the above result as a corollary.

**Corollary 3** If there is a subset of non-conformist and/or inconsistent agents in an organization with no leader, then the set of conformist and consistent agents converges to a common steady-state ideal action which is a weighted average of the initial ideal actions of the subset of non-conformist and/or inconsistent agents.

Therefore, the two main results obtained in the previous proposition in some strong sense still hold: the group tends to full homogeneity except obviously for the nonconformists agents, and this final and common ideal action is dictated by the initial distribution of preferences, but only from the subset of nonconformist and inconsistent agents! In some sense, nonconformist and inconsistent individuals act as pivotal agents who have a disproportional influence in the long run of the organization. There is still a tendency towards homogeneity but the nonconformists fully establish the common ideal action of the rest of the organization.

### 4 A charismatic leader.

In this section we assume the existence of a charismatic leader. As in the above section, for each member of the organization \( i \), the leader has a target ideal action, \( \bar{S}_i \) but now each member \( i \) perceives the leadership with a different strength \( c_i \). Hence, the leader is endowed with a given charisma vector \( (c_1, \ldots, c_N) \), where \( c_i \in [0, 1) \). The ideal action of agent \( i \) evolves attending the charisma of the leader, influencing in the agent \( i \) the “ideal” \( \bar{S}_i \) and when this charisma does not succeed, the preference of agent \( i \) changes in the direction of her actual (Nash equilibrium) behavior with an intensity \( \gamma_i \). Consequently, the dynamics of the preferred actions \( S_i^{(t)} \) of the members of the organization is now given by:

\[
S_i^{(t+1)} = c_i \bar{S}_i + (1 - c_i) \left( \gamma_i x_i^{(t)} + (1 - \gamma_i) S_i^{(t)} \right)
\]  

By substituting the Nash equilibrium expression (2) we get the following set of \( N \) coupled difference equations:

\[
S_i^{(t+1)} = c_i \bar{S}_i + (1 - c_i) \left( \gamma_i \frac{S_i^{(t)} + \bar{\omega}_i \langle S^{(t)} \rangle}{1 + \bar{\omega}_i} + (1 - \gamma_i) S_i^{(t)} \right)
\]  

The way to model the socialization influence of a “cultural parent” is similar to others in the preference transmission literature (see, for example, Bisin and Verdier,

Let us assume initially that all members of the organization display strictly positive levels of consistency and conformity and additionally, the leader has positive charisma with all them, i.e. \( \omega_i > 0, \gamma_i > 0 \) and \( c_i > 0, \forall i \). Next proposition states the steady state of the above dynamics (10).

**Proposition 4** Assume that \( \omega_i > 0, \gamma_i > 0 \) and \( c_i > 0, \forall i \) and that \( S_i \) is (statistically) independent of \( \omega_i, \gamma_i \) and \( c_i \), then the steady state ideal action of each member of the organization is a convex combination of the target of the leader for this agent \( S_i \) and the average target for the whole organization \( \langle S \rangle \), namely:

\[
S_i^{st} = \tilde{c}_i S_i + (1 - \tilde{c}_i) \langle S \rangle, \quad i = 1, \ldots, N
\]

where \( \tilde{c}_i = \frac{c_i}{c_i + (1 - c_i)\gamma_i} \) and \( \tilde{\gamma}_i = \frac{\gamma_i}{1 + \omega_i} \).

This Proposition (proven in Appendix 1) states that there is no influence at all of the initial condition of the organization \( (S_1^0, S_2^0, \ldots, S_N^0) \) in the steady state outcome and that there is no homogeneity in the long run unless the leader desires a completely homogeneous group of followers. This sharply contrasts with the results for an organization with no leader. Moreover, even when all the members of the organization perceive the leader with the same strength \( c_i = c, \forall i \), the organization does not tend to homogeneity. The steady state ideal action of each agent is a convex combination of the particular target of the leader for her \( S_i \) and the average target for the organization \( \langle S \rangle \). The steady state value \( S_i^{st} \) is closer to the target \( S_i \), the higher is the charisma \( c_i \) of the leader with agent \( i \), the smaller is the level of consistency \( \gamma_i \) of agent \( i \) and the smaller is the level of conformism \( \omega_i \). Nevertheless, the leader succeeds on average, because the average steady state preference coincides with his average target, i.e. \( \langle S^{st} \rangle = \langle S \rangle \).

Furthermore, if there exists some \( i \neq j \) such that \( S_i \neq S_j \), then the leader never fully succeeds. Therefore, only if the leader pursues a completely homogeneous organization with exactly the same target \( \tilde{S} \) for all its members, he will succeed in the long run provided he has positive levels of charisma with all the individuals, i.e., if \( \tilde{S}_i = \tilde{S} \) for all \( i \) then \( S_i^{st} = \tilde{S} \) for all \( i \). This result holds even if the organization has an initial homogeneous condition in the sense every member has the same conformism, consistency, initial ideal action and the charisma is also the same. The forces driven these results are related to the tension between the peer effects and the consistency effects of the agents and the charisma of the leader. We further discuss about this issue in more detail in the next subsection.
4.1 The success of the leader.

At the above subsection we have characterized the steady state of the dynamics (10) attending to the presence of a leader with a given charisma and followers with positive conformism and consistency. We have learned that (except he aims at a perfectly homogeneous organization) the leader can never achieve his objective in the sense that the steady state of each agent \(i\) never coincides with the targeted demanded ideal action \(S_i\). A natural question is then to analyze how well a charismatic leadership can perform, that is, on measuring the success of the leader and its determinants. This can be properly measured by the following average distance between the steady state preferences of the members of the organization and the targets of the leader.

**Definition 5** The success of the leader is defined as the distance,

\[
\rho^2 = \frac{1}{N} \sum_{i=1}^{N} (S_i^\text{st} - \bar{S}_i)^2 = \left\langle (S_i^\text{st} - \bar{S})^2 \right\rangle
\]  

The lower \(\rho^2\), the higher the leader’s success. A high distance indicates that the steady state composition of the group falls far away of the desired ideal composition for the leader.

Substituting the steady state values (11) we get

\[
\rho^2 = \left\langle (1 - \bar{c})^2 \left(\langle S \rangle - \bar{S}\right)^2 \right\rangle.
\]

But given the (statistical) independence assumption between the targets \(\bar{S}_i\) and the variables \(\omega_i, \gamma_i, c_i\), this expression simplifies to \(\rho^2 = \left\langle (1 - \bar{c})^2 \sigma^2[\bar{S}] \right\rangle\), being \(\sigma^2[\bar{S}] = \frac{1}{N} \sum_{i=1}^{N} ((\bar{S}_i - \bar{S})^2)\) the variance of the targets of the Leader for the organization. Given this expression we can explore this measure of the leader’s success by the following proposition.

**Proposition 6**

1. If \(\gamma_i, \omega_i, c_i > 0\), the leader fully succeeds only if his target is a completely homogeneous organization. That is, only if \(\sigma^2[\bar{S}] = 0\), i.e., \(\bar{S}_i = \bar{S}_j, \forall i, j\), it is \(\rho^2 = 0\). Otherwise, the ideal actions desired by the leader cannot be perfectly imposed, \(\rho^2 > 0\).

2. The more heterogeneous is the target of the leader for the organization, the less successful he is. That is, \(\rho^2\) increases with \(\sigma^2[\bar{S}]\).

As explained above, followers’ conformism and consistency create both a strong tendency to homogeneity in the organization and a heavy inertia of the average initial conditions. Only if the leader pursues himself a completely homogeneous organization then peer effects operate in the same direction as charisma, reinforcing
leadership. But if the leader desires some heterogeneity or diversity in the organization, then peer and consistency effects work against and define some limits on the success of charismatic leadership. Moreover, the leader’s success decreases with the dispersion of the targets of the leader for the organization. Therefore, there is a genuine tradeoff between the ideal targets of the leader for a group and what he can really obtain. In other words, if the leader wishes to obtain some level of success (some average distance of the organization to his targets), then he has to give up some heterogeneity or diversity in his originally preferred goals.

Another ingredients that may impact on the success of the leader are the determinants of $\rho^2$ besides $\sigma^2[\bar{s}]$, that is, the size and dispersion of the distributions of charisma $c_i$, the conformity $\omega_i$ and consistency $\gamma_i$ in the group. As for the set of followers we get a vector of those factors, we denote as a change on size, for instance in the levels of charisma $c_i$, a situation where some elements of the vector $c$ simultaneously increase (or simultaneously decrease) while the rest of the elements remain fixed. Then we can state unambiguously that charisma $c$ is higher (or lower respectively). Concerning changes in the dispersion of the parameters and in order to be able to make meaningful comparisons we work with mean preserving changes in the dispersion. Namely, we want to analyze the impact of a mean preserving spread (MPS henceforth) in the distribution of a parameter.

**Proposition 7**

i) For any given target of the leader with variance $\sigma^2[\bar{s}]$, the distance $\rho^2$ is lower, that is, the success of the leader is higher, the higher is charisma $c$ and the lower are the levels of conformity $\omega$ and consistency $\gamma$ of the followers.

ii) A MPS in the distribution of the levels of charisma $c_i$ of the leader, reduces his success or increases the distance $\rho^2$.

The intuition behind these results is quite simple. We already know that peer effects and consistency create a strong tendency towards homogeneity in the organization. Therefore, unless the leader desires a completely homogeneous organization, the strength of peer effects work against his success. However, with low levels of conformism and/or consistency the peer effects are weaker and thus the success is higher. It is interesting to notice how consistency works reinforcing the peer effect (the influence of average behavior). Even if an agent has a low level of conformism, if he has a high level of consistency then his ideal action will be strongly influenced by the average ideal action because it is one of the determinants of the Nash equilibrium actions. Therefore, while conformity reinforces the tendency to homogeneity directly, consistency does it indirectly through the change towards the Nash equilibrium.
Result ii) of the previous proposition is about the impact on the success of the leader of changes in the dispersion of the distribution of charisma, conformity and consistency. The impact of a mean preserving spread in the distribution of the levels of conformity or consistency depends on the relationship between the parameters. Yet, we have a clear prediction about a change in the dispersion of the levels of charisma.

A well-known result states that a MPS increases the expected value of any convex function. Define \( f = \sum_i (1-\tilde{c}_i)^2 = \left( \frac{(1-\tilde{c}_i)c_i}{c_i+(1-\tilde{c}_i)\gamma_i} \right)^2 \). It is easy to check that \( \frac{\partial f}{\partial c_i} < 0 \) and \( \frac{\partial^2 f}{\partial c_i^2} < 0 \). As the Hessian matrix has all zeros off the diagonal and all the elements of the diagonal are positive, it is definite positive. Therefore, for any given target of the leader with variance \( \sigma^2[\tilde{S}] \), a MPS of the distribution of the levels of charisma \( c_i \) results in an increase of \( \langle (1-\tilde{c})^2 \rangle \) that ends in an increase of \( \rho^2 \). The success of the leader diminishes when his charisma is more dispersedly distributed across the organization. Thus, if the leader compares two distributions of charisma with the same average, he should prefer the more concentrated distribution than the more dispersed one.

4.2 The diversity of ideal actions and behavior in the organization and the limits of leadership.

The desired heterogeneity from the point of view of the charismatic leader has a strong impact on the limits of leadership. Recall that \( S_i^{\text{st}} \) only reaches the ideal \( \tilde{S}_i \) when the leader wants the full homogeneity among the organization. An interesting question is thus to compare such level of heterogeneity with the resulting level of heterogeneity of the organization in the long run. In order to measure this discrepancy, we contrast the variance of the ideal actions in the steady state with the variance of the targets of the leader \( \sigma^2[\tilde{S}] \). The former is defined as:

\[
\sigma^2[S^{\text{st}}] = \frac{1}{N} \sum_{i=1}^{N} (S_i^{\text{st}} - \langle S^{\text{st}} \rangle)^2
\]  

Using \( \langle S^{\text{st}} \rangle = \langle \tilde{S} \rangle \) and substituting for the expression of \( S_i^{\text{st}} \), we obtain \( \sigma^2[S^{\text{st}}] = \langle \tilde{d}^2 \rangle \sigma^2[\tilde{S}] \). As \( \tilde{c}_i < 1 \), it is \( \sigma^2[S^{\text{st}}] \leq \sigma^2[\tilde{S}] \). Thus, the variance of the ideal actions (or preferences) in the steady state is always smaller than the variance of the desired composition of the organization for the leader. Next proposition states the above result.

**Proposition 8** The organization ends up always less heterogeneous than what a charismatic leader desires.
This result is caused again by the strength of the peer effects: to follow the average behavior is a strong force if followers are very conformist and/or are very consistent. Therefore, when either the levels of conformism or the levels of consistency are high, the contrast between the variance of the target of the leader and the variance of preferences in the long run is higher.

Another interesting comparison relates behavior, instead of preferences, with the targets of the leader. Therefore, we compute the aggregate distance between the Nash equilibrium actions in the steady state and the targets of the leader.

$$\psi^2 = \frac{1}{N} \sum_{i=1}^{N} (\hat{x}_i^{st} - \bar{S}_i)^2. \quad (14)$$

After substituting $\hat{x}_i^{st}$ by its value and given statistical independence we get $\psi^2 = \langle (1 - \frac{\hat{c}_i}{1+c_i})^2 \rangle \sigma^2 [S]$. It is easy to check that $\psi^2 > \rho^2$. The distance between the actions of the followers and what the leader desires is even higher than the distance with the followers’ preferences. Thus, the success of the leader is even smaller when it is measured with behavior.

### 4.3 An organization with irreducible agents.

We have focussed until now in the case of an organization without non-conformist and inconsistent agents and with a leader who has positive charisma towards all its members. Let us analyze next the more general case where the organization can have nonempty sets of these types of agents and where the leader might have no charisma or influence at all with some of its members. We want to determine their influence on the steady state values of the organization.

We begin by summarizing the possible cases given the characteristics of the agents. We split agents in two sets: set $A$ (cardinality $N_A$) and set $B$ (cardinality $N_B = N - N_A$) according to whether the leader has got charisma or not on them. Each set is further separated in two disjoint groups, $A = A_1 \cup A_2$, $B = B_1 \cup B_2$:

- $A_1$ is the set of agents such that $c_i > 0$ but $\hat{c}_i \omega_i = 0$. In this case $S_i^{st} = \bar{S}_i$, as derived from the dynamics of their ideal action $S_i^{(t+1)} = c_i \bar{S}_i + (1 - c_i)S_i^{(t)}$. Interestingly, a charismatic leader fully succeeds with non-conformist and inconsistent agents provided he has positive levels of charisma towards them.

- $A_2$ is the set of agents such that $c_i > 0$ and $\hat{c}_i \omega_i > 0$. For these we already know that $S_i^{st} = \frac{c_i \bar{S}_i + (1 - c_i) \langle S_i^{st} \rangle}{c_i + (1 - c_i) \bar{S}_i}$.

- $B_1$ is the set of agents such that $c_i = 0$ and $\hat{c}_i \omega_i = 0$. For those agents, $S_i^{(t+1)} = S_i^{(t)} = S_i^{0}$ and $S_i^{st} = S_i^{0}$. Members of this set will be referred to as the “irreducible agents” of the organization. The cardinality of the set is $N_{B_1}$.
Proposition 9 If there is a subset $B_1$ of $N_{B_1}$ irreducible agents in the organization, $N_A$ agents with $c_i > 0$ and $S_i$ is statistically independent of $\omega_i, \gamma_i$ and $c_i$, then the steady state of the dynamics is given by:

i) Irreducible agents: $S^s_{i} = S^0_i, \forall i \in B_1$.

ii) For all other $(N - N_{B_1})$ members of the organization (subset $B^c_1$):

$$S^s_{i} = \tilde{c}_i S_i + (1 - \tilde{c}_i) \left[ \frac{(N_A/N) \langle \frac{\tilde{c}}{1+\omega} \rangle_A \langle S \rangle_A + (N_{B_1}/N) \langle \frac{\tilde{S}}{1+\omega} \rangle_{B_1}}{((N - N_{B_1})/N) \langle \frac{\tilde{c}}{1+\omega} \rangle_{B^c_1} + (N_{B_1}/N) \langle \frac{1}{1+\omega} \rangle_{B_1}} \right]$$

Proposition 9 states that the presence of a subset of irreducible agents in the organization has an impact in the steady state ideal actions of the rest of the organization. The ideal action of an agent with positive levels of conformism and consistency and for whom the leader has positive levels of charisma is a convex combination of the target of the leader for that agent, $S_i$, the average target , $\langle S \rangle_A$ for the $N_A$ members of the organization with whom the leader has positive charisma ($c_i > 0$) and finally, a weighted average of the initial condition of the $N_{B_1}$ irreducible agents. The effect of the minority of irreducible agents in the rest of the organization depends on the size of its set, $N_{B_1}$, more precisely, on the proportion $(N_{B_1}/N)$. For those agents of the organization that show positive levels of conformism and consistency but such that $c_i = 0$, i.e. the leader has not charisma towards them, $S^s_{i} = \langle S^s \rangle$. Their ideal action in the steady state is a mix between the average target for the non-irreducible members of the organization, $\langle S \rangle_A$, and a weighted average of the initial condition of the irreducible agents. Finally, note that the leader only fully succeeds, $S^s_{i} = S_i$, with non-conformists agents but with whom he has positive levels of charisma.

5 A forward-looking Leader: socialization

We turn next to the situation where the leader can choose the levels of costly socialization efforts trying to indoctrinate the agents. The leader now is a shaper of preferences or a proselytizer. An equivalent interpretation is that he can invest in charisma or prestige.
The leader chooses a sequence of vectors of costly socialization efforts, \((c^t_1, \ldots, c^t_N)\), each \(c^t_i \in [0, 1]\), trying to reduce the average distance between the ideal actions \(S_i\) of the followers and his vector of targets \(\bar{S}_i\). His goal is to minimize \(\sum_{i=1}^N (\bar{S}_i - S^{(t+1)}_i)^2\) (or to maximize \(-\sum_{i=1}^N (\bar{S}_i - S^{(t+1)}_i)^2\)). But socialization effort or investment in charisma is costly, according to a cost function with two arguments: the level of effort \(c^t_i\) and the “cultural distance” between \(\bar{S}_i\) and \(S^{\ast(t)}_i\), where \(S^{\ast(t)}_i\) is agent \(i\)'s interim ideal action in period \(t\). That is, it depends on the “cultural” distance between the leader’s socialization target for agent \(i\) and the current ideal action of this agent. For the sake of simplicity, in this section we assume a common value of \(\bar{\gamma}_i = \bar{\gamma}\) for all the followers. We then represent the cost of socialization by an indirect cost function that depends on the choices of \(c^t_i\) and also on the distance between \(\bar{S}_i\) and \(S^{\ast(t)}_i\). Formally, the cost function is \(C(c^t_i, |\bar{S}_i - S^{\ast(t)}_i|) = \frac{\beta}{2} (c^t_i)^2 (\bar{S}_i - S^{\ast(t)}_i)^2\) with \(\beta > 0\). Note that the higher is the “cultural” distance for agent \(i\) and the leader, the more costly is an additional unit of socialization effort invested in agent \(i\). Recall that the difference between \(S^{\ast(t)}_i\) and \(S^t_i\) in each period \(t\) comes from the fact that agent \(i\) changes her preferences in the direction of her actual equilibrium behavior and prior to any influence from the leader. Summarizing, the instantaneous payoff function of the Leader is

\[
U(S^t, S^{(t+1)}) = -\sum_{i=1}^N \left[ (\bar{S}_i - S^{(t+1)}_i)^2 + \frac{\beta}{2} (c^t_i)^2 (\bar{S}_i - S^{\ast(t)}_i)^2 \right] 
\] (15)

The dynamic programming problem, Problem (1), faced by the Leader is then to determine the sequence of charisma’s vectors \(\{(c^t_1, \ldots, c^t_N)\}_{t=0}^\infty\) such that they maximize:

\[
\max \sum_{t=0}^\infty \delta^t \left\{ -\sum_{i=1}^N \left[ (\bar{S}_i - S^{(t+1)}_i)^2 + \frac{\beta}{2} (c^t_i)^2 (\bar{S}_i - S^{\ast(t)}_i)^2 \right] \right\}
\]

subject to the conditions

\[
c^t_i \in [0, 1], \ t \geq 0, \ i = 1, \ldots, N \tag{16}
\]

\[
S^{(t+1)}_i = c^t_i \bar{S}_i + (1 - c^t_i) S^{\ast(t)}_i \tag{17}
\]

\[
S^{\ast(t)}_i = S^t_i + \bar{\gamma}(\langle S^t \rangle - S^t_i) \tag{18}
\]

for given \(S^0_i\) and where \(\delta \in (0, 1)\) is the discount factor of the leader.

Replacing \(c^t_i\) and \(S^{\ast(t)}_i\) from (17-18), the instantaneous payoff function of the Leader becomes:

\[
U(S^t, S^{(t+1)}) = -\sum_{i=1}^N \left[ (\bar{S}_i - S^{(t+1)}_i)^2 + \frac{\beta}{2} (S^{(t+1)}_i) - S^{(t)}_i - \bar{\gamma}(\langle S^t \rangle - S^{(t)}_i)^2 \right], \tag{19}
\]
and Problem (1) is equivalent to the following Problem (2), where the control variables are \( \{S_i^{(t+1)}, \ldots, S_N^{(t+1)}\}_{t=0}^{\infty} \).

\[
\max \sum_{t=0}^{\infty} \delta^t \left\{ -\sum_{i=1}^{N} \left[ (\bar{S}_i - S_i^{(t+1)})^2 + \frac{\beta}{2} (S_i^{(t+1)} - S_i^{(t)})^2 - \gamma (\langle S_i^{(t)} \rangle - S_i^{(t+1)}) \right] \right\}
\]

\( \{S_i^{(t+1)}\}_{t=0}^{\infty} \) s.t. \( S_i^{(t+1)} \in [S_i^{(t)}, \bar{S}_i] \) if \( S_i^{(t)} \leq \bar{S}_i \)
\( S_i^{(t+1)} \in [\bar{S}_i, S_i^{(t)}] \) if \( S_i^{(t)} > \bar{S}_i \)

\( S_0^i \) given for \( i = 1, \ldots, N \). The constraints of Problem (2) define a correspondence \( G : \mathbb{Z} \times [0, 1]^N \Rightarrow [0, 1]^N \) where \( G(t, S_i^{(t)}) = S_i^{(t+1)} \). The properties of the instantaneous payoff function of the Leader and the constraint correspondence are presented in next Lemma.

**Lemma 10**

- (i) The function \( U(S_i^{(t)}, S_i^{(t+1)}) \) is bounded, continuous and differentiable on the interior of its domain.
- (ii) The function \( U(S_i^{(t)}, S_i^{(t+1)}) \) is jointly strictly concave.
- (iii) The constraint correspondence denoted by \( G(S_i^{(t)}) \) is non-empty for all \( S_i^{(t)} \in X \), compact valued, continuous, convex and monotone.

Given the properties of \( U(\cdot, \cdot) \) and the constraint correspondence, we can tackle Problem (2) by using the Bellman equation associated to the dynamic programming problem. Formally:

\[
V(S_i^{(t)}) = \max_{S_i^{(t+1)}} \{U(S_i^{(t)}, S_i^{(t+1)}) + \delta V(S_i^{(t+1)})\}_{\{t=0,1,\ldots\}} \tag{20}
\]

The objective is to obtain the optimal socialization policy function \( S_i^{(t+1)} = h(S_i^{(t)}) \) that maximizes the problem. A solution to that equation is also a solution of Problem (2).

### 5.1 Dynamic programming

In this subsection, we proof the existence of a solution of Problem (2) characterised by the unique value function \( V(\cdot) \) and the policy function \( h(S_i^{(t)}) \). Given the properties of the instantaneous payoff function \( U(S_i^{(t)}, S_i^{(t+1)}) \) and the constraint correspondence from Lemma 10, we know by standard arguments\(^3\) that there exist two functions \( V(S_i^{(t)}) \) and \( h(S_i^{(t)}) \) such that \( V(S_i^{(t)}) \) is uniquely defined, continuous,

\(^3\)(See for instance Acemouglou Theorem ...)

20
converse and differentiable which characterise the solution of Problem (2). Moreover, \( h(S^{(t)}) \) is single-valued. The optimal socialization plan \( h(S^{(t)}) \) must satisfy the following set of necessary and sufficient conditions whenever \( S^{(t+1)}_i \) are interior.

The first order condition with respect to \( S^{(t+1)}_i \) establishes that,

\[
2(\bar{S}_i - S^{(t+1)}_i) - \beta(S^{(t+1)}_i - S^{(t)}_i) - \tilde{\gamma}(\langle S^{(t)}_i \rangle - S^{(t)}_i) + \delta V'_i(S^{(t+1)}_i) = 0
\]

for \( i = 1, \ldots, N \). Since \( V(S^{(t)}) \) is differentiable, using the envelope theorem differentiating with respect to \( S^{(t)}_i \) we obtain:

\[
V'_i(S^{(t)}_i) = \beta \alpha(S^{(t+1)}_i - S^{(t)}_i) - \tilde{\gamma}(\langle S^{(t)}_i \rangle - S^{(t)}_i))
\]

for \( i = 1, \ldots, N \) and \( \alpha = (1 - \tilde{\gamma}(\frac{N-1}{N})) \). Therefore, we have the following set of Euler equations which solve for \( h(S^{(t)}) \) as the optimal socialization function:

\[
\beta(h_i(S^{(t)}_i) - S^{(t)}_i) - \tilde{\gamma}(\langle S^{(t)}_i \rangle - S^{(t)}_i)) - 2(\bar{S}_i - h_i(S^{(t)}_i)) = 0
\]

for \( i = 1, \ldots, N \).

Replacing \( S^{(t+1)}_i = h_i(S^{(t)}_i) \) and reordering we get a set of coupled second-order difference equations:

\[
S^{(t+2)}_i = \left(1 - \gamma + \frac{1}{\alpha \delta} \left( \frac{2}{\beta} + 1 \right) \right) S^{(t+1)}_i + \frac{\gamma}{N} \sum_{j=1}^{N} S^{(t+1)}_j - \frac{2}{\alpha \beta \delta} \bar{S}_i
\]

for \( i = 1, \ldots, N \) and given \( S^0_i \).

The next proposition characterizes the optimal socialization function that solves this problem and the optimal trajectory of the followers’ ideal actions \( \{S^{(t)}_i\}_{i=1,\ldots,N} \) and shows its convergence.

**Proposition 11** The optimal socialization plan \( h(S^{(t)}) \) is given by the set of recurrence relations

\[
S^{(t+1)}_i = S^{st}_i + (\lambda^+_2)(S^{0}_i - S^{st}_i) + ((\lambda^-_1) - (\lambda^-_2))(\langle S^{(t)}_i \rangle - \langle S^{st}_i \rangle)
\]

for \( i = 1, \ldots, N \) and where \( S^{st}_i \) is the steady state ideal action of individual \( i \).

For any initial condition \( \{S^0_i\} \), the optimal trajectory of the organization is given by:

\[
S^{(t)}_i = S^{st}_i + (\lambda^-_2)(S^{0}_i - S^{st}_i) + ((\lambda^-_1) - (\lambda^-_2))(\langle S^{(t)}_i \rangle - \langle S^{st}_i \rangle)
\]

for \( i = 1, \ldots, N \) and where \( \lambda^-_1 \) and \( \lambda^-_2 \) verify \( |\lambda^-_1| < 1 \).
We now compute the steady state solution of the organization. Next proposition states the ideal action in the long run of each member of the organization with a forward-looking socializing leader.

**Proposition 12** Assume that \( \omega > 0 \) and \( \gamma > 0 \) then the ideal action in the long run of each member of the organization with a forward-looking socializing leader is a convex combination between the target of the leader for this agent \( S_i \) and the average target for the whole organization \( \langle \bar{S} \rangle \). In particular, the steady state of the dynamics is given by:

\[
S_i^{st} = dS_i + (1 - d) \langle \bar{S} \rangle
\]

for \( d = \frac{\alpha}{2 + \beta \gamma (1 - \alpha \delta)} \) and \( \alpha = 1 - \gamma (1 - \frac{1}{N}) \).

See proof in the Appendix 1.

Similarly to the case of a charismatic leader analyzed in section 4, there is no influence at all of the initial condition of the organization in the steady state outcome of the socialization process and there is no homogeneity in the long run unless the leader desires a completely homogeneous group of followers. Nevertheless, the socializing leader succeeds in average, since the average steady state preference coincides with his average target, i.e., \( \langle S^{st} \rangle = \langle \bar{S} \rangle \). Moreover the distance between \( S_i^{st} \) and \( S_i \) is constant for each \( i \), that is the value \( d \). The steady state values \( S_i^{st} \) of socialization are closer to the target \( S_i \), the higher is the discount factor \( \delta \) of the leader, the lower is the marginal cost of socialization influenced by parameter \( \beta \), and the smaller are the levels of consistency \( \gamma \) and of conformism \( \omega \) of the followers.

As the steady state ideal action of each agent achieved with costly socialization is a convex combination of the particular target of the leader for this agent \( S_i \) and the average target for the organization \( \langle \bar{S} \rangle \) then there are limits in the effectiveness of a socializing leader.

**Socialization in the steady state.**

Once the steady state corporate culture has been reached, does the leader have to invest in socialization every period to maintain it? We can compute now the socialisation effort of he leader \( c_i^{st} \) in the steady state under two escenarios, when the leader looks for an homogenous and an heterogeneous organization.

If the leader pursues a completely homogeneous organization and the steady state is reached then the leader does not need to invest any additional amount on socialization, i.e., \( c_i^{st} = 0 \) for all \( i \). The peer effects maintain the homogeneous organizational culture that the leader aspires to.
Nevertheless, this result vanishes if the Leader desires some degree of diversity. In particular, if he does not invest in socialization one period then each follower \( \tilde{S}_i \) will move towards the average preference of the organization \( \langle \tilde{S} \rangle \) since as the peer effects, according to the equation:

\[
S_{i}^{\text{st}} = S_{i}^{\text{st}} + \tilde{\gamma}(\langle \tilde{S} \rangle - S_{i}^{\text{st}}).
\]

Notice that \( (S_{i}^{\text{st}} - \langle \tilde{S} \rangle) = (1 - \tilde{\gamma})d(\tilde{S}_i - \langle \tilde{S} \rangle) \) for all \( i \), while in the steady state this distance was \( (S_{i}^{\text{st}} - \langle \tilde{S} \rangle) = d(\tilde{S}_i - \langle \tilde{S} \rangle) \) for all \( i \). This implies that without the action of the leader, all followers move towards this average \( \langle \tilde{S} \rangle \) resulting in an organization less and less heterogeneous. The amount of socialization needed to keep each follower in the steady state is obtained from equation \( S_{i}^{\text{st}} = c_i^{\text{st}} \tilde{S}_i + (1 - c_i^{\text{st}})S_{i}^{\text{est}} \).

Solving this equation and after some algebra, we obtain that in the steady state

\[
c_i^{\text{st}} = \frac{2}{2 + \beta(1 - \alpha \delta)} = \frac{2}{2 + \beta(1 - \gamma(1 - \frac{1}{N}))}.
\]

independent of the agent index \( i \). Therefore, the leader has to invest at each period and with each follower a constant and identical amount of resources in order to maintain the organization in the configuration \( S^{\text{st}} \). This amount is independent of the level of desired diversity of the leader \( \sigma^2[\tilde{S}] \). This socialization effort per capita is increasing in the discount factor \( \delta \) and decreasing in the cost parameter \( \beta \) and the size of the organization \( N \). Paradoxically, it is also decreasing in the levels of consistency \( \gamma \) and of conformism \( \omega \) of the followers. In other words, for high levels of conformism and/or consistency of the members of the organization the needed level of socialization in order to keep the resulting corporate culture \( S^{\text{st}} \) is smaller. The reason is that for high levels of \( \tilde{\gamma} \), all the members of the organization are more far away of their leader’s individual target \( \tilde{S}_i \) and closer to the average ideal action \( \langle \tilde{S} \rangle \) \((d \ \text{diminishes with} \ \tilde{\gamma})\). Therefore, this less diverse corporate culture is less costly to maintain for a socializer leader in terms of the socialization effort per capita. In some sense once again the peer effects collaborate with the socialization efforts in the steady state.

The leader’s payoff in the steady state.

Recall that the leader’s instantaneous payoff function is a cost or disutility function with two components. On the one hand, the leader dislikes the distance between his targets for the followers and their steady state ideal actions. On the other hand, the other component captures the aggregate costs of socialization which in turn depend on this former distance. We already know that if the leader does not invest in socialization in the steady state the ideal action of a follower \( i \) would change in that period to: \( S_{i}^{\text{est}} = S_{i}^{\text{st}} + \tilde{\gamma}(\langle \tilde{S} \rangle - S_{i}^{\text{st}}) \). Thus, the cost of keeping the same distance with this follower is:

\[
\frac{\tilde{\gamma}}{2} (c_i^{\text{st}})^2 (\tilde{S}_i - S_{i}^{\text{est}})^2.
\]

It is easy to check that the leader’s payoff in the steady state with these two
components is

\[ U_{st} = -\left( \sum_{i=1}^{N} [(\bar{S}_i - S_i^{st})^2] + \sum_{i=1}^{N} \left[ \frac{\beta}{2} (\bar{\gamma})^2 (\bar{S} - S_i^{st})^2 \right] \right) \]

Substituting the steady state preferences by its values the instantaneous payoff of the leader (19) in the steady state becomes:

\[ U_{st} = -N \sigma^2[\bar{S}][(1 - d)^2] + \left[ \frac{\beta}{2} (\bar{\gamma})^2 d^2 \right] = -N \sigma^2[\bar{S}] \frac{\beta \gamma^2 (2 + \beta(1 - \alpha \delta)^2)}{(2 + \beta \gamma^2 (1 - \alpha \delta))^2} \]  \hspace{1cm} (28)

Notice again that only when \( \sigma^2[\bar{S}] = 0 \), that is, the leader pursues a completely homogeneous organization, then he attains the highest possible payoff (the lowest cost) of zero. The payoff of the leader decreases in the size of the organization \( N \), his target on diversity \( \sigma^2[\bar{S}] \) and \( \bar{\gamma} \).

**The success of socialization.**

We are interested on analyzing how well a socializing leadership can do, that is, on measuring the success of the process of socialization and its determinants. This can be properly measured by the following average distance between the steady state preferences of the members of the organization and the targets of the leader.

**Definition 13** The success of socialization \( \rho_s^2 \) is defined as the distance,

\[ \rho_s^2 = \frac{1}{N} \sum_{i=1}^{N} (S_i^{st} - \bar{S}_i)^2 = \left( \langle S^{st} - \bar{S} \rangle \right)^2 \]  \hspace{1cm} (29)

The lower is this distance \( \rho_s^2 \), the higher is the socialization’s success. A high distance indicates that the steady state composition of the group falls far away of the desired ideal composition for the socializer leader. Substituting for the steady state values we get, \( \rho_s^2 = \left( \frac{\beta \gamma^2 (1 - \alpha \delta)}{2 + \beta \gamma^2 (1 - \alpha \delta)} \right)^2 \sigma^2[\bar{S}] \). Given this expression we can explore this measure of socialization’s success by the following proposition.

**Proposition 14**

1. Socialization fully succeeds only if its target is a completely homogeneous organization. Otherwise, it never fully succeeds. That is, only if \( \sigma^2[\bar{S}] = 0 \), i.e., \( \bar{S}_i = \bar{S} \) for all \( i \) then \( \rho_s^2 = 0 \), and if \( \sigma^2[\bar{S}] > 0 \), the ideal actions desired by the leader cannot be perfectly imposed, \( \rho_s^2 > 0 \).

2. The more heterogeneous is the target of the leader for the organization, the less successful socialization is. That is, \( \rho_s^2 \) increases with \( \sigma^2[\bar{S}] \).

Another ingredients that may impact on the success of socialization are the determinants of \( \rho_s^2 \) besides \( \sigma^2[\bar{S}] \), that is, the conformity \( \omega_i \) and consistency \( \gamma_i \) in the group, the cost of socialization \( \beta \) and the discount factor of the socializer \( \delta \).
Proposition 15  For any given target of the socializer with variance $\sigma^2[S]$, the distance $\rho_s^2$ is lower, that is, the success of socialization is higher, the higher is the discount factor $\delta$, the lower is the cost of socialization $\beta$ and the lower are the levels of conformity $\omega$ and consistency $\gamma$ of the followers.

5.2 Charismatic leadership versus socialization.

Now we are ready to compare the levels of success of these two different types of leadership: charisma versus costly socialization. Suppose that the target for the organization is the same, as summarized by a common $\sigma^2[S]$, then the success of socialization is higher than the success of a charismatic leader (i.e. $\rho_s^2 < \rho^2$) if $c_i < c_{st}^i = [2/(2 + \beta(1 - \alpha\delta))] = c_{st}$ for $i = 1, 2, \ldots, N$.

Costly socialization is better than a charismatic leader if the leader’s individual charisma with all followers is below some critical bound that depends on parameters $\alpha$, $\beta$ and $\delta$. Notice that this critical threshold is precisely the level of socialization of the leader in the steady state. The higher is this critical value the more likely is that socialization is more successful than charismatic leadership, in the sense that for the latter being better it is needed relatively higher levels of charisma with all the followers. In other words, we can take this bound as a good proxy of the comparative advantages of a charismatic leadership or a socializer one. This critical bound depends positively on the discount factor $\delta$ and negatively on the cost factor $\beta$ and on the strength of peer effects $\gamma$, that is consistency and conformism. A low value of $c_{st}$, because of high levels of conformism and consistency for example, would imply that a very charismatic leader is more likely to perform better than a socializer.

Proposition 16  A sufficiently charismatic leader is better than socialization in groups with strong peer effects.

Consequently socialization is likely to perform better than charisma, in groups with lower levels of consistency and conformity and thus, low peer effects. Paradoxically, in these groups a socializer leader would have to invest in the steady state higher levels of socialization efforts $c_{st}$ in order to maintain the long-run organizational culture.

Hermalin (2012) pointed out that the idea that in groups with strong peer effects charismatic leadership performs very well was pointed out many years ago by the notion of "Asabiyah" or "group feeling": the effectiveness of the leader may depend on the strength of the corporate culture.
References


Appendix 1: Proofs of propositions and lemmas

Proof of Lemma 1

Given our assumptions on the choice set $X$ and on the payoff functions, the Nash equilibrium is computed by solving the set of first-order conditions:

$$
\frac{\partial u_i^{(t)}(x_i)}{\partial x_i} \bigg|_{(x_j = \hat{x}_j^{(t)})_{j=1,...,N}} = 0, \quad i = 1, \ldots, N. \quad (30)
$$

After performing the partial derivatives in (30), we obtain \(^4\) then that the best response function of agent $i$ at time $t$ satisfies:

$$
\hat{x}_i^{(t)} = \frac{S_i^{(t)} + \tilde{\omega}_i \langle \hat{x}^{(t)} \rangle}{1 + \tilde{\omega}_i},
$$

where, for brevity in notation, we define $\tilde{\omega}_i \equiv \omega_i \left(1 - \frac{1}{N}\right)$.

We have not yet found the Nash equilibrium actions $\hat{x}_i^{(t)}$ as the previous expression (31) constitutes actually a set of $N$ coupled equations since $\hat{x}_j^{(t)}$ appear in the definition of the average $\langle \hat{x}^{(t)} \rangle$. The explicit solution is found by taking averages on both sides of this equation,

$$
\langle \hat{x}^{(t)} \rangle = \left\langle \frac{S^{(t)}}{1 + \tilde{\omega}} \right\rangle + \left\langle \frac{\tilde{\omega}}{1 + \tilde{\omega}} \right\rangle \langle \hat{x}^{(t)} \rangle,
$$

or

$$
\langle \hat{x}^{(t)} \rangle = \left\langle \frac{S^{(t)}}{1 + \tilde{\omega}} \right\rangle = \langle S^{(t)} \rangle,
$$

where the notation $\langle f \rangle$ indicates a weighted average $\langle f \rangle = \frac{1}{N} \sum_{j=1}^{N} p_j f_j$, with $p_j = \frac{(1 + \tilde{\omega}_j)^{-1}}{(1 + \tilde{\omega})^{-1}}$. Therefore, combining (31) and (33), the result (1) follows.

**Proof of proposition 4**

The stationary solution satisfies:

$$
S_i^{st} = c_i \tilde{S}_i + (1 - c_i) \left( \gamma_i S_i^{st} + \tilde{\omega}_i \langle S^{st} \rangle \right) + (1 - \gamma_i) S_i^{st}.
$$

Which, after some straightforward algebra, leads to:

$$
S_i^{st} = \frac{c_i \tilde{S}_i + (1 - c_i) \tilde{\gamma}_i \langle S^{st} \rangle}{c_i + (1 - c_i) \tilde{\gamma}_i},
$$

where we have defined $\tilde{\gamma}_i = \frac{\gamma_i \tilde{\omega}_i}{1 + \tilde{\omega}_i}$.

\(^4\)One must note that $x_i^{(t)}$ appears as well in the average $\langle x^{(t)} \rangle$. 29
We rewrite this expression as

\[ S_i^{st} = a_i \bar{S}_i + (1 - a_i) \langle \langle S^{st} \rangle \rangle \]  \tag{35} \]

where \( a_i = \frac{c_i}{c_i + (1 - c_i) \gamma_i} \). We then multiply by \( \frac{1}{1 + \bar{\omega}_i} \) and take averages in both sides, to obtain

\[ \left\langle \frac{S^{st}}{1 + \bar{\omega}_i} \right\rangle = \left\langle \frac{a \bar{S}}{1 + \bar{\omega}_i} \right\rangle + \left\langle \frac{1 - a}{1 + \bar{\omega}_i} \right\rangle \langle \langle S^{st} \rangle \rangle, \]

or

\[ \left\langle \frac{1}{1 + \bar{\omega}_i} \right\rangle \langle \langle S^{st} \rangle \rangle = \left\langle \frac{a \bar{S}}{1 + \bar{\omega}_i} \right\rangle + \left\langle \frac{1 - a}{1 + \bar{\omega}_i} \right\rangle \langle \langle S^{st} \rangle \rangle. \]

Thus,

\[ \langle \langle S^{st} \rangle \rangle = \left\langle \frac{a \bar{S}}{1 + \bar{\omega}_i} \right\rangle = \left\langle \frac{\frac{c}{(1 + \bar{\omega}_i)(c + (1 - c_i) \gamma_i)} \bar{S}^{st}}{1 + \bar{\omega}_i} \right\rangle. \]

We can now assume that \( \bar{S}_i \) is statistically independent of \( \omega_i, \gamma_i \) and \( c_i \), and approximate\(^5\)

\[ \left\langle \frac{c}{(1 + \bar{\omega}_i)(c + (1 - c_i) \gamma_i)} \bar{S}^{st} \right\rangle \approx \left\langle \frac{c}{(1 + \bar{\omega}_i)(c + (1 - c_i) \gamma_i)} \right\rangle \bar{S}. \] \tag{36} \]

Replacing in (34), we obtain that the stationary values are given by:

\[ S_i^{st} = \frac{c_i \bar{S}_i + (1 - c_i) \gamma_i \bar{S}}{c_i + (1 - c_i) \gamma_i}. \]

**Proof or proposition 9**

Let us compute:

\[
\left\langle \frac{S^{st}}{1 + \bar{\omega}_i} \right\rangle = \frac{1}{N} \sum_{i=1}^{N} \frac{S_i^{st}}{1 + \bar{\omega}_i} \\
= \frac{1}{N} \left[ \sum_{i \in A_1} \frac{S_i^{st}}{1 + \bar{\omega}_i} + \sum_{i \in B_2} \frac{S_i^{st}}{1 + \bar{\omega}_i} + \sum_{i \in A_2} \frac{S_i^{st}}{1 + \bar{\omega}_i} + \sum_{i \in B_1} \frac{S_i^{st}}{1 + \bar{\omega}_i} \right] \\
= \frac{1}{N} \left[ \sum_{i \in A_1} \frac{c_i \bar{S}_i + (1 - c_i) \gamma_i \langle \langle S^{st} \rangle \rangle}{(c_i + (1 - c_i) \gamma_i)(1 + \bar{\omega}_i)} + \sum_{i \in B_2} \langle \langle S^{st} \rangle \rangle \frac{1}{1 + \bar{\omega}_i} + \sum_{i \in B_1} \frac{S_i^0}{1 + \bar{\omega}_i} \right].
\]

\(^5\)Numerical simulations raul03:/Users/raul/LEADERSHIP/test4.f90 indicate that the approximation is very good already for \( N = 100 \).
We obtain:

\[
\langle S^{st} \rangle \left\langle \frac{1}{1 + \omega} \right\rangle = \frac{1}{N} \sum_{i \in A_1} \frac{c_i \tilde{S}_i}{(c_i + (1 - c_i) \tilde{\gamma}_i)(1 + \hat{\omega}_i)} + \frac{1}{N} \sum_{i \in A_2} \frac{\tilde{S}_i}{1 + \hat{\omega}_i} + \langle S^{st} \rangle \left\langle \frac{1}{N} \sum_{i \in A_1} \frac{(1 - c_i) \tilde{\gamma}_i}{(c_i + (1 - c_i) \tilde{\gamma}_i)(1 + \hat{\omega}_i)} + \sum_{i \in A_2} \frac{1}{1 + \hat{\omega}_i} \right\rangle
\]

The first and the second addends are equivalent to \( T_1 = \frac{1}{N} \sum_{i \in A} \frac{c_i \tilde{S}_i}{(c_i + (1 - c_i) \tilde{\gamma}_i)(1 + \hat{\omega}_i)} \).

The third and the fourth addends is \( T_2 \) that is equivalent to \( \frac{1}{N} \sum_{i \in A_1 \cup B_2} \frac{(1 - c_i) \tilde{\gamma}_i}{(c_i + (1 - c_i) \tilde{\gamma}_i)(1 + \hat{\omega}_i)} \).

The last term is named as \( T_3 \). Therefore, we write the above expression as:

\[
\langle S^{st} \rangle \left\langle \frac{1}{1 + \omega} \right\rangle = T_1 + T_2 \langle S^{st} \rangle + T_3 \text{ getting}
\]

\[
\langle S^{st} \rangle = \frac{T_1 + T_3}{\langle \frac{1}{1 + \omega} \rangle} - T_2 \equiv \bar{S}
\]

Now is better to group sets \( A_1, B_2 \) and \( A_2 \) as the complementary of set \( B_1 \), i.e. \( B_1^c = A_1 \cup B_2 \cup A_2 \). Let \( N_1 \) be the number of agents with \( c_i > 0 \), that is, agents from sets \( A_1 \) and \( A_2 \). Therefore,

\[
\langle S^{st} \rangle \left\langle \frac{1}{1 + \omega} \right\rangle = \frac{1}{N} \sum_{i \in A} \frac{c_i \tilde{S}_i}{(c_i + (1 - c_i) \tilde{\gamma}_i)(1 + \hat{\omega}_i)} + \langle S^{st} \rangle \left\langle \frac{1}{N} \sum_{i \in A_1} \frac{(1 - c_i) \tilde{\gamma}_i}{(c_i + (1 - c_i) \tilde{\gamma}_i)(1 + \hat{\omega}_i)} + \frac{1}{N} \sum_{i \in B_1} \frac{\tilde{S}_i^0}{1 + \hat{\omega}_i} \right\rangle
\]

Taking into account the statistical independence assumption we obtain:

\[
\langle S^{st} \rangle = \frac{(N_1/N) \left\langle \frac{d}{(d+(1-d) \tilde{\gamma}) (1+\hat{\omega})} \right\rangle_A \langle \bar{S} \rangle_A + (N_0/N) \left\langle \frac{\tilde{S}_i^0}{1 + \hat{\omega}_i} \right\rangle_{B_1}}{(N - N_0)/N} \left\langle \frac{d}{(d+(1-d) \tilde{\gamma}) (1+\hat{\omega})} \right\rangle_{B_1} + \langle N_0/N \left\langle \frac{1}{1 + \hat{\omega}} \right\rangle_{B_1} \right\rangle
\]

where \( \left\langle \frac{d}{(d+(1-d) \tilde{\gamma}) (1+\hat{\omega})} \right\rangle_{B_1} = \frac{1}{N - N_0} \sum_{i \in B_1} \frac{s_i}{(c_i + (1 - c_i) \tilde{\gamma}_i)(1 + \hat{\omega}_i)} \); \( \langle \bar{S} \rangle_A = \frac{1}{N_1} \sum_{i \in A} \tilde{S}_i \); \( \left\langle \frac{\tilde{S}_i^0}{1 + \hat{\omega}_i} \right\rangle_{B_1} = \frac{1}{N_0} \sum_{i \in B_1} \frac{s_i^0}{1 + \hat{\omega}_i} \) and \( \langle \frac{1}{1 + \hat{\omega}} \rangle_{B_1} = \frac{1}{N_0} \sum_{i \in B_1} \frac{1}{1 + \hat{\omega}_i} \).

**Proof of lemma 10**

**Proof of (i) Notice that** \( U(\cdot, \cdot) \) **is the sum of quadratic forms with respect to the state and control variables, then it is clearly continuous, differentiable on the interior of its domain. As** \( S^{(t)} \) **and** \( S^{(t+1)} \) **are defined in the compact set** \( [0, 1] \) **then** \( U(\cdot, \cdot) \) **is bounded.**
Proof of (ii) In order to study the concavity condition, we have to compute the Hessian matrix 
\[
\left( \frac{\partial^2 U(S^{(t)}, S^{(t+1)})}{\partial S_i \partial S_j} \right)_{i,j=1,\ldots,N; k \in \{t,t+1\}}.
\]

Doing some computation we obtain:

\[
H_{2N} \equiv \begin{pmatrix}
M_N(-\beta \alpha^2, -\beta \frac{\gamma}{N}) & M_N(\beta \alpha, 0) \\
M_N(\beta \alpha, \beta \frac{\gamma}{N}) & M_N(- (\beta + 2), 0)
\end{pmatrix}
\]  \hspace{1cm} (37)

where \( \alpha = 1 - \gamma(1 - \frac{1}{N}) \). Let \( H_k \) be the principal minor of size \( k \in \{1, \ldots, N, N + 1, \ldots, 2N\} \) of the Hessian matrix \( H \). In order to prove that the Hessian matrix \( H \) is negative we compute the determinant of all principal minors of \( H \). For \( n \in \{1, \ldots, N\} \), from the results in Appendix 5.2 we obtain the determinant of \( H_n = (a-b)^{n-1} (a+(n-1)b) \) where \( a = -\beta \alpha^2 \) and \( b = -\beta \frac{\gamma}{N} \).

Note that \( (a+(n-1)b) = -\beta \alpha^2 -(n-1)\beta \alpha \frac{\gamma}{N} < 0 \) for all \( N \). Then the sign of \( \det(H_n) \) depends on the sign of \( (a-b)^{n-1} \). As \( (a-b)^{n-1} = \beta \alpha(\frac{\gamma}{N} - \alpha) = (\gamma - 1) \) then \( (a-b)^{n-1} < 0 \) when \( n \) is odd and positive when \( n \) is even.

Consequently, for \( k \leq N \) odd, the minor \( H_k \) has negative sign and positive sign when \( k \) is even.

For \( k \in \{N + 1, \ldots, 2N\} \) let us proceed by induction. We claim that \( \det(H_k) = (-2(\beta-1))^{k-N} \det(H_N) \) for \( 2N \geq k > N \). Consider the first case by fixing \( k = N + 1 \). Then the principal minor is:

\[
H_{N+1} \equiv \begin{pmatrix}
-\beta \alpha^2 & -\beta \alpha \frac{\gamma}{N} & -\beta \alpha \frac{\gamma}{N} & \cdots & \beta \alpha \\
-\beta \alpha \frac{\gamma}{N} & -\beta \alpha^2 & -\beta \alpha \frac{\gamma}{N} & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots & \cdots \\
-\beta \alpha & -\beta \frac{\gamma}{N} & -\beta \frac{\gamma}{N} & \cdots & -(\beta + 2)
\end{pmatrix}
\]  \hspace{1cm} (38)

By computing the determinant by adjoints we get that \( \det(H_{N+1}) \) is equal to \( (-1)^{N+2} \beta \alpha (-1)^{N-1} \frac{\gamma}{\alpha} \det(H_N) + (-1)^{2N+2} (\beta + 2) \det(H_N) = -2(\beta - 1) \det(H_N) \).

Suppose that the claim holds for \( k \). Then \( \det(H_k) = (-2(\beta - 1))^{k-N} \det(H_N) \). By making the same computations than the first case, we get the case \( k + 1 \) where \( \det(H_{k+1}) = (-1)^{N+k+1} \beta \alpha (-1)^{N-1} \frac{\gamma}{\alpha} \det(H_k) + (-1)^{2(N+k)} (\beta + 2) \det(H_k) = -2(\beta - 1) \det(H_k) = -2(\beta - 1)(-2(\beta - 1))^{k-N} \det(H_N) = (-2(\beta - 1))^{k+1-N} \det(H_N) \).

Therefore the sign changes every stage from the case \( N \). Hence, \( U(\cdot, \cdot) \) is strictly concave.

Proof of (iii)

Proof of Proposition 12

By 24 and given that \( S^{(t+2)} = S^{(t+1)} = S^{(t)} \), we get the set of the following equations for all \( i \): \( 2(\tilde{S}_i - \tilde{S}_{i^*}) + \beta \tilde{\gamma} (\langle S_{\text{at}} \rangle - \tilde{S}_{i^*}) - \delta \beta \tilde{\gamma} (\langle S_{\text{at}} \rangle - \tilde{S}_{i^*}) = 0 \). Getting

\[
2(\tilde{S}_i - \tilde{S}_{i^*}) + (\beta \tilde{\gamma}(1 - \delta \tilde{\gamma}))(\langle S_{\text{at}} \rangle - \tilde{S}_{i^*}) = 0. \hspace{1cm} (39)
\]
Taking averages:

\[ 2(\langle S \rangle - \langle S^{st} \rangle) + (\beta \gamma (1 - \delta \gamma))(\langle S^{st} \rangle - \langle S^{st} \rangle) = 0 \]

Therefore \( \langle S \rangle = \langle S^{st} \rangle \). By use \( \langle S \rangle \) in place of \( \langle S^{st} \rangle \) in equation 39 we obtain \( S^{st}_i (2 + \beta \gamma (1 - \delta \gamma)) = 2\tilde{S}_i + \langle \tilde{S} \rangle (\beta \gamma (1 - \delta \gamma)) \). Then,

\[ S^{st}_i = \frac{2\tilde{S}_i + \langle \tilde{S} \rangle (\beta \gamma (1 - \delta \gamma))}{2 + \beta \gamma (1 - \delta \gamma)}. \]

**Proof of proposition 11**

In Appendix 2 it is introduced the family of \( N \times N \) matrices denoted by \( M_N(a,b) \) which are characterised by having a common value, \( a \), at the diagonal and another common value, \( b \), at all non-diagonal entries. We then define:

\[ A_1 = M_N(a_1, b_1), \quad A_2 = M_N(a_2, b_2) \tag{40} \]

with \( a_1 = \alpha + \frac{1}{\alpha \delta} \left( \frac{2}{\beta} + 1 \right), b_1 = \frac{\gamma}{N}, a_2 = -\frac{1}{\delta}, \) and \( b_2 = -\frac{\gamma}{\alpha \delta N} \). With this notation and defining the vectors:

\[ S^{(t)} = \begin{pmatrix} S_1^{(t)} \\ \vdots \\ S_N^{(t)} \end{pmatrix}, \quad \tilde{S} = \begin{pmatrix} \tilde{S}_1 \\ \vdots \\ \tilde{S}_N \end{pmatrix}, \tag{41} \]

the recurrence relation 24 reads:

\[ S^{(t+2)} = A_1 S^{(t+1)} + A_2 S^{(t)} - \frac{2}{\alpha \beta \delta} \tilde{S}. \tag{42} \]

We first convert it into a homogeneous relation by defining \( Z^{(t)} = S^{(t)} - S^{st} \), with

\[ S^{st} = -\frac{2}{\alpha \beta \delta} (1 - A_1 - A_2)^{-1} \tilde{S}. \tag{43} \]

As \( 1 - A_1 - A_2 = M_N(1 - a_1 - a_2, -b_1 - b_2) \), it is possible after some algebra, and using the results of Appendix 2, to derive the explicit expression Eq.(43). The homogeneous relation reads

\[ Z^{(t+2)} = A_1 Z^{(t+1)} + A_2 Z^{(t)}. \tag{44} \]

It is clear that given \( Z^{(0)} \) and \( Z^{(1)} \) this linear recursion relation is uniquely determined. By trying \( Z^{(t)} = \Pi^t C \), being \( C \) a vector, we derive that the matrix \( \Pi \) must satisfy \( \Pi^2 = A_1 \Pi + A_2 \). If we now make the ansatz that \( \Pi = M_N(a,b) \), the two
eigenvalues of \( \Pi \), \( \lambda_1, \lambda_2 \) are given by solving the pair of quadratic algebraic equations whose coefficients are the eigenvalues of \( A_1 \) and \( A_2 \):

\[
\begin{align*}
\lambda_1^2 &= \lambda_1(b_1(N-1) + a_1) + b_2(N-1) + a_2, \\
\lambda_2^2 &= \lambda_2(a_1 - b_1) + a_2 - b_2.
\end{align*}
\]

(45) (46)

One can solve for \( \lambda_1 \) and \( \lambda_2 \) independently. The coefficients for both equations are:

\[
\begin{align*}
b_1(N-1) + a_1 &= 1 + \frac{2}{\alpha} + 1, \\
b_2(N-1) + a_2 &= -\frac{1}{\alpha}, \\
a_1 - b_1 &= 1 - \gamma + \frac{2}{\alpha} + 1, \\
a_2 - b_2 &= -\frac{1}{\alpha}.
\end{align*}
\]

(47) (48) (49) (50)

We prove now a lemma:

**Lemma:** Let \( \lambda^-, \lambda^+ \) be the two roots of the equation \( \lambda^2 - (a + AB)\lambda + aA = 0 \) with \( 0 < a < 1, A > 1, B > 1 \). Then the roots are real and satisfy \( 0 < \lambda^- < a, \lambda^+ > A \).

**Proof:** The roots are:

\[
\lambda^\pm = \frac{1}{2} \left[ a + AB \pm \sqrt{(a + AB)^2 - 4aA} \right]
\]

(51)

Both roots are real as \( (a + AB)^2 > (a + A)^2 \) and \( (a + A)^2 > 4aA \) as this inequality is equivalent to \( (A - a)^2 > 0 \). As we are subtracting from \( a + AB \) a smaller number it turns out that \( \lambda^- > 0 \). We now prove that \( \lambda^- < a \) or \( AB - a < \sqrt{(a + AB)^2 - 4aA} \). As \( AB > a \) we can square safely to prove that \( (AB - a)^2 < (a + AB)^2 - 4aA \) or \( aA < aAB \) which is true. As the product of the roots is \( aA \) and \( \lambda^- < a \) it follows that \( \lambda^+ > A \).

Introducing \( B = 1 + 2/\beta \) and \( A = 1/(\alpha\delta) \) and \( a = 1 \) or \( a = 1 - \gamma \) this lemma proves that \( 0 < \lambda^-_1 < 1, \lambda^+_1 > 1/(\alpha\delta) \) and \( 0 < \lambda^-_2 < 1 - \gamma, \lambda^+_2 > 1/(\alpha\delta) \). Explicit expressions are:

\[
\begin{align*}
\lambda^+_1 &= \frac{1}{2} \left( 1 + AB \pm \sqrt{(1 + AB)^2 - 4A} \right), \\
\lambda^+_2 &= \frac{1}{2} \left( 1 - \gamma + AB \pm \sqrt{(1 - \gamma + AB)^2 - 4(1 - \gamma)A} \right)
\end{align*}
\]

(52) (53)

We construct \( \Pi_+ = M_N(a^+, b^+) \) using the two eigenvalues \( \lambda^+_1, \lambda^+_2 \) larger than 1, and \( \Pi_- = M_N(a^-, b^-) \) using the two eigenvalues \( \lambda^-_1, \lambda^-_2 \) smaller than 1. Therefore
The corresponding values of $a^\pm$ and $b^\pm$ follow from

\begin{align}
a^\pm &= \frac{\lambda_1^\pm + (N - 1)\lambda_2^\pm}{N} \\
b^\pm &= \frac{\lambda_1^\pm - \lambda_2^\pm}{N}
\end{align}

(54) (55)

The (unique) solution can be written as

$$Z^{(t)} = \Pi^+ C_1 + \Pi^- C_2,$$

where vectors $C_1$ and $C_2$ are chosen to satisfy the initial conditions for $t = 0, 1$, namely $C_1 = (\Pi^+ - \Pi^-)^{-1}(Z^{(1)} - \Pi^- Z^{(0)})$ and $C_2 = Z^{(0)} - C_1$.

As the eigenvalues $\lambda_1^{\pm,2} > 1$, it turns out that the coefficients of $\Pi^+$ diverge for $t \to \infty$. Next lemma shows that the transversality condition holds when the only possibility is that the vector $C_1$ is equal to 0. Consequently, undoing the change of variables, the solution of the recurrence relation for vector $S^{(t)}$ is:

$$S^{(t)} = S^{st} + \Pi^+ (S^{(0)} - S^{st}).$$

(56) (57)

which can be written in the form of Eq.(25).

**Lemma 17** The condition of transversality holds if and only if the vector of parameters $C_1$ is the vector 0.

As consequence of the above lemma the solution of the recurrence relation for vector $S^{(t)}$ is:

$$S^{(t)} = S^{st} + \Pi^- (S^{(0)} - S^{st}).$$

(58)

**Proof**

The condition of transversality says:

$$\lim_{t \to \infty} \delta^t D_{S^{(t)}} (U(S^{st}), U(S^{(t+1)})S^{st}) = 0.$$

In our context we derive:

$$\lim_{t \to \infty} \delta^t D_{S^{(t)}} (U(S^{(t)}), U(S^{(t+1)})S^{(t)}) = \lim_{t \to \infty} \delta^t [\frac{\gamma}{N} \sum_{i=1}^{N} [S_i^{(t+1)} - S_i^{(t)} - \gamma((S^{(t)} - S^{st})_i)] \sum_{i=1}^{N} S_i^{(t)}$$

$$- [\beta(\gamma - 1) \sum_{i=1}^{N} S_i^{(t)}(S_i^{(t+1)} - S_i^{(t)} - \gamma((S^{(t)} - S^{st})_i))]

$$

Doing some algebra we get the following expression:

$$\lim_{t \to \infty} \delta^t [\beta(1 - \gamma)(S^{(t+1)}S^{(t)} - (1 - \gamma)S^{st}S^{st})$$

$$+ \frac{\beta\gamma}{N} \sum_{i=1}^{N} (\sum_{i=1}^{N} S_i^{(t+1)})S_i^{(t)} + \frac{\beta\gamma^2}{N} \sum_{i=1}^{N} (\sum_{i=1}^{N} S_i^{(t)})S_i^{(t)}].$$

35
As the three components of the above limit are positive then each term must have a limit equals to 0. Formally:

\[
\lim_{t \to \infty} \delta^t \left[ \beta (1 - \gamma) (S^* (t+1) S^* (t) - (1 - \gamma) S^* (t) S^* (t)) \right] = 0
\]

\[
\lim_{t \to \infty} \delta^t \left[ \frac{\beta \gamma}{N} \sum_{i=1}^{N} \left( \sum_{i=1}^{N} S_i^{*(t+1)} S_i^{*(t)} \right) \right] = 0
\]

\[
\lim_{t \to \infty} \delta^t \left[ \frac{\beta \gamma^2}{N} \sum_{i=1}^{N} \left( \sum_{i=1}^{N} S_i^{*(t)} S_i^{*(t)} \right) \right] = 0
\]

We know that \( S(t) = S^{st} + \sum_{i=1}^{N} \Pi_i^{t+1} C_1 + \sum_{i=1}^{N} \Pi_i^{t+1} C_2 \) where \( \Pi_i^+ = M_N (a^+, b^+) \) with eigenvalues \( \lambda_1^+, \lambda_2^+ \) larger than 1, and \( \Pi_i^- = M_N (a^-, b^-) \) with eigenvalues \( \lambda_1^-, \lambda_2^- \) smaller than 1. Moreover, the corresponding values of \( a^\pm \) and \( b^\pm \) follow

\[
a^\pm = \frac{\lambda_1^\pm + (N-1) \lambda_2^\pm}{N}
\]

\[
b^\pm = \frac{\lambda_1^\pm - \lambda_2^\pm}{N}
\]

Therefore

\[
\sum_{i=1}^{N} S_i^{*(t+1)} = \sum_{i=1}^{N} S_i^{st} + \sum_{i=1}^{N} \Pi_i^{t+1} C_1 + \sum_{i=1}^{N} \Pi_i^{t+1} C_2 = \sum_{i=1}^{N} S_i^{st} + (\lambda_1^+)^{t+1} \sum_{i=1}^{N} (C_1)_i + (\lambda_1^-)^{t+1} \sum_{i=1}^{N} (C_2)_i
\]

Analogously,

\[
\sum_{i=1}^{N} S_i^{st(t)} = \sum_{i=1}^{N} S_i^{st} + \sum_{i=1}^{N} \Pi_i^t C_1 + \sum_{i=1}^{N} \Pi_i^t C_2 = \sum_{i=1}^{N} S_i^{st} + (\lambda_1^+)^{t+1} \sum_{i=1}^{N} (C_1)_i + (\lambda_1^-)^{t+1} \sum_{i=1}^{N} (C_2)_i
\]

Let us compute now \( \lim_{t \to \infty} \delta^t \left[ \frac{\beta \gamma}{N} \sum_{i=1}^{N} S_i^{*(t+1)} \right] \left( \sum_{i=1}^{N} S_i^{*(t)} \right) \):
\[
\lim_{t \to \infty} \delta^t \left[ \frac{\beta\gamma}{N} \sum_{i=1}^{N} \left( \sum_{i=1}^{N} S_i^{(t+1)} \right) S_i^{(t)} \right] =
\]

\[
\lim_{t \to \infty} \delta^t \left[ \frac{\beta\gamma}{N} \sum_{i=1}^{N} S_i^t + \sum_{t=1}^{N} (C_1) + (\lambda^0_t) + \sum_{t=1}^{N} (C_2) \right] =
\]

\[
\lim_{t \to \infty} \left( \frac{\beta\gamma}{N} \lambda^0_t \right) \delta(\lambda^0_t)^2 \left[ \sum_{i=1}^{N} S_i^t + \sum_{i=1}^{N} (C_1) + (\lambda^0_t) + \sum_{i=1}^{N} (C_2) \right] =
\]

As \((\delta(\lambda^0_t)^2)^t > 1 \text{ since } \lambda^0_t > \frac{1}{\lambda_0} \) then the above limit is equal to 0 if and only if \((C_1)_i = 0 \) for all \(i\). Similarly we get the same condition for the limit \(\lim_{t \to \infty} \delta^t \left( \frac{\beta\gamma}{N} \sum_{i=1}^{N} S_i^{(t+1)} \right) S_i^{(t)} = 0\).

The last term is computed in the same way. Recall that \(S^t = \sum_{i=1}^{N} S_i^t\).

\[
\lim_{t \to \infty} \delta^t \left[ \beta (1 - \gamma)(S^t + \Pi^t + C_1) - (1 - \gamma)(S^t + S^t) \right] = 0
\]

\[
\lim_{t \to \infty} \delta^t \beta (1 - \gamma)(S^t + \Pi^t + C_1 + \Pi^t + C_2 - (1 - \gamma) \Pi^t C_1 + \Pi^t C_2) =
\]

\[
\lim_{t \to \infty} \delta^t (1 - \gamma)(\Pi^t + \Pi^t \Pi^{-1}_t + \Pi^t \Pi^{-1}_t)^{-1} S^t + C_1 + \Pi^t \Pi^{-1}_t \Pi^t C_2 - (1 - \gamma) (\Pi^t)^{-1} S^t + C_1 + \Pi^t (\Pi^t)^{-1} C_2 = 0
\]

In order to get the above limit it is necessary and sufficient that the vector \(C_1\) must be equal to 0.

**Appendix 2: Properties of matrices** \(M_N(a, b)\)

Consider the family of \(N \times N\) matrices denoted by \(M_N(a, b)\) which are characterised by having a common value, \(a\), at the diagonal and another common value, \(b\), at all non-diagonal entries. Namely,

\[
M_N(a, b) \equiv \begin{pmatrix}
    a & b & b & \ldots & b & b \\
    b & a & b & \ldots & b & b \\
    \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\
    b & b & b & \ldots & a & b \\
    b & b & b & \ldots & b & a
\end{pmatrix}
\]

(59)
The following summarises some properties of such set of matrices:

- **Eigenvalues:**
  \[
  \lambda_1 = a + (N - 1)b, \quad \text{multiplicity 1}, \\
  \lambda_2 = a - b, \quad \text{multiplicity } N - 1.
  \]

The eigenvector matrix is:

\[
\varphi_N = \begin{pmatrix}
1 & 1 & 1 & \ldots & 1 & 1 \\
1 & -1 & 0 & \ldots & 0 & 0 \\
1 & 0 & -1 & \ldots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
1 & 0 & 0 & \ldots & -1 & 0 \\
1 & 0 & 0 & \ldots & 0 & -1
\end{pmatrix}
\]

(60)

and all eigenvectors are independent of \(a\) and \(b\).

- **Determinant:**
  \[
  \text{Det}[M_N(a, b)] = (a - b)^{N-1}(a + (N - 1)b).
  \]

- **Product**
  \[
  M_N(a_1, b_1)M_N(a_2, b_2) = M_N(a_2, b_2)M_N(a_1, b_1) \\
  = M_N(a_1a_2 + (N - 1)b_1b_2, a_1b_2 + a_2b_1 + (N - 2)b_1b_2)
  \]
  (61)

- **Powers.** If \(t \in \mathbb{Z}\), \([M_N(a, b)]^t = M_N(a^{(t)}, b^{(t)})\) with
  \[
  a^{(t)} = \frac{(a + (N - 1)b)^t + (N - 1)(a - b)^t}{\frac{N}{N}} = \frac{1}{N} (\lambda_1^t + (N - 1)\lambda_2^t)
  \]
  (62)

  \[
  b^{(t)} = \frac{(a + (N - 1)b)^t - (a - b)^t}{\frac{N}{N}} = \frac{1}{N} (\lambda_1^t - \lambda_2^t)
  \]
  (63)

- **Particular case, inverse matrix**
  \[
  [M_N(a, b)]^{-1} = M_N(a^{-1}, b^{-1})
  \]
  (64)

with

\[
 a^{-1} = \frac{a + (N - 2)b}{(a - b)(a + (N - 1)b)}
\]

(65)

\[
 b^{-1} = \frac{-b}{(a - b)(a + (N - 1)b)}
\]

(66)

- **Multiplication by a vector:**
  \[
  [M_N(a, b)x]_i = (a - b)x_i + Nb \langle x \rangle.
  \]