Slow Moving Debt Crises*

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We study slow moving debt crises, defined as self-fulfilling equilibria where a sudden increase in interest rates due to fears of higher future default leads to a gradual but faster accumulation of debt, ultimately validating investors’ fears. Such slow moving debt crises may be contrasted with rollover crises, which prompt immediate default. We show that slow moving crises coexist with low-interest equilibria in a variety of settings, in which fiscal policy either follows a fixed rule or is discretionarily optimized. We discuss how multiplicity can be avoided depending on the initial debt level, the shape of the fiscal policy rule, and the maturity of debt. We also find that debt dynamics can be characterized by a “tipping point,” below which debt is stationary and above which debt grows towards certain default. Multiplicity arises when instead of a tipping point we have a “tipping region”, i.e., an interval of debt values from which both outcomes are possible in equilibrium.

1 Introduction

Yields on sovereign bonds for Italy, Spain and Portugal shot up dramatically in late 2010 with nervous investors suddenly casting the debt sustainability of these countries into doubt. An important concern for policy makers was the possibility that higher interest rates were self-fulfilling. High interest rates, the argument goes, contribute to the rise in debt over time, eventually driving countries into insolvency, thus justifying higher interest rates in the first place. News coverage illustrates how future debt dynamics were at

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Thomson Reuters offered a simple web application, under the title “Italian Debt Spiral”, that computed the primary surplus needed to stabilize the debt-to-GDP ratio under different scenarios.

Yields subsided in the late summer of 2012 after the European Central Bank’s president, Mario Draghi, unveiled plans to purchase sovereign bonds to help sustain their market price. A view based on self-fulfilling crises was explicitly used to justify such interventions during Draghi’s news conference announcing the Outright Monetary Transactions (OMT) bond-purchasing program (September 6th, 2012),

“The assessment of the Governing Council is that we are in a situation now where you have large parts of the Euro Area in what we call a bad equilibrium, namely an equilibrium where you have self-fulfilling expectations. You may have self-fulfilling expectations that generate, that feed upon themselves, and generate adverse, very adverse scenarios. So there is a case for intervening to, in a sense, break these expectations [...]”

If this view is correct, a credible announcement to do “whatever it takes” is all it takes to rule out bad equilibria, no bond purchases need actually be carried out. To date, this is exactly how it seems to have played out: there have been no purchases by the ECB and no country has applied to the OMT program.

In this paper, we build a dynamic model that formalizes this multiple-equilibria view of debt crises. We then use this model to explore how the initial debt level, the fiscal policy regime, and the maturity structure affect the possibility of a crisis.

In our model, the government sells bonds to a large group of investors to finance and smooth shocks to its funding needs. Investors are risk neutral and price bonds according to their expected payoffs, forming expectations of future default probabilities. Given bond prices and fiscal policy, one can compute the path for debt. This, in turn, affects future default probabilities and, through investors’ expectations, bond prices. This creates a feedback loop between interest rates and debt accumulation, opening the door to multiple equilibria.

When a crisis occurs—an unexpected switch from a good to a bad equilibrium path—bond prices jump in response to changes in future default probabilities, but the crisis can

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1For example, a Financial Times’ report on the Italian bond market cites a pessimistic observer expecting “Italian bonds to perform worse than Spanish debt this summer, as investors focus on the sustainability of Italy’s debt burden,” given Italy’s high initial debt-to-GDP ratio. A more optimistic investor in the same report argues that Italy “can cope with elevated borrowing costs for some time particularly when shorter-dated bond yields remain anchored” and that “it’s critical to bring these yields down, but there is time for Italy to establish that its policies are working.” (“Investors wary of Italy’s borrowing test”, Financial Times, July 16, 2012.)

2See the widget here.
play out for a while before default actually occurs. We label this episode a slow moving crisis to distinguish it from a rollover crisis, which is essentially a run on the country’s debt leading to a failed bond auction and immediate default. Both slow moving crises and run-like episodes seem relevant to interpret observed turbulence in sovereign debt markets. However, the academic literature has mostly focused on the latter. Our aim here is to bring attention to the slow moving mechanism, so the paper focuses on it.

The multiplicity of equilibria in which we are interested, is due to a coordination problem across investors in sovereign debt markets, within and across time. The precise way in which the borrower is modeled is less crucial for this purpose. In most of the paper, we assume that the borrower is committed to follow a fixed fiscal rule. The fiscal rule determines the primary surplus as a function of the debt level and is subject to exogenous shocks. Default occurs mechanically when the borrower is unable to finance coupon payments on the existing stock of debt. However, we also consider versions of our model in which fiscal policy and default are chosen by an optimizing government, under discretion.

**Short-term debt, long-term debt and tipping points.** Our first contribution is to provide a general recursive construction of bond price schedules and debt dynamics when the stochastic process for the primary surplus is given by a fiscal policy rule. This construction extends the standard analysis of debt sustainability (e.g., Bohn (1995), Hall (2014)) to the case of defaultable debt.

We then analyze the case of short-term debt. When all debt is in the form of one-period bonds, the bond price schedule and the maximum revenue from debt issuance are uniquely determined at each point in time. This does not imply that the equilibrium path is unique: multiplicity is possible because the revenue from debt issuance is non-monotone in the amount of bonds issued—we call this revenue function a Laffer curve. For a given level of revenue there are generally multiple bond prices that clear the market, lying on different sides of the Laffer curve. However, equilibria on the downward sloping part of the Laffer curve are unstable, in a sense that we discuss. A natural refinement is then to require stability. Multiple interior stable equilibria are still possible, but require primitives that deliver Laffer curves with multiple peaks. Whether or not there exist multiple equilibria that are both interior and stable there is always an extreme equilibrium “at infinity”, with a bond price of zero and immediate default. This equilibrium is also stable and can be interpreted as a form of rollover crisis.

We then turn to long-term debt and show that the bond price function is no longer

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3The relation with this literature is discussed below.
uniquely determined. Future bond prices now feed back into the current bond price schedule. Now the coordination problem among investors takes an intertemporal dimension, so that the bad equilibrium cannot be prevented even if all current investors were to coordinate at a given point in time.

To say more, we specialize the model with long term debt in two directions. First, we consider a model in which all uncertainty is resolved at a fixed date $T$. This model allows us to extend the characterization of an equilibrium in terms of a debt Laffer curve. We show that multiple stable equilibria can exist even if this Laffer curve is single peaked. Second, we consider a stationary environment, in which uncertainty is resolved at a stochastic date with a constant Poisson arrival rate and study Markov equilibria, with debt serving as the state variable.

The stationary model provides a clean characterization of the state space that captures the idea a “tipping point” for debt dynamics, which has been floating in the public debate on debt sustainability (see, in particular, Greenlaw et al. (2013)). In our model there is a cutoff level of debt such that if we start below it there exists an equilibrium path with decreasing debt converging towards a stable steady state. There is also a second cutoff such that if we start above it there exists an equilibrium path with ever rising debt and default probability converging to one. Depending on parameters, the two cutoffs can coincide, in which case we have a unique equilibrium with a tipping point. Alternatively, the second cutoff can be strictly lower than the first, in which case we have a “tipping interval,” in between the two cutoffs, where both the decreasing path and the increasing path are equilibria.

**Policy rules and maturity.** Using the specialized long-term debt models, we investigate the role of the policy rule and of the maturity structure.

If the fiscal rule is sufficiently active at reducing deficits when debt rises, it can sustain a unique equilibrium. The slope of the fiscal rule is acting to directly offset the negative feedback coming from endogenous interest rates. High interest rates induce a rise in debt, but that cannot lead to a self-fulfilling crisis if the borrower is expected to take strong actions to repay when the debt level rises. Note that when this is the case we may never observe the need for such efforts—it is out-of-equilibrium efforts that rule out the loss in investors’ confidence and avoid the crises even starting. This may explain why some countries are more prone to these kind of crises than others and why credibility appears hard to earn.

In the context of the stationary model, we investigate what properties of the policy rule ensure that a steady state is saddle-path stable. In a model without default the usual
requirement for stability is that the slope of the rule be larger than the interest rate (see, e.g., Hall (2014)). We show that the presence of default requires a more aggressive rule, not just because the interest rate is higher but because it responds endogenously to an increase in debt. The stationary model also helps to clarify that a bad equilibrium is possible even if we start at a point in the state space where the rule is sufficiently aggressive and in which a good equilibrium converging to a “good”, low debt steady state, exists. The crucial problem is that the policy rule cannot be equally aggressive for all levels of debt, as the primary surplus must be eventually bounded above by the government taxation capacity. But then it is possible that, on top of the good equilibrium, we have a bad equilibrium in which debt is expected to reach levels at which the rule is less responsive and the responsiveness of the fiscal rule today may be insufficient to counteract the drop in bond prices today, which factors in expected future fiscal policy responses.

We also show that a longer debt maturity contributes towards uniqueness. A short maturity requires constant refinancing, exposing the borrower to increases in interest rates and potentiating the feedback effects, whereas long maturity mutes these forces. The mechanism for this result is different from that found in rollover crises, where short debt affects the strategic decision to default, by making repayment more costly in the event of a rollover crisis.

A noteworthy property is that, in our model, the self-fulfilling nature of crises is transitory. If the economy slips along the bad path long enough, debt eventually crosses a cutoff beyond which the good path is no longer accessible. Although initially triggered by self-fulfilling pessimism, the crises eventually damages fundamentals. This highlights the importance of applying policies aimed at avoiding or escaping a bad equilibrium swiftly, avoiding delays that lead to irreparable damage and allow the bad equilibrium to settle in.

**Optimizing government and timing issues.** In the last part of the paper, we show that multiple equilibria with slow moving features also arise in models that include an optimizing government, maximizing a discounted sum of utility under discretion.

First, we consider an optimizing version of our Poisson model and show that a crucial constraint for the presence of multiplicity is a non-negativity constraint on spending. Absent this non-negativity constraint, the equilibrium is unique for the following reason. Consider a borrower with debt just above the threshold separating the good and the bad paths. The borrower considers making an effort to save enough to reach the threshold and switch to the good path. Because the effort is proportional to the distance from the

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4Ghosh et al. (2011) refer to this problem as “fiscal fatigue.”
threshold, if there were a discontinuous jump in utility from crossing the threshold, then borrowers near the threshold would find it optimal to reach the threshold. This would unravel the proposed equilibrium. By implication the threshold must be set to equalize utilities, which pins down the equilibrium uniquely. This logic fails when we impose a non-negativity constraint on spending as long as reducing debt below the threshold would require negative spending, due to the high interest rates along the bad path. When this is the case the proposed equilibrium cannot be broken by the effort of those just above the threshold. This extra freedom in determining the threshold leads to multiple equilibria. It is important to point out that in our example the non-negativity constraint does not bind in equilibrium, only off the equilibrium path were the borrower to attempt to cross the threshold.

This example illustrates a more general point. One may impose other constraints on governments, say, a lower bounds different from zero or a limit on the rate of change of spending. Realistically, governments face a host of such restrictions. Such constraints will generally widen the opening for equilibrium multiplicity to sneak in. Once again, these constraints may not even bind along any equilibrium paths, but they condition deviations off equilibrium.

We also use two models with an optimizing government to discuss issues of timing and equilibrium selection. In most of the paper, we take the following approach: past debt obligations and the current surplus determine the government’s financing needs on the bond market; any combination of price and quantity of bonds issued that satisfies these financing needs is considered part of a possible equilibrium.\(^5\) Why can’t the government choose the quantity of bonds issued and thus select the equilibrium price? Our argument is that allowing the government to choose the quantity of bonds amounts to a strong commitment assumption: if the revenue of the bond issuance falls short of the planned financing needs, the government commits to adjust the surplus instantaneously rather than issuing more bonds. This commitment assumption seems incredible, as it requires an extremely elastic policy rule in the very short run. We prefer to assume that the government has limited room for primary surplus adjustment in the short run. The two models in Section 5 provide a formalization of this argument and a microfoundation for the approach followed in the rest of the paper. Namely, we formulate two explicit games describing an optimizing government that can issue bonds in repeated rounds. We assume the government can commit to a certain level of bond issuance for the current round, but not for future rounds.

The first game features a potentially unlimited number of bond auctions within each

\(^5\)Modulo the stability considerations mentioned above.
“period”. As a result, the government loses all ability to commit to its bond issuance, since it can always reverse or increase issuances in the next round. Subgame perfect equilibrium of this game provides an explicit microfoundation for the timing assumption used in the rest of the paper.

The second game allows for a finite number of auctions, but introduces preferences that are not additively separable across periods. Lower funds acquired in the market today increase the need for funds tomorrow. This seems directly relevant for infrastructure investments, but it may also capture elements of the government payroll—a temporary shortfall in payments is possible, incurring a debt with government employees and pensioners, but the accumulated balances must be repaid eventually. For this game, we show that there may be multiple subgame-perfect equilibria with different bond prices, similar to the equilibria we isolate in this paper. The second game shows that our form of multiplicity can arise even in environments where the government has some ability to commit to bond issuances, as long as one captures intertemporal linkages.

**Related literature.** The main precursor to our paper is Calvo (1988), who models the feedback between interest rates and the debt burden in a two-period model in which the government chooses its financing needs and debt issuances and interest rates are jointly determined on the bond market. Our main goal is to reinvigorate this approach and expand it to a dynamic setting, more appropriate for the study of slow moving crises. A dynamic setting allows us to explore the conditions for multiple equilibria with respect to debt levels, debt maturity and fiscal policy rules. We also find that the nature of the multiplicity, coupled with stability refinements, is somewhat different in a more dynamic setting with long term bonds. A handful of recent papers also go back to the Calvo (1988) approach. Corsetti and Dedola (2011) and Corsetti and Dedola (2013) extend the Calvo (1988) model to investigate whether independent monetary policy is sufficient to insulate a government from confidence crises. Navarro et al. (2014) apply the same approach to a sovereign-default model with short-term debt similar to Arellano (2008), to investigate how likely are sovereign borrowers to enter the multiple equilibrium region.

Our use of a fiscal rule in the baseline model follows the literature on debt sustainability (Bohn, 1995, 2005) and and the literature on the interaction of fiscal and monetary policy e.g. Leeper (1991). The idea of sustainability is to ask what properties of a fiscal policy regime ensures that holders of government debt are expected to be repaid with certainty at all future dates (Hall (2014)). Here, we extend the analysis to the case in which default is possible. The crucial novelty is that then bond prices are endogenous to future debt developments, requiring more aggressive rules to ensure stability.
Rollover crises have been extensively studied by Giavazzi and Pagano (1989), Alesina et al. (1992), Cole and Kehoe (2000).\(^6\) Aguiar and Amador, 2013 provide a recent survey. The latter paper provides the workhorse model for the more recent literature, including Conesa and Kehoe (2012) and Aguiar et al. (2013). We see rollover crises and slow moving crises as complementary explanations for turbulent sovereign debt markets. Indeed, rollover crises are also possible in our model, and slow moving crises may be thought of as less extreme versions of the same phenomena. For most of the paper we set rollover crises aside, so as to focus on slow moving crises.

2 Bond Price Schedules and Debt Dynamics

In this section, we show how to construct bond price functions and maximum debt revenue levels in a setup where the government is committed to follow a fixed fiscal rule and default occurs when the government is unable to raise enough debt revenue to cover its current financing needs.

Consider a discrete-time environments, with periods \( t = 0, 1, 2, \ldots \) To simplify, we assume that all uncertainty is revealed at some finite date \( T < \infty \). This assumption will be relaxed later, but it allows us to solve the model by backward induction, which is both revealing and simple, and ensures that multiplicity is not driven by an infinite horizon.

The government generates a sequence of primary fiscal surpluses \( \{s_t\} \), representing total taxes collected minus total outlays on government purchases and transfers. A negative realization of \( s_t \) corresponds to a primary deficit.

The government issues non-contingent bonds in a competitive credit market to a continuum of risk-neutral investors with discount factor \( \beta = 1 / (1 + r) \). Bonds have geometrically decreasing coupons: a bond issued at \( t \) promises to pay the sequence of coupons

\[
\kappa, (1 - \delta) \kappa, (1 - \delta)^2 \kappa, \ldots
\]

where \( \delta \in (0, 1) \) and \( \kappa > 0 \). We normalize and set \( \kappa = \delta + r \), so that the bond price equals 1 when the risk of default is zero at all future dates. This well-known formulation of long-term bonds is useful because it avoids having to carry the entire distribution of bonds of different maturities (see Hatchondo and Martinez, 2009). A bond issued at \( t - j \) is equivalent to \( (1 - \delta)^j \) bonds issued at \( t \), so the vector of outstanding bonds can be summarized

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\(^6\)Chamon (2007) argues that the coordination problem in rollover crises can be avoided by appropriately designing the way in which bonds are underwritten and offered for purchase to investors by investment banks.
by a single state variable $b_t$, which is equal to total debt in terms of equivalent newly issued bonds. The parameter $\delta$ controls the maturity of debt, with $\delta = 1$ corresponding to the case of a short-term bond and $\delta = 0$ corresponding to the case of a consol.

Absent default, the government budget constraint is

$$s_t + q_t(s^t) \cdot \left( b_{t+1}(s^t) - (1 - \delta) b_t(s^{t-1}) \right) = \kappa b_t(s^{t-1}),$$

(1)

where $q_t$ is the price of a newly issued bond. Coupon payments on outstanding bonds are covered either by the primary surplus or by sales of newly issued bonds.

The fiscal policy rule and shocks to spending and taxes are all embedded in the stochastic process for the primary surplus, governed by

$$F(s_t \mid s^{t-1}, b_t),$$

a conditional cumulative distribution function, where $s^{t-1} = (s_1, s_2, \ldots, s_{t-1})$ denotes a history up to period $t - 1$. We assume that $s_t$ is bounded above: $s_t \leq s < \infty$. At date $T$ all uncertainty is resolved and the surplus is constant at $s_T$ from then on. We allow the distribution of $s_t$ to depend on current debt $b_t$ to capture policy rules where the government responds to higher debt levels with fiscal efforts to cut spending or raise taxes. Fiscal rules of this kind are commonly adopted in the literature studying solvency (e.g. Bohn, 2005; Ghosh et al., 2011).

We assume that the government honors its debts whenever possible, so that default occurs only if the surplus and borrowing are insufficient to refinance outstanding debt. Let $\chi(s^t) = 1$ denote full repayment and $\chi(s^t) = 0$ denote default. We assume that after a default event debtors receive a recovery value $v_t(s^t) \geq 0$. Therefore, bond prices at date $t$ satisfy

$$q(s^t) = \beta \mathbb{E}\left[ \chi(s^{t+1}) \left( \kappa + (1 - \delta) q(s^{t+1}) \right) + \left( 1 - \chi(s^{t+1}) \right) \frac{v(s^{t+1})}{b_{t+1}(s^t)} \mid s^t \right].$$

(2)

Our focus is on debt dynamics preceding default. Consequently, we characterize the equilibrium up to the first default episode and only derive the path for debt and prices $b_{t+1}(s^t)$ and $q_t(s^t)$ along histories for which $\chi(s^\tau) = 1$ for all $\tau \leq t$.

**Equilibrium.** We derive the equilibrium conditions by backward induction. After period $T$ the surplus is constant, so the government repays if and only if the present value
exceeds the debt burden, or

\[ s_T \geq r b_T. \]

Let the repayment function \( X_T(b_T, s^T) \) equal 1 if this condition is met and 0 otherwise. Note that we have written the last period budget constraint as an inequality, instead of an equality, to allow larger surpluses than those needed to service the debt.\(^7\)

We now have all the elements to compute the price of debt in period \( T - 1 \) for every possible value of \( b_T \),

\[
Q_{T-1}(b_T, s^{T-1}) \equiv \beta \mathbb{E} \left[ X_T(b_T, s^T) (\kappa + 1 - \delta) + \left( 1 - X_T(b_T, s^T) \right) \frac{v(s^T)}{b_T} | s^{T-1}, b_T \right].
\]

The maximal revenue from new debt issuances in period \( T - 1 \) equals

\[
m_{T-1}(b_{T-1}, s^{T-1}) \equiv \max_{b_T} \left\{ Q_{T-1}(b_T, s^{T-1}) \cdot (b_T - (1 - \delta)b_{T-1}) \right\}.
\]

Default can be avoided if and only if

\[
\kappa b_{T-1} - s_{T-1} \leq m_{T-1}(b_{T-1}, s_{T-1}).
\]

Let the repayment function \( X_{T-1}(b_{T-1}, s^{T-1}) \) equal 1 if this inequality is met and 0 otherwise. Whenever \( X_{T-1}(b_{T-1}, s^{T-1}) = 1 \) we select a value of \( b_T \) that solves

\[
s_{T-1} + Q_{T-1}(b_T, s^{T-1}) \cdot (b_T - (1 - \delta)b_{T-1}) = \kappa b_{T-1},
\]

and denote the selected value as \( B_T(b_{T-1}, s^{T-1}) \).

The same procedure can be applied for earlier periods. Suppose we have \( Q_{t+1}, m_{t+1}, X_{t+1} \) and \( B_{t+1} \). We can construct \( Q_t, m_t, X_t \) and \( B_t \) as follows. In period \( t \) the price function \( Q_t \) is given by

\[
Q_t(b_{t+1}, s^t) \equiv \beta \mathbb{E} \left[ X_{t+1}(b_{t+1}, s^{t+1}) (\kappa + 1 - \delta) Q_{t+1}(B_{t+2}(b_{t+1}, s^{t+1}), s^{t+1}))
\]

\[
+ (1 - X_{t+1}(b_{t+1}, s^{t+1})) \frac{v(s^{t+1})}{b_{t+1}} | s^t, b_{t+1} \right], (3)
\]

\(^7\)One should hence interpret \( s_T \) as the potential maximal surplus, not the actual surplus. Of course, if \( s_T > rb_T \), then any slack would be redirected towards lower taxes or increased spending and transfers. We abstract from describing this adjustment.
the maximal revenue function $m_t$ is given by
\[
m_t(b_t, s^t) \equiv \max_{b_{t+1}} \{Q_t(b_{t+1}, s^t) \cdot (b_{t+1} - (1 - \delta)b_t)\},
\] (4)
and let the repayment function $X_t(b_t, s^t) = 1$ if $s_t + m_t(b_t, s^t) \geq \kappa b_t$ and 0 otherwise. Whenever $X_t(b_t, s^t) = 1$ we select a value of $b_{t+1}$ solving
\[
s_t + Q_t(b_{t+1}, s^t) (b_{t+1} - (1 - \delta)b_t) = \kappa b_t.
\] (5)
and denote the selected value by $B_{t+1}(b_t, s^t)$.

Continuing in this way, we can solve iteratively for the functions $m_t, Q_t, X_t, B_{t+1}$ in all earlier periods. In general, equation (5) admits multiple solutions, so there may be multiple sequences of functions \{m_t, Q_t, X_t, B_{t+1}\} satisfying this construction. For any given one, we can then construct equilibrium outcome paths by iterating on $b_{t+1}(s^t) = B_{t+1}(b_t(s^t), s^t)$ starting with the given level of initial debt $b_0$; evaluating $q_t(s^t) = Q_t(b_t(s^t), s^t)$ then gives the sequence of bond prices.

**Multiplicity, Selection and Timing Assumptions.** If at each juncture in the backward induction one always selects the lowest $b_{t+1}$ solving equation (5) the recursion pins down a unique sequence \{m_t, Q_t, X_t, B_{t+1}\} and a unique outcome $b_t(s^t)$ and $q_t(s^t)$.

**Proposition 1.** There is a unique equilibrium satisfying the requirement that $B_{t+1}(b_t, s^t)$ always equal the lowest possible solution to (5).

If, instead, we do not impose a selection criterion, multiple equilibria arise whenever (5) admits multiple solutions for some realizations of $s^t$ and $b_t$. This will be the approach we follow in the rest of the paper.

Let us conclude this section discussing briefly alternative approaches to selection and multiplicity. The sovereign-debt literature following Eaton and Gersovitz (1981)—and the subsequent quantitative literature started by Arellano (2008)—make the assumption that the government chooses the quantity of bonds it auctions each period and then investors bid and determine the bond price. This timing assumption acts as a selection criterion picking a unique value of $b_{t+1}$ in each period, and thus leads to uniqueness.\(^8\) Implicit in this timing assumption there is an assumption about the ability of the government

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\(^8\)In finite horizon versions of the models in Eaton and Gersovitz (1981) and Arellano (2008) the backward induction argument behind Proposition 1 can be easily extended to prove uniqueness. In the infinite horizon case things are more difficult, since a backward induction argument is unavailable. Auclert and Rognlie (2014) show uniqueness in the infinite horizon case under some conditions.
to commit to a given bond issuance, on and off the equilibrium path. Committing to a given bond issuance requires a credible commitment to cut spending or raise taxes if bond prices turn out lower than expected. In practice, the credibility of an announcement of this kind seems questionable.\footnote{Given a non-negativity constraints on spending and an upper bound on tax revenues, one also has to worry about the very feasibility of committing to a given level of bond issuances for any realization of the price \( q_t \) off the equilibrium path. Consider a setup with short-term debt, where the budget constraint is \( q_t b_{t+1} \geq b_t - s_t \). Given a committed level \( b_{t+1} \), if the price is low enough and the maximum achievable primary surplus is too low to fully repay \( b_t \), then it is unfeasible for the government to fulfill its commitment: either \( b_{t+1} \) has to be increased, or the government has to default. The timing in Cole and Kehoe (2000), by allowing for default triggered by low bond prices, has the virtue of ensuring that payoffs are well defined for all possible strategies.}

Suppose a government is running a bond auction every week. On a given week, the government announces it will sell bonds with a face value of $100bn, expecting a price of $1, and will use the revenue to finance its current deficit needs of $100bn. If the bond price turns out to be 90 cents, the government is short $10bn. Should we expect the government to immediately implement tax increases or spending cuts for $10bn, or should we expect it to increase its bond issuances in the following weeks? We see the second form of adjustment as more plausible in the short run.

In the model presented here, in which the government follows a fiscal rule, the primary surplus is fixed at the beginning of each period. In this context, we are naturally led to choose the timing assumption in which spending and taxes are fixed in the short run and, given bond prices, the government adjusts bond issuances to satisfy the budget constraint ((5)). Therefore, multiple solutions of (5) are considered legitimate equilibrium outcomes.

A more general approach would be to allow for both margins of adjustment to be active: after off-the-equilibrium changes in bond prices, the government can adjust both the primary surplus and bond issuances. This would amount to allow for a fiscal rule that responds to the bond price today. The question would then be: how elastic is the rule in the very short run? A sufficiently elastic rule will rule out multiplicity. The logic of this argument is very close to the logic of the argument that rules out multiplicity with a policy rule that is sufficiently elastic to the accumulated stock of debt, which is the main focus of this paper.

We shall return to timing and commitment issues towards the end of the paper, in Section 5.
3 Short and Long Debt

In this section we specialize the model laid out in the previous section. We first consider the case of short-term debt. We then consider a specific environment with long-term debt. In both setups we introduce a debt Laffer curve and investigate the relation between the shape of this curve and multiplicity.

3.1 Short-Term Debt

With short term debt, \( \delta = 1 \), the price functions are uniquely determined by backward induction.

**Proposition 2.** With short-term debt \( \delta = 1 \) the functions \( m_t, Q_t \), and \( X_t \) are uniquely defined and the function \( m_t(b_t, s_t) \) does not depend on \( b_t \).

**Laffer Curves and Multiplicity.** Proposition 2 does not imply that the equilibrium path is unique. Multiple equilibria exist whenever

\[
Q_t(b_t + 1, s_t) = (1 + r) b_t - s_t
\]

admits multiple solutions for \( b_t + 1 \). The expression \( Q_t(b_t + 1, s_t) b_t + 1 \) defines a relation between the debt level \( b_t + 1 \) and its market value. We call this relation the debt Laffer curve. This curve is in general non-monotone because an increase in \( b_t + 1 \) reduces the probability of repayment at \( t + 1 \), reducing the price at investors are willing to pay for bonds.

Consider first the case of zero recovery, \( v_t(s^t) = 0 \) for all \( s^t \). The bond price \( Q_t(b_{t+1}, s^t) \) is then zero for \( b_{t+1} \) sufficiently large.\(^{10}\) This ensures that the Laffer curve has at least two sides, implying the existence of multiple equilibrium paths.

**Proposition 3 (Multiplicity).** Suppose there is no recovery value, \( v(s^t) = 0 \). If \( (1 + r)b_0 - s_0 < m_0(b_0, s^0) \) then there are at least two equilibrium paths for debt and interest rates.

The left panel of Figure 1 plots the Laffer curve. The amount the government must finance \( (1 + r)b_t - s_t \) is represented by the dashed horizontal line. There are two equilibrium values for \( b_{t+1} \). The high-debt equilibrium is sustained by a higher interest rate that is self fulfilling: a lower bond price forces the government to sell more bonds to meet its financial obligations; this higher debt leads to a higher probability of default in the future, lowering the price of the bond, which justifies the pessimistic outlook. This two-way

\(^{10}\)This follows because default at \( t + 1 \) is certain if \( m_t(b_{t+1}, s_{t+1}) + s_{t+1} < (1 + r)b_{t+1} \) for \( b_{t+1} \) large enough, given a bounded support for \( s_{t+1} \).
feedback between high interest rates and high debt opens the door to multiple equilibria.

With a positive recovery value the Laffer curve converges to the recovery value for high debt. This implies a unique equilibrium for low enough financing needs. This case is illustrated in Figure 2.

**Proposition 4.** Suppose the recovery value is bounded away from zero, \( \nu(s^t) \geq \phi > 0 \). Then for any history \( s^t \), there is a unique \( b_{t+1} \) that solves (6) if \( (1 + r)b_t - s_t \) is low enough.

Figure 1 illustrates that a sufficient reduction in the initial debt level \( b_t \) shifts the dashed red line downwards, eliminating the bad equilibrium.

What are the effects of inherited debt \( b_t \)? Along the good side of the Laffer curve an increase in \( b_t \) raises the current interest rate as well and the entire path of bond issuances and interest rates. Higher debt also increases the potential for multiple equilibria. This creates a feedback, from past equilibria selection, into the future. For example, if a bad equilibrium is selected at \( t \), this raises \( b_{t+1} \) and the interest rate at \( t + 1 \) even if the good equilibrium is expected to be played at \( t + 1 \). It also raises the potential for equilibrium multiplicity. Notably, with short debt, there is no feedback running in the opposite direction, from the future to the present: as implied by Proposition 2 the Laffer curve \( Q_t(b', s^t)b' \) is uniquely determined and independent of the equilibrium selection in future periods. Thus, expectations of bad equilibrium selection in the future have no effects in the current period. As we shall see in Section 3.2, this conclusion is no longer true with long-term debt.

**Stable Equilibrium Selection.** Equilibrium points where the Laffer curve is locally decreasing are “unstable” in the Walrasian sense that a small increase in the price of bonds reduces the supply by more than the demand, creating excess demand. These equilibria are also pathological on other grounds. First, they are also unlikely to be stable with respect to most forms of learning dynamics. Second, Frankel, Morris and Pauzner (2003)
show that global games do not select such equilibria. Finally, these equilibria lead to counterintuitive comparative statics: an increase in the current debt level $b_t$ increases the bond price $q_t$, i.e. higher financing needs lower the equilibrium interest rate.

For all these reasons, we discard unstable equilibria and only consider stable equilibria, which in the present context requires the Laffer curve to be locally increasing.

With short term debt, multiple stable equilibria can only be obtained with primitives that generate a Laffer curve with multiple peaks. If the surplus process is fully exogenous (i.e., does not respond to $b_t$) and there is no recovery, one requires assumptions on the surplus distribution. Our next result shows that one needs a non-monotone hazard rate.

**Proposition 5.** Suppose the recovery value is zero $v(s) = 0$ and the surplus process is independent of the debt level. If the hazard rate $\frac{f(s_t|s_t-1)}{1-F(s_t|s_t-1)}$ is monotone then the Laffer curve is single peaked and there is a unique equilibrium path that is interior and stable.

One way to obtain multiple steady state equilibria is to specify distributions with non-monotone hazard rates. Navarro et al. (2014) show that a specification with a form of disaster risk can yield multi peaked Laffer curves and explore their implications. Another approach is to consider responsive policy rules. When debt affects the distribution for surplus this may generate multiple peaks, even if the conditional distribution satisfies the monotone hazard condition. As we shall see, once we introduce long term debt, multiple stable equilibria are possible even if the Laffer curve is single-peaked.

**Rollover Crises.** The logic of Walrasian tatonnement also leads us to consider the possibility of a different type of equilibrium, which is similar to a rollover crisis a la Cole and Kehoe (2000). Consider the left panel of Figure 1. Suppose we start immediately to the right of the bad, unstable equilibrium. At that point, there is an excess supply of bonds so bond prices should fall, pushing us further to the right. This continues until we reach a level of bond issuance associated with a zero probability of repayment and zero bond prices. This suggests the presence of a third, stable equilibrium, with a zero bond price.
and default in period $t$; a similar logic applies with a positive recovery value, for sufficiently high financing needs $(1 + r)b_t - s_t$. To allow for this possibility, we must relax the assumption that default only happens when there is no issuance $b_{t+1}$ that allows the government to finance $b_t - s_t$, and introduce the possibility of default triggered by zero bond prices today. This possibility can be interpreted as a rollover crisis, where pessimistic investors force immediate default and this default makes future repayments impossible, justifying investors’ expectations. As discussed in the introduction, rollover crises are the focus of Cole and Kehoe (2000) and a large subsequent literature. For the remainder of this paper we set rollover crises aside to focus on slow moving crises. However, it is important to keep in mind that when one takes rollover crises into account, then multiplicity is relatively pervasive with short term debt, even when the Laffer curve is single peaked.

3.2 Long-Term Debt

We now turn to the case of long-term bonds. Long-term debt allows us to fully develop our idea of slow-moving crises, in which higher spreads lead to a gradual increase in debt servicing costs, as the government only replaces a fraction of maturing bonds with newly issued ones.

Allowing for long-term bonds is important for a number of reasons. First, in most advanced economies the bulk of the debt is in the form of relatively long maturities, rather than three-month or one-year bonds (Arellano and Ramanarayanan, 2012). For example, the average maturity of sovereign debt for Greece, Spain, Portugal and Italy was 5-7 years over the 2000-2009 period. Second, it is widely believed that short-term borrowing exposes a sovereign borrower to debt crises and that longer maturities provides some protection. In the context of a rollover crises, Cole and Kehoe (1996) find support for this view. How does debt maturity affect the possibility of slow moving debt crises in our model?

Our first objective is to illustrate the dynamics of a slow moving crisis when long-term bonds are present. In particular, we characterize the debt levels for which multiplicity is present and show that multiplicity emerges in the early stages of a crisis, but vanishes once the bad path has played out long enough. The second objective is to show how the fiscal rule and debt maturity affect the presence of multiplicity.

A Special Case with Uncertainty at $T$. We first consider a case where all uncertainty is concentrated at some date $T > 0$. This special case is insightful because it can be mapped into the Laffer curve diagram, providing a bridge to the results with short term debt.
For $t \geq T$ the surplus is constant at $s_t = rS$, with $S$ drawn from a continuous distribution $F(S)$. For $t < T$, the surplus is given by
\begin{equation}
s_t = (1 - \lambda) s_{t-1} + \lambda (a_0 + a_1b_t).
\end{equation}
Here $a_0 + a_1b_t$ is a target surplus, which is increasing in the debt level if $a_1 > 0$. The surplus adjusts to its target at a speed determined by $\lambda$. The parameter $a_1$ plays an important role: larger values represent a more aggressive fiscal response to debt.

Given the concentration of news at $T$, default during the first stage can only occur at $t = 0$. We proceed by assuming that default does not occur during the first stage. If this construction does not yield an equilibrium, default is unavoidable and occurs at $t = 0$. Assuming no default during the first stage, for $t < T - 1$, the bond price satisfies the difference equation (using the normalization for $\kappa$)
\begin{equation}
q_{t} - 1 = \frac{1 - \delta}{1 + r} (q_{t+1} - 1),
\end{equation}
In period $T - 1$ just before the resolution of uncertainty
\begin{equation}
q_{T-1} = 1 - F(b_T) + \beta \frac{\phi}{b_T} \int_{0}^{b_T} S dF(S).
\end{equation}
The government’s budget constraint for $t < T$ can be written as (using, once again, the normalization for $\kappa$)
\begin{equation}
s_t + (q_t - 1) (b_{t+1} - (1 - \delta)b_t) + b_{t+1} = (1 + r)b_t.
\end{equation}

Equations (7), (8) and (10) describe the dynamics for $s_t$, $q_t$ and $b_t$, respectively. The initial values for debt and surplus, $b_0$ and $s_0$, are given, while equation (9) provides a boundary condition for $q_{T-1}$.

To find an equilibrium, we proceed as follows. For each candidate value for the initial price $q_0 \in (0, \infty)$, we solve (7), (8) and (10) forward and find terminal values $b_T$ and $q_{T-1}$. If these values satisfy (9) we have an equilibrium. Equation (9) is equivalent to
\begin{equation}
q_{T-1}b_T = (1 - F(b_T))b_T + \beta \phi \int_{0}^{b_T} S dF(S),
\end{equation}
This is a Laffer curve similar to the one analyzed in the case of short-term debt.

This construction is represented graphically in Figure 3 plotting $b_T$ against $q_{T-1}b_T$. The single peaked curve is the Laffer curve. The downward sloping curve plots the values of
Figure 3: Solid green line shows $t = 0$, dashed green line $t = 14$, dotted green line $t = 35$. $b_T$ and $q_{T-1}b_T$ obtained by solving (7), (8) and (10) forward for all values of $q_0 \in (0, \infty)$. Equilibria are represented by intersections of these two curves.

In the figure, the following parameters were employed to roughly set to mimic the case of Italy:

$$T = 120, \quad \delta = \frac{1}{7} \cdot \Delta, \quad \beta = 1.02^{-\Delta}, \quad \phi = 0.7, \quad \log S \sim N(0.3, 0.1^2),$$

where $\Delta$ represents the length of a time period, measured in annual terms. We set $\Delta = \frac{1}{12}$ to represent a month. In our example, all uncertainty is resolved in 10 years and average debt maturity is 7 years. The risk-free interest rate is 2% and the recovery rate in case of default is 70%. The distribution of the present value of surplus, after uncertainty is resolved has mean 1.357 and standard deviation 0.136. It is useful to interpret all these numbers as being relative to a country’s GDP. The initial conditions are set to

$$s_0 = -0.1 \cdot \Delta, \quad b_0 = 1,$$

and the fiscal policy parameters are

$$\lambda = \Delta, \quad \alpha_0 = 0, \quad \alpha_1 = 0.02,$$

so that a unit increase in the debt stock implies a fiscal effort to increase the surplus by 0.02 in annual terms; note that this would exactly stabilize the debt with a risk free interest of 2%.

Figure 3 shows the presence of three equilibria as the intersection of the two solid lines (the dashed lines will be described below). Importantly, both the first and third equilibrium are stable, in the sense discussed earlier. Thus, in the presence of long-term
debt it is possible to obtain multiple equilibria that are interior and stable even with a Laffer curve that is single peaked. Figure 4 shows the dynamics for the primary surplus, debt and bond prices for the two stable equilibria, which we term “good” (solid lines) and “bad” (dashed lines).

This stylized example is able to capture various features of recent episodes of sovereign market turbulence. Sovereign bond yields experience a sudden and unexpected jump, when shifting unexpectedly from the good to the bad equilibrium path. The debt-to-GDP ratio increases slowly but steadily. Auctions for new debt issues continue without any signs of illiquidity, but interest rates climb along with the level of debt. Large differences in debt dynamics appears gradually, as bond prices diverge and a larger fraction of debt is issued at crisis prices.

**Eventual Uniqueness.** A characteristic feature of our model is that multiplicity only plays out in the early phase of a crisis. Indeed, along both equilibrium paths, multiplicity eventually vanishes.

A formal result about eventual uniqueness can be derived by considering a sequence of economies with $\Delta \to 0$, so that we can consider a period as close as we want to the final period $T$. In particular, consider a family of economies parametrized by $\Delta$. Each economy has a given distribution $F$, given values of $\phi, \alpha_0, \alpha_1$, given initial condition $b_{T-1}$ and values of the remaining parameters satisfying $\beta = \tilde{\beta}^\Delta, \kappa = \tilde{\kappa}\Delta, \delta = \tilde{\delta}\Delta, \lambda = \tilde{\lambda}\Delta, s_{T-2} = \tilde{s}\Delta$, for some given values of the parameters with a tilde.

**Proposition 6.** For any $\bar{b} > 0$, there exists a $\bar{\Delta}$ such that for $\Delta < \bar{\Delta}$ the equilibrium is unique at $T - 1$ for all $b_{T-1} \in [0, \bar{b}]$.

**Proof.** At time $T - 1$ an equilibrium can be found finding a $b_T$ that satisfies
\[
\left( 1 - F(b_T) + \beta \frac{\phi}{b_T} \int_0^{b_T} S dF(S) \right) \cdot (b_T - (1 - \delta) b_{T-1}) = \kappa b_{T-1} - s_{T-1}. \tag{11}
\]

For a sequence of economies with \( \Delta \to 0 \), select a sequence of equilibrium values for \( s_{T-1}, b_T \) and denote them by \( s^\Delta_{T-1}, b^\Delta_T \). It is easy to show that \( \kappa^\Delta b^\Delta_{T-1} - s^\Delta_{T-1} \to 0 \), so the left-hand side of 11 must also converge to zero.

\[
\begin{align*}
b_T &= (1 - \delta \Delta) b_{T-1} + \frac{(r + \delta) b_{T-1} - s_{T-1} \Delta}{q} \\
V &= (1 - \delta \Delta) qb_{T-\Delta} + [(r + \delta) b_{T-\Delta} - s_{T-\Delta}] \Delta \\
q &= \frac{(r + \delta) b_{T-\Delta} - s_{T-\Delta} \Delta}{b_T - (1 - \delta \Delta) b_{T-\Delta}} \\
V &= (1 - \delta \Delta) \frac{(r + \delta) b_{T-\Delta} - s_{T-\Delta} \Delta b_{T-\Delta} + [(r + \delta) b_{T-\Delta} - s_{T-\Delta}] \Delta}{b_T - (1 - \delta \Delta) b_{T-\Delta}}
\end{align*}
\]
as \( \Delta \to 0 \)

Figure 3 illustrates this point. Each dashed and dotted line corresponds to a different time horizon and initial debt condition. In particular, we plot them for \( t = 14 \), and \( t = 35 \) and use as initial conditions the values of \( s_t \) and \( b_t \) reached under the good and the bad equilibrium paths from Figure 4. At \( t = 14 \) the two dashed lines show that multiplicity is still present. For example, the top dashed line indicates that it would be possible to snap out of a crisis after enduring the bad path for 14 months and switch to a good path.\(^{11}\) However, after the crises has gone on for 35 months a switch is no longer possible, the dotted line representing \( t = 35 \) features only one intersection.

There are two reasons why multiplicity disappears as we approach \( T \). First, there is a direct or mechanical effect, since the remaining time for debt to accumulate or decumulate shortens, weakening the feedback loop that gives rise to multiplicity. Second, along the bad path debt may grow to a high enough level making the bad path the only possible outcome; conversely, along the good path debt may fall to a low enough level making the good path the only possible outcome. In Section 4.1 we consider a stationary model where this first effect is not present, but the second effect still implies that the intermediate levels of debt where multiplicity is possible is transitory.

Note that even if we switch from the bad path to the good path, unexpectedly, the economy inherits a higher debt level and higher interest rates with it. This can be seen in the figure: the new leftmost intersection with the upper dashed line is further to the right.

\(^{11}\) Clearly, the switch needs to be unexpected for prices to be in equilibrium between \( t = 0 \) and \( t = 14 \).
Figure 5: Left Panel: solid green line $\alpha_1 = 0.02 \cdot \Delta$, dashed green line $\alpha_1 = 0.03 \cdot \Delta$, dotted green line $\alpha_1 = 0.05 \cdot \Delta$. Right panel: solid green line $\delta = \frac{1}{7} \Delta$, dashed green line $\delta = \frac{1}{10} \Delta$, dotted green line $\delta = \frac{1}{5} \Delta$.

up the good side of the Laffer curve, relative to the intersection of the two solid lines, representing the outcome if the economy had never ventured off the good equilibrium path.

**Fiscal Rules.** How does the fiscal policy rule affect the equilibrium or the existence of multiple equilibria? Figure 5 shows the effects of increasing $\alpha_1$, while adjusting $\alpha_0$ to keep the good equilibrium unchanged. As shown, a high enough value for $\alpha_1$ rules out the bad equilibrium. As investors contemplate the effect of lower bond prices and increased debt issuances that entails, they also realize that the government will make a greater fiscal adjustment in the future. This helps counter the feedback loop between interest rates and debt.

Of course, an extremely responsive fiscal rule could, in principle, ensure that debt is stabilized for any sequence of interest rates. Or, even more extreme, prevent default altogether. It is important to emphasize that nothing as drastic as this is required to obtain a unique equilibrium. In our parameterization, lower bond prices do lead to higher debt accumulation; higher values of $\alpha_1$ mitigate the rise in debt, but never offset it completely.

**Initial Debt.** What are the effects of initial debt? Figure 6 shows the parameter space $(\alpha_1, b_0)$ and divide it into four regions. We now make no adjustment to $\alpha_0$. In the red region there is a single equilibrium, in the bottom portion debt is low and on the good side of the Laffer curve, while in the upper portion (above pink region) the unique equilibrium lies on the bad side of the Laffer curve. There are three equilibria in the pink region, just as in our calibrated example. In the yellow region no equilibrium with debt exists, implying
Consider for example, the case $\alpha_1 = 0.01$ in the graph, in which four cases are possible. For low levels of $b_0$, we get a unique equilibrium on the increasing portion of the Laffer curve (lower portion of the red region). For higher levels of $b_0$, we have three equilibria, as depicted in Figure 6 (pink region). For even higher levels of $b_0$, we have a unique equilibrium again, but this time on the bad side of the Laffer curve. Finally, for very high values of $b_0$, there is no equilibrium without default.

**Debt Maturity.** Consider next the impact of debt maturity, captured by $\delta$. Figure 5 shows the effects of varying $\delta$ around our benchmark value, while adjusting $\alpha_0$ to keep the good equilibrium unchanged. A longer maturity, with a low enough value for $\delta$, leads to a unique equilibrium. Intuitively, shorter maturities require greater refinancing, increasing the exposure to self-fulfilling high interest rates. The debt burden of longer maturities, in contrast, is less sensitive to the interest rate, mitigating the feedback loop that leads to equilibrium multiplicity.

The right panel in Figure 6 is similar to the left panel, but over the parameter space $(\delta, b_0)$ instead of $(\alpha, b_0)$. Again, we divide the figure into four regions. There are three equilibria in the pink region, just as in our calibrated example. In the red region there is a single equilibrium. In the bottom portion of the red region the equilibrium lies on the good side of the Laffer curve, while in the upper portion (above the pink region) it is on the bad side of the Laffer curve. In the yellow region no equilibrium exists, implying immediate default at $t = 0$.

For given $\delta$, we see the same effects with respect the level of initial debt. Turning to the effects of $\delta$, shorter maturities, higher values for $\delta$, place the economy in a “danger zone” (pink region) with 3 equilibrium values for the interest rate. Still higher values for $\delta$ may lead to a unique bad equilibrium (upper red region) or to non-existence prompting immediate default (upper right, yellow region). These last two conclusions depend on the fact that $\alpha_0$ was not adjusted here, unlike in Figure 5.

### 4 Stationary Environments: Fiscal Rules and Optimizing Governments

This section explores stationary environments where the date where all uncertainty is resolved is itself stochastic, arriving at a constant Poisson rate. This is the main distinction with the previous section. A more superficial difference is that, for convenience, we
now work with continuous time. Together with stationarity, this allows us to display the model’s dynamics using a phase diagram.

We first consider fiscal policy that is committed to a fiscal rule and later consider discretionary policy chosen by an optimizing government with additive preferences.

4.1 Fiscal Rules

Time is continuous and runs forever. Investors are risk neutral discounting at rate $r$. Bonds issued at time $t$ pay a coupon $\kappa e^{-\delta(t-1)}$ for all $\tau > t$, the continuous-time analog of the bonds used in previous sections. We adopt the normalization $\kappa = r + \delta$ again, so that the bond price equals 1 in the absence of default risk.

The primary surplus evolves in two stages. In the first stage, it is a deterministic function of the stock of bonds outstanding

$$s = h(b),$$

where $h$ is a weakly increasing function with $h(b) = \bar{s} > 0$ for $b \geq \bar{b}$. Note that this fiscal rule imposes a direct relationship between $b$ and $s$ in levels, rather than of a relation between $b$ and the rate of change of $s$ as in Section 3.2.

At a Poisson arrival rate $\lambda$ we reach the second stage, where all uncertainty is resolved. The present value of future surpluses, $S$, is drawn from a continuous distribution $F(S)$ over $[\bar{S}, \tilde{S}]$. If $S \geq b$ default is avoided and the bond price equals 1. If $S < b$ bond holders obtain a recovery value $\phi S$, with $\phi < 1$. Thus, the bond price upon entering the second stage

![Figure 6: Regions with unique equilibrium (red), three equilibria (pink) or immediate default (yellow).](image)
stage but immediately before the resolution of uncertainty is

\[ q = \Psi(b) \equiv 1 - F(b) + \frac{\Phi}{b} \int_b^\infty S \, dF(S). \]

We now derive the dynamics for \( q \) and \( b \) during the first stage. The assumption that \( \bar{s} > 0 \) will ensure that default does not occur within the first stage, as we discuss further below. Thus, the bond price solves

\[ rq = \kappa - \delta q + \dot{q} + \lambda (\Psi(b) - q), \tag{12} \]

Bonds decay at rate \( \delta \), pay coupon \( \kappa \), earn a capital gain \( \dot{q} \) before uncertainty is revealed, and have an expected capital gain \( \Psi(b) - q \) with Poisson arrival \( \lambda \).

The government budget constraint is

\[ h(b) + q(b) = \kappa b, \tag{13} \]

The dynamics of \( q \) and \( b \) solve these two ODEs. We develop boundary conditions below.

**Steady States.** Steady states are found at the intersection of the loci for \( \dot{q} = 0 \) and \( \dot{b} = 0 \). The locus \( \dot{q} = 0 \) is given by

\[ q = \frac{\kappa + \lambda \Psi(b)}{r + \delta + \lambda} \tag{14} \]

and is downward sloping since \( \Psi'(b) < 0 \). A larger stock of bonds implies a smaller probability of repayment after the resolution of uncertainty, lowering bond prices.

The locus \( \dot{b} = 0 \) is given by

\[ q = \frac{\kappa b - h(b)}{\delta b} \tag{15} \]

and is decreasing if and only if

\[ h'(b) > \kappa - \delta q. \tag{16} \]

This condition is violated, for instance, if the fiscal rule is insensitive to debt, \( h'(b) = 0 \). A more typical assumption is that fiscal rules stabilize debt. In models without default risk, it is common to assume \( h'(b) > r \), which amounts to a “Ricardian regime” or a “passive” fiscal policy in the terminology of Leeper (1991).

The average cost of servicing a dollar of debt at a steady state is \( \kappa - \delta q \), since the government must pay the coupon but can issue \( \delta \) new bonds to replace depreciated old bonds. Therefore, inequality (16) ensures that an increase in the stock of debt leads to an increase in the primary surplus greater than the increase in the average cost of servicing
the debt. The relative slopes of the two loci play an important role when we turn to the
dynamics.

Figure 7 displays an example with two steady states. In the low-debt steady state,
bond prices are high, debt issuances \( \delta q b \) cover a large fraction of the coupon payments \( \kappa b \)
and the government can run a low primary surplus, consistent with its policy rule \( h(b) \). In
the high-debt steady state bond prices are low and debt issuances cover a smaller portion
of coupon payments, so the government needs to run a larger surplus.

**Boundary Conditions.** The ODEs must be complemented by boundary conditions. We
consider two types of conditions: either the economy converges to the low-debt steady
state or it converges to a path with ever growing debt. The high-debt steady state turns
out to be unstable and, thus, can never serve as an adequate boundary condition.

The path with ever growing debt is characterized analytically using the fact that \( h(b) \)
and \( S \) are bounded above and that these bounds become binding for large enough \( b \).
Indeed, along a path with \( b \to \infty \) we must have \( q \to 0 \) and a constant value for debt
\[ bq = \vartheta \equiv \frac{s + \lambda \Psi(S) \bar{S}}{r + \lambda}. \]
This is the unique path available for \( b \geq \hat{b} \) for some \( \hat{b} \). This provides a boundary con-
dition at \( \hat{b} \). As long as \( \vartheta > 0 \) there is an equilibrium without default in stage 1. The
assumption that \( \bar{s} > 0 \) ensure that \( \vartheta > 0 \). If \( \vartheta < 0 \) then default is unavoidable and occurs
right away at \( t = 0 \).

It is simple to impose alternative boundary conditions. For example, assume that
whenever debt reaches some high arbitrary threshold \( \hat{b} \) this triggers a renegotiation be-
tween investors and the borrower, perhaps intermediated by the IMF or some other or-
ganization. Suppose we can predict the outcome of such a renegotiation. Then this pins
down the value of debt at \( \hat{b} \), providing a boundary condition to solve the ODE system.

**Dynamic Stability.** Next we provide a necessary and sufficient condition for saddle-
path stability of a steady state.

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12 Along the explosive debt path, the primary surplus is constant at \( \bar{s} > 0 \) and lenders anticipate default
as soon as stage 2 is reached. Effectively, an investor that always buys new issuances gets the expected
value of surpluses and the recovery value. The value at any moment is constant and equals \( \vartheta \).

13 The cutoff is
\[ \hat{b} = \max \left\{ \bar{s}, \bar{s}, \frac{1}{\kappa} \left[ \bar{s} + \delta \bar{s} + \lambda \Psi(S) \bar{S} \right] \frac{1}{r + \lambda} \right\}. \]
Lemma 1. A steady state with positive debt is locally saddle-path stable if and only if the $\dot{b} = 0$ locus is downward sloping and steeper than the $\dot{q} = 0$ locus, or equivalently,

$$h'(b) > \kappa - \delta q - \frac{\delta \lambda}{r + \delta + \lambda} \Psi'(b) b.$$  \hspace{1cm} (17)

As mentioned above, in a steady state with no default risk $q = 1$ and $\lambda \Psi'(b) = 0$, so that condition (17) reduces to $h'(b) > r$. This ensures that the total secondary surplus (primary minus the cost of servicing the debt) is increasing in $b$, a standard stability condition for fiscal rules in the literature on fiscal and monetary policy (Leeper, 1991). This condition is stronger than condition (16) precisely because it includes the induced change in bond prices. Condition (16) ensures the $\dot{b} = 0$ locus is decreasing; whereas condition (17) ensures it is decreasing and steeper than the $\dot{q} = 0$ locus.

Presumably, saddle-path stability is a desirable feature. However, given the model’s non-linearities, having a saddle-path stable equilibrium is not enough to rule out multiple steady states or multiple equilibria. Indeed, multiple steady states are ensured if any steady state is saddle path stable.

Proposition 7. If there is a saddle-path stable steady state with positive debt, then there is another steady state with higher debt that is not stable.

The result follows from the fact that the surplus is bounded above by $\bar{s} > 0$, implying that the fiscal policy rule cannot be responsive at high debt levels. Indeed, for high enough debt the locus for $\dot{b} = 0$ is increasing and would certainly intersect the decreasing locus for $\dot{q} = 0$. 

Figure 7: Phase diagrams with fiscal rule.
Multiple Equilibria. Figure 7 shows the phase diagram for our example. The blue and green solid lines represent the \( \dot{q} = 0 \) and \( \dot{b} = 0 \) loci. Also shown are two paths: one converging to the low-debt steady state boundary and another converging to the exploding-debt boundary.

In this example, the high-debt steady state has local spiral-like dynamics and the two paths unwind outwards from the high-debt steady state.\(^{14}\) For the path converging to the low-debt steady state we focus on its uppermost segment; likewise, for the path converging to the explosive debt boundary we focus on the lowermost segment. The two paths overlap over an interval containing the high-debt steady state. Within this interval there are two feasible paths satisfying the ODEs and boundary conditions.

We construct Markov equilibria as follows. Select any threshold \( \tilde{b} \) for debt within the interval of overlap. For \( b \leq \tilde{b} \), let \( Q(b) \) equal the price \( q(t) \) associated with the upper path converging to the low-debt steady state when \( b(t) = b \). For \( b > \tilde{b} \), let \( Q(b) \) equal the price \( q(t) \) associated with the lower path converging to the upper boundary when \( b(t) = b \).\(^{15}\) The price function \( Q(b) \) is decreasing and necessarily features a single point of discontinuity at \( \tilde{b} \). There are a continuum of such Markov equilibria indexed by the threshold \( \tilde{b} \).

Eventual Uniqueness. Along the bad path debt eventually exits the interval of multiplicity. For some time the good path may remain available, but there is a point of no return. In other words, a crisis initially driven by self-fulfilling expectations eventually turns into an insolvency crisis based on fundamentals.

This is similar to the result for the non-stationary model of Section 3.2. There were two forces at work there, a shrinking time horizon and the accumulated level of debt. In the present stationary setting, only the latter force is present.

Tipping Points with Uniqueness. Multiple steady states do not imply multiple equilibria. The right panel of Figure 7 shows a case with two steady states with a unique equilibrium. At the high-debt steady state both eigenvalues are real and positive, allowing the good and bad path not to overlap.

Although the equilibrium is unique, long-run dynamics may be very sensitive to initial conditions. Just below the high steady state, debt converge to the low steady state; just above the high steady state, debt explodes. This formalizes the notion that debt dy-

\(^{14}\)i.e. the Jacobian has complex eigenvalues.

\(^{15}\)The value for \( Q(\tilde{b}) \) was arbitrarily set to at the uppermost value. It could also be set at the lowermost value.
namics may display a “tipping point” where debt-sustainability concerns drastically alter
debt dynamics, as suggested by Greenlaw et al. (2013).

The following proposition gives sufficient conditions for tipping point dynamics and
for equilibrium multiplicity.

**Proposition 8.** Consider a steady-state \((b, q)\) that is not a saddle. If

\[
h'(b) < \kappa + q(r + \lambda)
\]

the steady state is a source (both eigenvalues have positive real part). If

\[
4\delta \lambda \Psi'(b) qb + (\delta q + h'(b) + \lambda \Psi(b))^2 < 0,
\]

then the steady state is a spiral (the eigenvalues’ imaginary part is non-zero) and there exist mul-
tiple Markov equilibria.

The condition for multiplicity in the proposition is sufficient but not necessary. Mul-
tiple equilibria can also arise when both eigenvalues are real and positive if the non-
linearity of the system generates an overlapping interval for both paths.

**An Example Based on Italy.** We now adapt the model and choose parameters to capture
the dynamics of bond prices and government debt for Italy in the Summer of 2012.

First, we adapt the model slightly by introducing growth. It is easy to reinterpret our
model in terms of debt-to-GDP and primary-surplus-to-GDP ratios. The only equation
that needs to be modified is the government budget constraint which becomes

\[
q \left(\dot{b} + (\delta + g) b\right) = \kappa b - h(b),
\]

where \(g\) is the growth rate of GDP. We also allow the long run interest rate to be a random
variable, drawn after the realization of uncertainty. This only affects the calculation of the
\(\Psi\) function which becomes

\[
\Psi(b) \equiv \int_{S > b} \frac{\kappa}{\tilde{r} + \delta} dF(\tilde{r}, S) + \phi \int_{S < b} \frac{S}{\tilde{b}} dF(\tilde{r}, S),
\]

where \(F\) is the joint distribution of \(\tilde{r}\) and \(S\).

The fiscal rule is chosen to fit the observed relation between the debt-to-GDP ratio and
the primary surplus in the period 1988-2012. A linear regression of the primary surplus
on the debt-to-GDP ratio yields \(a_0 = -0.13\) and \(a_1 = 0.135\). We also assume the fiscal
surplus is bounded above at $\bar{s} = 6\%$ and use the rule

$$s = \min\{\alpha_0 + \alpha_1 b, \bar{s}\}.$$  

We choose $\delta = 1/7$ to match the average maturity of Italian government debt, which is about 7 years. We choose $r = 3\%$ to match the 10 year nominal bond yield in Germany in 2011 and $g = 2\%$ to capture nominal growth equal to the ECB inflation target of 2\% plus zero real growth. We assume that the interest rate after the resolution of uncertainty is $\tilde{r} = 5\%$. We choose $\phi = 0.5$ and assume that $S$ is normally distributed. The mean and variance of $S$ are chosen so that in the low-debt steady state debt is essentially safe (with a spread of 10 basis points) and so that the high-debt steady state is $b = 1.3$.

We start the economy at $b = 1.2$, which corresponds to Italy’s debt-to-GDP ratio in 2011. Given the parameters above, the model features multiple equilibria and $b = 1.2$ is in the multiplicity region. The dynamics of bond spreads, of the primary surplus and of debt-to-GDP are plotted in Figure 8. We can then imagine Italy following a path of slow debt reduction, as in the purple-line equilibrium. At date 0, if investors’ sentiment shift to the bad equilibrium path, spreads jump from 50bp to 220bp, which is roughly the order of magnitude of the increase in spreads in the Summer of 2011.\textsuperscript{16} Therefore, this simple model is able to account for a crisis in which spreads increase suddenly, but not to the point of shutting off Italy from financial markets, and in which debt dynamics only slowly incorporate the effect of the higher spreads.

\textsuperscript{16}Yields on 10-year Italian bonds went from 4.75 in May 2011 to 7.6 in November of the same year. Similar magnitudes of “sentiment” shocks can be read off the estimate of the effects of OMT announcements in Krishnamurthy et al. (2013).
4.2  An Optimizing Government

In this section, we consider a model analogous to the stationary model from the previous subsection, but we derive fiscal policy endogenously from an optimizing government with additively separable preferences lacking commitment. The goal is to show that the similar multiplicity of equilibrium is present here.

The government now has additively separable preferences

\[ \int_0^\infty e^{-\rho t} u(c(t)) dt \]

where \( c \) is government spending, \( u(c) = \frac{c^{1-\sigma}}{1-\sigma} \) and \( \sigma, \rho > 0 \).

As in Section 4.1, we work in continuous time with an infinite horizon. Lenders are risk neutral with discount rate \( r \), the country issues long-term bonds with coupon \( \kappa \) that decays at rate \( \delta \), and we set \( \kappa = r + \delta \) so the bond price is 1 if there is no default risk.

Uncertainty is fully resolved at some random date with Poisson arrival rate \( \lambda \), at which point we enter stage 2 and the government receives a constant stream of tax revenue \( \tilde{y} \) drawn from the continuous distribution \( H(\tilde{y}) \). The government then decides to repay or default. In the latter case tax revenue is reduced to \( \eta \tilde{y} \) forever, with \( \eta < 1 \). We assume the country is not excluded from financial markets upon default.\(^{17}\) The country repays if and only if the present value of tax revenue net of repayment is greater than that after default,

\[ \tilde{y} - rb \geq \eta \tilde{y}. \]  \( \text{(18)} \)

The value to the government immediately before uncertainty is resolved is

\[ W(b) = \frac{1}{\hat{\rho}} \int_0^{\infty} u \left( \frac{\hat{\rho}}{r} \max \langle \tilde{y} - rb, \eta \tilde{y} \rangle \right) dH(\tilde{y}), \]

for \( \hat{\rho} = \rho + \left( \frac{1}{\sigma} - 1 \right) (\rho - r) \).\(^{18}\)

If default occurs, investors recover a proportion of the tax revenue \( \zeta \tilde{y} \) and we assume \( \eta + \zeta < 1 \). Define

\[ S \equiv \frac{1 - \eta}{r} \tilde{y} \quad \text{and} \quad F(S) = H \left( \frac{r}{1 - \eta} S \right), \]

where \( F \) is the c.d.f. for \( S \). Condition (18) is then equivalent to \( S \geq b \) and investors recover

\(^{17}\)The exact same results obtain if they are, provided \( r = \rho \).

\(^{18}\)In general \( \hat{\rho} \neq \rho \) because the government does not consume a constant path, unless \( \rho = r \). We also have \( \hat{\rho} = \rho \) in the logarithmic case \( \sigma = 1 \).
in present value $\phi S$, where $\phi \equiv \frac{\xi}{1-\eta} < 1$. The pricing condition before the Poisson event is then identical to equation (12) from Section 4.1, which we rewrite here for convenience:

$$q = (r + \delta + \lambda) q - \kappa - \lambda \Psi(b),$$

(19)

where, just as before, $\Psi(b) \equiv 1 - F(b) + \frac{\phi}{b} \int_b^\infty SdF(S)$.

During stage 1, before the resolution of uncertainty the government receives a constant $y$ and chooses the consumption flow $c(t) \geq 0$. The government budget constraint is

$$y - c + q(b + \delta b) = \kappa b,$$

identical to equation (13) except that the primary surplus $y - c$ is now chosen by the government.

Given the presence of a positive recovery rate, we must introduce some limit on debt issuance to ensure the borrower’s problem is well defined. We do so by assuming that if $b$ reaches some upper bound $\bar{b}$ before the resolution of uncertainty, renegotiation takes place between the borrower and its creditors. Following renegotiation, the government agrees to receive a net transfer $\tau$ (possibly negative) in all future periods before the resolution of uncertainty and not to issue any additional debt, so the creditors will receive the expected value $\Psi(\bar{b}) \bar{b}$ when uncertainty is resolved.

**Markov Equilibria.** A Markov equilibrium is a price function $Q(b)$ and a government consumption function $C(b)$ such that: (i) government behavior is optimal taking the price function as given; (ii) the price function provides a fair price to investors given government behavior. We also require $Q$ to be piecewise differentiable. Just as in Section 4.1 the function $Q$ may have a point of discontinuity at a threshold that divides a path with falling and rising debt.

Let $V(b)$ denote the value function before the resolution of uncertainty. The Hamilton-Jacobi-Bellman equation associated to the government’s optimization problem is

$$0 = \max_{c \geq 0} \left\{ u(c) + V'(b) \left( \frac{\kappa b - y + c}{Q(b)} - \delta b \right) + \lambda (W(b) - V(b)) - \rho V(b), \right\}$$

(20)

with first-order condition

$$Q(b)u'(c) = -V'(b),$$

for and interior solution $c > 0$ consumption. As we shall, despite the Inada condition $u'(0) = \infty$, the corner solution $c = 0$ is a possibility we cannot exclude.

The bond price must satisfy (19) along the path induced by government policy $C(b)$. 31
Differentiating $q(t) = Q(b(t))$ along an equilibrium path gives $\dot{q} = Q'(b) \dot{b}$ where $Q'(b)$ can be interpreted as the left derivative if $\dot{b} < 0$ and the right derivative if $\dot{b} > 0$. Rewriting condition (19) in terms of $Q$ gives

$$Q'(b) \left( \frac{\kappa b - y + c}{Q(b)} - \delta b \right) = (r + \delta + \lambda) Q(b) - \kappa - \lambda \Psi(b).$$

ODEs and Boundary Conditions. Equations (20) and (21) provide a system of ordinary differential equations (ODEs) for the pair of functions $V(b)$ and $Q(b)$. Just as before we require boundary conditions.

There are two alternative boundary conditions. The first is at $b = S$, the lowest value in the support of $S$. This represents the safe level of debt where default is avoided. We assume that $r = \rho$

The second boundary condition is obtained by assuming that there exists a high enough level of debt $\hat{b}$ where renegotiation is triggered. We assume this delivers some given values to the borrower and investors, pinning down $V(\hat{b})$ and $Q(\hat{b})$.

An Example with Multiple Equilibria. We construct a numerical example displaying multiple equilibria. The example is meant as an illustrative proof of concept, not a calibration. The parameters are set to

$$r = \rho = 0.1, \quad \delta = 2, \quad \lambda = 0.1, \quad \eta = 0.5, \quad \zeta = 0.1, \quad \sigma = 4.$$

The distribution of $\tilde{y}$ is chosen such that $S$ is uniformly distributed on $[10, 100]$. The value of $y$ is set so that the model admits a steady state at $b = 0$ with no default and $q = 1$. The upper bound for debt is $\bar{b} = 30$ and we assume that upon reaching this level, investors provide a transfer to the borrower equal to $\tau = \frac{1}{2} \lambda \Psi(\hat{b}) \hat{b}$. In other words, they split the residual value of debt equally.

We want to characterize equilibria with initial conditions $b_0 \in [0, \hat{b}]$. To do so, it is useful to first separately characterize two possible equilibrium paths, one displaying $\dot{b} < 0$ and the other $\dot{b} > 0$. We will then study multiplicity by asking whether both equilibrium paths can be reached from the same initial condition $b_0$.

Figure 9a plots consumption and bond price functions $C^-(b)$ and $Q^-(b)$ consistent with a path with $\dot{b} < 0$ converging to $b = 0$. We solve the HBJ equation (20) and the pricing equation (21) moving upwards from the steady state at $b = 0$. Also shown, in the top panel, is a dashed green line representing the consumption level that would yield $\dot{b} = 0$, which is above $C^-(b)$ since $\dot{b} < 0$. 

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Figure 9: Consumption and bond price functions.

Figure 9b is analogous, but for the path with $\dot{b} > 0$ converging to the renegotiation boundary $\hat{b}$, solved using the ODEs starting at $\hat{b}$ and moving downwards. Importantly, for debt above $b''$ setting $\dot{b} = 0$ requires negative consumption. In other words, given the bond price in the lower panel, feasibility requires $\dot{b} > 0$ for all $b > b''$. In that region, bond prices are so low that the revenue from replacing old bonds $q\delta b$, plus current income $y$, are not sufficient to cover the coupon payment $\kappa b$. The borrower needs to go into further debt to finance the difference even it made the maximal effort of setting $c = 0$.

In Figure 10, we plot the value function $V(b)$ both paths. Whenever the borrower can select which path to be on, it chooses the one with the highest value: $\dot{b} < 0$ to the left of $b'$ and $\dot{b} > 0$ to the right of $b'$. Indeed, this constitutes a Markov equilibrium, with a cutoff of $b'$. If it weren’t for the non-negativity constraint on consumption, this would be the unique equilibrium.

However, the borrower may lack the power to select equilibria in the region $(b'', b')$: if investors price bonds according to $Q^+$ the borrower must choose a path with $\dot{b} > 0$ leading to $\hat{b}$. But conditional on doing so, the borrower chooses the optimal path leading to $\hat{b}$, validating investors’ expectations. This implies that we can choose any cutoff $\hat{b} \in (b'', b')$ and construct a Markov equilibrium, as we did in Section 4.1.
Figure 10: Value functions: $\dot{b} < 0$ (blue line) and $\dot{b} > 0$ (red line).

Proposition 9. For any $\hat{b} \in (b'', b']$, there is a Markov equilibrium with

$$Q(b) = Q^-(b) \text{ and } C(b) = C^-(b) \text{ for } b \leq \hat{b},$$
$$Q(b) = Q^+(b) \text{ and } C(b) = C^+(b) \text{ for } b > \hat{b}. $$

Equilibria are Pareto ranked according to $\hat{b}$, with the best equilibrium being $\hat{b} = b'$.

Within the region $[0, b'']$ the equilibrium is unique. To understand why, suppose the we have a threshold equilibrium with $b' < b''$. Then for a borrower just above $b'$ there is a discrete drop in utility. This is not consistent with optimal behavior: a borrower would willingly sacrifice lower consumption for a small amount of time to set $\dot{b} < 0$ and reach $b'$ in order to experience a discrete improvement in utility. This argument breaks down, however, in the region $(b'', b')$, precisely because $\dot{b} < 0$ is no longer feasible.

There is an interesting similarity and difference with models featuring “poverty traps”, as in optimal growth models with increasing returns (Skiba, 1978) or occupational choice models of entrepreneurship with financial constraints (Buera, 2003). In these models the optimization problem is non-convex and there are multiple steady states. The optimal path converges to the highest steady state if the state finds itself above some threshold, and vice versa. However, this threshold is uniquely pinned down by optimization arguments, at the intersection of the two value functions, since the non-negativity of consumption never prevents saving.

5 Commitment and Multiplicity

In previous sections, we assumed that whenever the government budget constraint was satisfied at multiple bond prices, all of these constituted potential equilibria. This implicitly assumes that the government cannot select the best price by committing to the
quantity of bonds issued.

In this section, we study simple game-theoretic models where the government can commit to bond issuances in the very short run, yet multiple sub-game perfect equilibria arise. We view these results as providing microfoundations for our timing assumption.

We endow the government with partial commitment powers: it can commit to sell a fixed amount of bonds in the present auction, but cannot make binding commitments regarding future auctions. The key idea is that if a bond auction delivers a price that falls short of the government’s expectations, then the government is expected to make further bond issuances in the next auction to make up for the shortfall. In other words, its lack of commitment across auctions renders its commitment powers within an auction useless. Non-additive preferences are crucial for this mechanism, otherwise past funding outcomes would not affect current funding needs—bygones would be bygones. Non-additive preferences are especially realistic over short time horizons and may also capture investment in durable goods or capital.

As discussed towards the end of Section 2, letting the government commit to bond issuances is simply not feasible in our fiscal rules context. However, the question becomes relevant when one considers other settings, such as the discretionary optimizing government in Section 4.2. Given the purposes of this section, we will work in a setting where fiscal policy is chosen by an optimizing government.

We present two models. The first one features a potentially unbounded number of bond auction rounds within each period. This delivers a very sharp and unambiguous result: equilibrium outcomes coincide with those studied in previous sections. The second model assumes a single auction per period, but assumes preferences are non-separable over time. The results are slightly more nuanced and depend on parameters, however we show multiple equilibria of the sort studied in previous sections is also possible. Overall, our results show that the opportunity to raise funds in future rounds may jeopardize a borrower’s attempt to stay away from the wrong side of the Laffer curve. An intertemporal commitment problem leads to an intertemporal coordination failure.

5.1 A Game with No Commitment

The idea of the first model is to split a period of the models in the previous sections into shorter subperiods and to assume that the government can only commit to bond issuances in a subperiod. For example, a period in previous sections might be taken to be a month or a quarter, within which the government’s funding needs are determined by fiscal policy decisions that adjust slowly, while the subperiods may be different days
in which auctions of Treasury bonds can take place.

Consider a two-period model in which the government’s objective function is

\[ u(c) + \beta V(b), \]

where \( c \) is current spending and \( b \) is the stock of bonds issued in the first period, to be repaid in the second period. Both \( u \) and \( V \) are decreasing, differentiable and concave functions. We could interpret \( u \) as the payoff resulting from a full specification of the benefits of public expenditure and the costs of taxation and \( V \) as as the expectation of a value function in an optimizing model with an infinite horizon.

The government receives a given tax revenue \( y \) and has a stock of bonds \( b \) inherited from the past that it needs to repay at the end of the first period. Thus, in the first period it must borrow to finance \( c - y + b \).

There is a continuum of risk neutral atomistic investors with discount factor \( \beta \). Because of risk neutrality and because all bond holders are treated equally, only the expected payment by the government in the second period need be specified to price bonds. If the total debt owed to investors is \( b \), the expected repayment per bond is given by the non-increasing function \( \Psi(b) \). This function encapsulates all the relevant considerations regarding repayment, including the probability of default as well as the recovery value in the case of default and how these vary with the level of indebtedness. Note that this framework could capture strategic default or moral hazard by the government, as all these consideration can be embedded in \( V \) and \( \Psi \).

The first period is divided into infinite rounds \( i = 1, 2, \ldots \) and the government can run an auction in each round. Think of auctions taking place in real time at \( t = 0, 1/2, 2/3, 3/4, \ldots \). At \( t = 1 \), the government collects the revenue from all these auctions, repays \( b \), buys \( c \), and the payoff \( u(c) \) is realized. Finally, at \( t = 2 \) the payoff \( V(b) \) is realized. Letting \( d_i \) denote bond issuances in round \( i \), total bonds issued in period 1 are then

\[ b = \sum_{i=0}^{\infty} d_i. \]

At each round \( i \) the investors bid price \( q_i \) for the issuance \( d_i \).

The crucial assumption we make is that in each auction the government cannot commit to the size of debt issuances in future auctions. Strategies are described by functions \( d_i = D_i(d^{i-1}, q^{i-1}) \) and \( q_i = Q_i(d^i, q^{i-1}) \), where superscripts denote sequences up to round \( i \). A subgame perfect equilibrium requires that:

i. In round \( i \), after any history \((d^{i-1}, q^{i-1})\), the government strategy \( D_j \) for the remain-
ing rounds \( j = i, i + 1, \ldots \) is optimal, given that future prices satisfy \( q_j = Q(d^j, q^{i-1}) \) at \( j = i, i + 1, \ldots \).

ii. The price in round \( i \) after history \((d^i, q^{i-1})\) satisfies \( Q(d^i, q^{i-1}) = \Psi(\sum_{i=0}^{\infty} d_i) \) where \( \{d_i\} \) is computed using the government strategy \( D_j \) for \( j = i, i + 1, \ldots \) and future bond prices \( Q_j \) for \( j = i + 1, \ldots \).

For an equilibrium to be well defined, the sequence \( \{d_i\} \) must be summable in equilibrium and after any possible deviation. Moreover, since investors are atomistic, the only restriction on prices is that they be consistent with expected repayment, which in turn is determined by total debt issued. Observe that along an equilibrium path the bond price is constant across rounds \( q_i = q^* \). We can then denote by \((c^*, b^*, q^*)\) an equilibrium outcome of the game in terms of government spending, total debt issued and bond price. The main result of this section is a tight characterization of all possible equilibrium outcomes.

**Proposition 10.** A triplet \((c^*, b^*, q^*)\) is the outcome of a subgame perfect, pure strategy equilibrium if and only if

\[
(c^*, b^*) \in \arg \max_{c, b} u(c) + \beta V(b) \quad \text{s.t.} \quad c + b = y + q^* b
\]

and

\[
q^* = \beta \Psi(b^*).
\]

The assumption that a further round is always available delivers equilibria with outcomes that are equivalent to that of a price-taking government. The government solving the maximization problem in Proposition 10 is the polar opposite of a government that can fully commit to \( b' \) and solves

\[
\max_{c, b'} u(c) + \beta V(b) \quad \text{s.t.} \quad c + b' = y + \beta \Psi(b) b.
\]

This is the assumption typically adopted in the literature. Instead, we assume that the government can commit to bond issuances in each round, but find that the outcome is as if it lacked any such commitment.

Notice the discontinuity between a bond-auction game with infinitely many rounds and a game with a large but finite number of rounds. With finitely many rounds, one can use backward induction to

It is of interest to look at intermediate cases in which some degree of commitment is available. In the remainder of this section, we explore one such case using a simple example.
5.2 A Game with Limited Commitment

We now develop a simple three-period model, with one auction per period.

The Game There are three periods, \( t = 0, 1, 2 \). The government wishes to finance consumption in all three periods with income (taxes) available only in the last period. Debt is long-term, a promise to pay in period 2.

In period 0, the government chooses how many bonds \( b_1 \) to sell. Next, an auction takes place and risk neutral investors bid \( q_0 \) for the bonds,\(^\text{19}\) the government receives \( q_0 b_1 \) from investors and uses it to finance spending

\[
c_0 = q_0 b_1.
\]

In period 1, the government chooses \( b_2 \), the investors bid \( q_1 \), and the government raises \( q_1 (b_2 - b_1) \) financing spending

\[
c_1 = q_1 (b_2 - b_1).
\]

Finally, in period 2 the random income \( y \) is drawn from a cumulative distribution function \( F(y) \) on \([0, \infty)\). The government then decides to repay or default on the debt. In the event of default all income \( y \) is lost: there is no recovery value for investors and government consumption is zero. Thus, the government repays if \( y \geq b_2 \) and defaults otherwise.

Investors are risk neutral and do not discount future payoffs. The government objective is to maximize

\[
\mathbb{E}U(c_0, c_1, c_2).
\]

Key to our analysis is that \( U \) not be additively separable between \( t = 0 \) and \( t = 1 \).

Strategies and Equilibrium The government’s strategy is given by a choice for \( b_1 \) and a function \( B_2(b_1, q_0) \) that gives \( b_2 \) for each past history \((b_1, q_0)\). The investors’ strategy is given by two functions \( Q_0(b_1) \) and \( Q_1(b_1, q_0, b_2) \).

We study subgame perfect equilibria, proceeding by backward induction. In period 1, investors bid

\[
Q_1(b_1, q_0, b_2) = 1 - F(b_2),
\]

so the price is actually independent of \( b_1 \) and \( q_0 \). In period 1, given \( b_1 \) and \( q_0 \), the govern-

\(^{19}\)The particular auction protocol is not important, but for concreteness we can assume investors play a second price auction.
ment solves

$$\max_{s_1,b_2} \mathbb{E}U(c_0,c_1,c_2) \quad (22)$$

subject to

$$c_0 = q_0 b_1,$$
$$c_1 = (1 - F(b_2))(b_2 - b_1),$$
$$c_2 = \max\{y - b_2, 0\}.$$ 

The solution gives the best response $B_2(b_1,q_0)$, which depends, in general, on both $b_1$ and $q_0$. In period 0, investors bid

$$q_0 = 1 - F(B_2(b_1,q_0)). \quad (23)$$

The fact that $q_0$ appears on both sides of this equation is crucial for equilibrium multiplicity. Specifically, a necessary condition for multiplicity is that $B_2(b_1,q_0)$ be decreasing in $q_0$ for some value of $b_1$. Since the only place where $q_0$ enters the maximization problem (22) is in determining the value of $c_0$, we see that we need non-separability of $U$ in $c_0$ and $c_1$. Namely, we need a high realized bond price $q_0$, by allowing the government to finance a high level of spending in period 0, reduces its incentive to finance high-spending in period 1. Below, we work out a specific example in which this effect is present.

A lower $q_0$ leads the government to issue more bonds in the next period, to make up for lower spending in period 0 (for low enough $b_1$ so that $c_0 < \bar{c}$). This makes the right hand side increasing in $q_0$.

Depending on the value of $b_1$, there may be multiple values of $q_0$ that solve (23). Let $Q(b_1)$ be a map that selects a solution of (23) for each $b_1$ and let $B_2(b_1) = B_2(b_1,Q(b_1))$ denote the associated value of $b_2$. The choice of $b_1$ at date 0 must maximize

$$\mathbb{E}U(Q(b_1)b_1, Q(b_1)B_2(b_1), \max y - B_2(b_1), 0).$$

To have multiplicity we need many solutions to ... for some $b_1$ and that both

**Special Assumptions** For concreteness we make several special assumptions that allow us to solve the model. We pick an exponential distribution

$$F(y) = 1 - e^{-\lambda y}$$

\[^{20}\text{We could easily extend the analysis to allow a stochastic selection of equilibria.}\]
with $\lambda > 0$. We adopt the utility function

$$U(g_0, g_1, g_2) = \alpha \min\{g_0, \bar{g}\} + \theta \min\{g_0 + g_1, \bar{g}\} + g_2$$

In other words, within the two first periods, the government has a target level of spending $\bar{g}$ and has a preference for early spending. The parameter $\theta > 1$ captures the loss from not meeting the target $\bar{g}$, while $\alpha > 0$ captures a desire for early spending.\(^{21}\)

We now make some assumptions that simplify the analysis. First, we assume that

$$\bar{c} < \max_b e^{-\lambda b} b = (\lambda e)^{-1}$$

to ensure that the government can reach the target $\bar{g}$. Under this assumption there are two solutions to

$$e^{-\lambda b} b = \bar{g},$$

which we label $\underline{b} < 1/\lambda < \bar{b}$. Next, assume that

$$\theta (1 - \lambda \underline{b}) > 1,$$

so that the government has a sufficiently strong incentive to spend in periods 0 and 1. Define

$$\hat{\alpha} \equiv \frac{1}{\lambda} \frac{e^{-\lambda \underline{b}} - e^{-\lambda \bar{b}}}{\bar{g} - e^{-\lambda \bar{b}} \left( \bar{b} - \frac{1}{\lambda} \left( 1 - \frac{1}{\bar{b}} \right) \right)}$$

whenever the denominator is positive and $\hat{\alpha} = \infty$ otherwise.\(^{22}\)

**Proposition 11.** There is always a good equilibrium with strategy $B_2(b_1) = \underline{b}$ and outcome $b_1 = b_2 = \underline{b}$. If $\alpha > \hat{\alpha}$ there is also a bad equilibrium with outcome $b_1 = b_2 = \bar{b}$. In both equilibria $g_0 = \bar{g}$.

Spending is identical in both the good and bad equilibrium, but the bad equilibrium is on the wrong side of a Laffer curve. In both cases all bond issuances take place in the first period. The possibility of bond issuance in period 1 does matter off the equilibrium path.

At $t = 1$, the government can commit to the number of bonds it issues, so it will never

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\(^{21}\)Both assumptions seem reasonable. For example, investment spending on infrastructure requires some total outlay over an extended time horizon, but with a preference for early completion. As another example consider the payment of government wages. Suppose payment can be delayed, if needed, but at a cost, because workers are impatient and demand compensation.

\(^{22}\)It is easy to find combinations of model parameters that ensure $\hat{\alpha} < \infty$. 

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reach a $b_2$ such that a reduction in $b_2$ increases current revenues—it will always be on the increasing side of the Laffer curve for new issuances

$\left( 1 - F (b_2) \right) (b_2 - b_1)$.

Given our functional form assumption, this implies $1 - \lambda (b_2 - b_1) \geq 0$. However, the Laffer curve for total debt issued is

$\left( 1 - F (b_2) \right) b_2$

and the last inequality is not enough to rule out an equilibrium with total debt on the wrong side of the this Laffer curve, because the slope of this curve is $1 - \lambda b$, which can be negative in spite of the above inequality if $b_2 - b_1$ is small. Moreover, the government at date 0 cannot try to move away from the bad equilibrium by reducing $b_1$ below $\bar{b}$, because, if it does, the market expects the government to issue $\bar{b} - b_1 > 0$ at date 1, and therefore the pricing function $Q_0 (b_1)$ is flat for $b_1$ near $\bar{b}$. The only option to eliminate the bad equilibrium would be to reduce $b_1$ below some cutoff $\hat{b}_1 < \bar{b}$, which we derive in the appendix. But if $\alpha$ is large enough, reducing $b_1$ below $\hat{b}_1$ is too costly in terms of delayed spending.

6 Conclusions

Our analysis supports the suspicion that borrowing countries may become entrapped in a self-fulfilling equilibrium with high interest rates. Fortunately, our results also indicate that such crises are far from inevitable, highlighting the importance of fiscal policy rules and debt maturity in ensuring the good equilibrium is unique.

References


7 Appendix

7.1 Proof of Proposition 2
The argument is by backward induction. The functions $X_T$, $Q_{T-1}$ and $m_{T-1}$ are uniquely defined. The first step at which multiple equilibria can arise is in the selection of $b_T$ when constructing the bond issuance function $B_T(b_{T-1}, s^{T-1})$. However, when $\delta = 1$, the bond issuance function $B_T$ does not affect the construction of the repayment function $X_{T-1}$ and of the pricing function $Q_{T-2}$, as repayment only depends on the maximum of the function $Q_{T-1} (b_T, s^{T-1}) b_T$ and the term $(1 - \delta) Q_{T-1} (B_T(b_{T-1}, s^{T-1}), s^{T-1})$ in (3) disappears when $\delta = 1$. The same argument applies in all previous periods.

7.2 Proof of Proposition 5
In the case considered, the Laffer curve takes the form $[1 - F ((1 + r) b - m)] b$ (omitting time subscripts and dependence on $s^t$ to simplify notation). The slope of the Laffer curve is

$$1 - F ((1 + r) b - m) - (1 + r) f ((1 + r) b - m) b$$

which has the same sign of

$$1 - (1 + r) \frac{f ((1 + r) b - m)}{1 - F ((1 + r) b - m)} b.$$ 

So if $f / (1 - F)$ is monotone non-decreasing, the derivative can only change sign once.

7.3 Proof of Lemma 1
Using steady-state conditions, the Jacobian can be written as

$$J = \begin{bmatrix} \frac{\kappa - \kappa'}{q} & -\delta & -\frac{gb}{q} \\ -\lambda \Psi' (b) & r + \delta + \lambda \end{bmatrix}.$$ 

A necessary and sufficient condition for a saddle is a negative determinant of $J$, i.e., $J_{11} J_{22} < J_{12} J_{21}$. Since $J_{12} < 0$ and $J_{22} > 0$, this is equivalent to $-J_{11} / J_{12} < -J_{21} / J_{22}$, which means that the $b = 0$ locus is downward sloping and steeper than the $q = 0$ locus. Condition (17) then follows.
7.4 Proof of Proposition 7

Consider the functions on the right-hand sides of (14) and (15), which are both continuous for \( b > 0 \). If there is a saddle-path stable steady state at \( b' \), the second function is steeper, from Lemma 1, and so is below the first function at \( b' + \epsilon \) for some \( \epsilon > 0 \). Taking limits for \( b \to \infty \) the the second function yields \( q \to \kappa / \delta \) and the first yields

\[
q \to \frac{\kappa + \lambda \Psi(S)}{r + \delta + \lambda} < \frac{\kappa}{\delta},
\]

where the inequality can be proved using \( \Psi(S) < 1 \) and \( \kappa = r + \delta \). Therefore, the second function is above the first for some \( b'' \) large enough. The intermediate value theorem implies that a second steady state exists in \((b' + \epsilon, b'')\).

7.5 Proof of Proposition 10

We start with the sufficiency part, by constructing an equilibrium which implements the desired outcome. The equilibrium pricing function sets the price \( Q(d^i, q^i-1) = q^* \) for any history \((d^i, q^i-1)\) with \( q^i-1 = \{q^*, ..., q^*\} \). The strategy of the government is to issue \( b^* - \sum_{j=0}^i d_j \) after any history with \( q^i-1 = \{q^*, ..., q^*\} \). The resulting equilibrium play is that the government issues \( b^* \) in the first auction and no further auction takes place. Since at each round the price is independent of the amount of bonds issued, the government cannot gain by changing its bonds issuances. Investors, on the other hand, expect that if the government deviates and offers anything other than \( b^* - \sum_{j=0}^i d_j \) in the current round, it will adjust its issuances in the next round so as to reach the debt level \( b^* \). This justifies their bid being independent of the amount of bonds issued in the current round.

Turning to the necessity part, suppose we have an equilibrium with outcome \((s,b)\) and define \( q = (b_\infty - c)/b \). We want to prove that \( q = \text{MRS} \equiv V'(b)/u'(s) \). Suppose, towards a contradiction, that we have a proposed equilibrium where instead \( q \neq \text{MRS} \). For concreteness suppose \( q > \text{MRS} \). The other case is symmetric.

In equilibrium the borrower is supposed to exit with \((b,c)\) at some round. Suppose that instead, upon reaching this round, the government considers a deviation, does not exit and instead issues a small extra \( \epsilon > 0 \) amount of debt in the next round, for a current total of

\[
\tilde{b} = b + \epsilon.
\]

\(^{23}\)It is not difficult to complete the description of the equilibrium constructing continuation strategies after histories with \( q_i \neq q^* \). However, given the atomistic nature of investors, these off-equilibrium paths are irrelevant for the borrower’s maximization problem.
The market must then respond with a price $\tilde{q}$ for this round. The current price yields a current revised consumption

$$\tilde{c} = c + \tilde{q}\epsilon.$$ 

In the equilibrium of the ensuing sub-game, the price in all future rounds must be constant and given by $\tilde{q} = \tilde{q}^* = G(\tilde{b}^*)$ where $\tilde{b}^*$ is the end outcome for debt following this sub-game. The associated end outcome for consumption is then $\tilde{c}^* = c + \tilde{q}(\tilde{b}^* - b)$.

The following inequalities hold

$$u(c) + V(b) \geq u(\tilde{c}^*) + V(\tilde{b}^*) \geq u(\tilde{c}) + V(\tilde{b}).$$  \hspace{1cm} (24)

The first inequality follows because $(b, c)$ is an equilibrium outcome. The second because otherwise in the sub-game the borrower would prefer to stop after the initial deviation. We can now prove that the end outcome of the sub-game cannot have more total debt than the initial deviation, that is, $\tilde{b}^* \leq b$. Suppose, by contradiction, that $\tilde{b}^* > b$. Let

$$\lambda = \frac{\tilde{b} - b}{\tilde{b}^* - b} \in (0, 1),$$

and notice that

$$(\tilde{b}, \tilde{c}) = (1 - \lambda) (b, c) + \lambda (\tilde{b}^*, \tilde{c}^*).$$

Strict concavity and the first inequality in (24) then imply $u(\tilde{c}^*) + V(\tilde{b}^*) < u(\tilde{c}) + V(\tilde{b})$, which contradicts the second inequality in (24).

Since the function $G(b)/b$ is non-increasing in $b$, $\tilde{b}^* \leq \tilde{b}$ implies

$$\tilde{q} = \frac{G(\tilde{b}^*)}{\tilde{b}^*} \geq \frac{G(\tilde{b})}{\tilde{b}}.$$ 

By choosing an initial deviation with $\epsilon > 0$ small enough, the borrower can ensure that the lower bound on $\tilde{q}$ is arbitrarily close to $q$, since $G(\tilde{b})/\tilde{b} \to q$ as $\tilde{b} \to b$. But then, since $q > MRS$, this implies that along this deviation the borrower can sell bonds at a price $\tilde{q} > MRS$, which implies $u(\tilde{c}) + V(\tilde{b}) > u(c) + V(b)$. Therefore, if the proposed equilibrium satisfies $q > MRS$, there is a profitable deviation by the borrower, a contradiction.

### 7.6 Proof of Proposition 11

We start by characterizing the government’s optimal choice for debt $B_2(b_1, q_0)$ at $t = 1$, given values of $b_1$ and $q_0$. 

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Lemma 2. Given $q_0$ and $b_1$, the optimal choice of $b_2$ must satisfy either
\[
q_0 b_1 + (1 - F(b_2)) (b_2 - b_1) < \overline{g} \\
\theta (1 - \lambda (b_2 - b_1)) = 1
\]
or
\[
q_0 b_1 + (1 - F(b_2)) (b_2 - b_1) = \overline{g} \\
\theta (1 - \lambda (b_2 - b_1)) \geq 1.
\]

Proof. In equilibrium we always have $q_0 b_1 \leq \overline{g}$. Therefore, the marginal benefit of increasing $b_2$ is
\[
\theta (1 - F(b_2) - f(b_2)(b_2 - b_1)) - (1 - F(b_2)) = (1 - F(b_2)) [\theta (1 - \lambda (b_2 - b_1)) - 1]
\]
if $g_0 + (1 - F(b_2)) (b_2 - b_1) < \overline{g}$ and 0 otherwise. The statement follows immediately. \qed

Define the cutoff
\[
\hat{b}_1 = \overline{b} - \frac{1}{\lambda} \left( 1 - \frac{1}{\theta} \right) \in \left( 0, \overline{b} \right).
\]  

We can now characterize the continuation equilibria that arise after the choice of $b_1$ by the government at date 0, that is, we look for candidates for the equilibrium selections $Q_0(b_1)$ and $B_2(b_1)$. We first consider the case in which $b_1$ is below the cutoff $\hat{b}_1$.

Lemma 3. If $b_1 < \hat{b}_1$ there is a unique continuation equilibrium, with $b_2 = b$. If $b_1 \geq \hat{b}_1$ there are two continuation equilibria, one with $b_2 = b$ and one with $b_2 = \overline{b}$.

Proof. The equilibrium exists because $(1 - F(b_2)) b_2 = \overline{g}$ at $b_2 = \overline{b}$ and the assumption that $\theta (1 - \lambda b) > 1$ implies $\theta (1 - \lambda (b_2 - b_1)) > 1$ for any $b_1 \geq 0$. To prove uniqueness notice that we cannot have $b_2 \in (b, \overline{b})$ in equilibrium, otherwise $e^{-\lambda b_2} b_2 > \overline{g}$, we cannot have $b_2 \geq \overline{b}$, otherwise $\theta (1 - \lambda (b_2 - b_1)) < 1$, and we cannot have $b_2 < b$, otherwise $e^{-\lambda b_2} b_2 < \overline{g}$ and $\theta (1 - \lambda (b_2 - b_1)) > 1$ (always using Lemma 2). The second equilibrium exists because $b_1 \geq \hat{b}_1$ is equivalent to
\[
\theta \left( 1 - \lambda \left( \overline{b} - b_1 \right) \right) \geq 1.
\]
\qed
The lemma implies that a possible selection for continuation equilibria is

\[ B_2(b_1) = \begin{cases} b & \text{if } b_1 \leq \hat{b}_1 \\ \bar{b} & \text{if } b_1 > \hat{b}_1 \end{cases}. \]

We go back to period 0 and study the government’s optimization problem with this selection. The government chooses \( b_1 \) to maximize

\[ \alpha e^{-\lambda B_2(b_1)} b_1 + \theta \min \left\{ e^{-\lambda B_2(b_1)} B_2(b_1), \hat{B} \right\} + \frac{1}{\lambda} e^{-\lambda B_2(b_1)}. \]

The government faces a trade-off here. If it chooses \( b_1 \leq \hat{b}_1 \) it ensures that in the continuation game investors will expect low issuance of bonds in period 1 and so only \( b \) bonds will be eventually issued, keeping the government on the good side of the Laffer curve. However, to keep \( b_1 \) low the government foregoes the benefits from early spending \( \alpha \). In particular, choosing \( 0 \leq b_1 \leq \hat{b}_1 \) we have

\[ \alpha e^{-\lambda \hat{b}_1} b_1 + \theta \hat{B} + \frac{1}{\lambda} e^{-\lambda b}. \]

While choosing \( \hat{b}_1 < b_1 \leq \bar{b} \) we have

\[ \alpha e^{-\lambda \bar{b}_1} b_1 + \theta \bar{B} + \frac{1}{\lambda} e^{-\lambda \bar{b}}. \]

Clearly, the only possible optimal choices are \( b_1 = \hat{b}_1 \) and \( b_1 = \bar{b} \). It is optimal to choose \( b_1 = \bar{b} \) if

\[ \alpha e^{-\lambda \bar{b}} b + \frac{1}{\lambda} e^{-\lambda \bar{b}} > \alpha e^{-\lambda \hat{b}} b + \frac{1}{\lambda} e^{-\lambda \hat{b}}. \]

Using (25) to substitute for \( \hat{b}_1 \) in this inequality completes the proof.