Nonlinear Gravity*

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Abstract

We consider a static trade model with simple production and general non-homothetic preferences, requiring only that utility be increasing, concave and twice differentiable. In this framework, we provide an explicit formula for the elasticity of import intensity with respect to trade costs (trade elasticity). We show that the trade elasticity is a function, not a constant, implying a nonlinear version of the gravity relationship commonly studied in the empirical trade literature. We demonstrate that, even in this environment, the elasticity of welfare with respect to trade intensity (welfare elasticity) is equal to the reciprocal of the trade elasticity. We provide several examples of models that are analytically intractable, yet where the welfare elasticity can be solved in closed form using our procedure. We also provide sufficient conditions to compare the gains from trade implied by non-homothetic models to those implied by CES models.

1 Introduction

Trade models are typically divided into two types: those that are analytically tractable, which allow for easy computation of the gains from trade reform, and those that replicate features of the trade data. The first group includes many of the workhorse models of the trade literature, such as Krugman (1980), Eaton and Kortum (2002), and Melitz (2003). These models can typically be solved in closed form. They accomplish this by utilizing some form of a constant elasticity of substitution aggregator, and hereafter we will refer to them as CES models. However, they fail to match many of the well-known patterns of international trade, such as how import patterns differ by country-level income. The second class of models is able to match rich patterns of trade by introducing complexities to the

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preference or production structure of the economy. These include Markusen (1986), Fieler (2011), and Simonovska (2014). The ability to capture these patterns of trade comes at the cost of analytical tractability. These models typically cannot be solved in closed form, making it difficult to do basic comparative statics on trade costs. Therefore, these models typically are not used to answer such questions, even though we know that they incorporate features that better fit the data.

The analytical tractability of the CES models comes in part from the simple relationship that the models have between import demand and trade costs. These models exhibit a linear gravity form, in which the traditional linear gravity model employed by the empirical trade literature is correctly specified within these CES models, such as that used by Bergstrand (1985). In trade models in which trade costs have a constant marginal effect on the intensity of imports, the gains from reducing trade costs can be measured easily. As demonstrated in Arkolakis, Costinot and Rodriguez-Clare (2012), which will hereafter be referred to as ACR, these models imply gains from trade equal to:

\[
\log \left( \frac{W^{\text{TRADE}}}{W^{\text{AUT}}} \right) = \frac{1}{\varepsilon_T} \log(\lambda_0)
\]

where \( W^{\text{TRADE}} \) is real income at the observed level of trade, \( W^{\text{AUT}} \) is real income in autarky, \( \varepsilon_T \) is the trade elasticity, and \( \lambda_0 \) is one minus the ratio of imports to domestic gross output. This is true for any static trade model in which factors are in fixed supply, trade is balanced, profits are a constant fraction of firm revenues, and the gravity model is correctly specified. As they show, this includes all the CES models previously mentioned. Moreover, the means of measuring the gains from trade here is simple, requiring only information on imports, and the commonly measured constant trade elasticity.

Clearly the procedure in ACR cannot be applied to the non-homothetic models described above, since those models do not exhibit the simple, linear gravity relationship needed for their results. The difficulty comes from the fact that the trade elasticity is not a constant. As trade costs are varied, the real income of the importing country varies. Because the composition of their consumption varies with their real income level, the marginal effect of changes in trade costs on import behavior for countries in these models varies, making it a challenge to characterize the effects of changes in trade costs in a tractable way.

In this paper, we introduce a procedure to bridge the gap between these two parts of the trade literature. We consider a static trade model with simple, competitive production and a representative household with general, non-homothetic preferences. We require only

\[1\] Throughout the paper, by gains from trade we mean how much higher is real income at the observed level of trade than it would be in autarky.
that the household’s utility function be increasing, strictly concave and twice differentiable. These non-homothetic preferences mean that the linear gravity approach described above is inapplicable. Instead, we derive two main results that allow for analytical tractability, even in this non-homothetic environment. First, we derive an explicit formula for the trade elasticity in this environment. Instead of being a constant, as in the CES models, in the non-homothetic environment the trade elasticity is a function of trade costs. That is, the gravity relationship in the model is nonlinear. We provide several examples showing how our formula can be used to solve for the trade elasticity in closed form. Second, we show that the logic of the ACR result, that the gains in welfare are tied to the trade elasticity, generalizes to this environment. That is, the elasticity of real income to $\lambda_0$ is still the reciprocal of the trade elasticity. However, in this nonlinear environment, both the trade and real income elasticities are functions, so this relationship is true point-by-point.

Therefore, the essential logic of the ACR result generalizes to this more complex environment. The gains from trade are fundamentally tied to how changes in trade costs affect import behavior. This is true whether the effect of trade costs on imports is constant (linear gravity relationship) or varying (nonlinear gravity relationship).

This procedure allows us to measure gains from trade easily even in complex, non-homothetic models. Section 2.5 gives several examples of models that cannot be solved in closed form, yet can have their trade and real income elasticities computed in closed form using our results. This provides the tractability of the CES models in this richer, non-homothetic environment. Moreover, this approach is an appealing alternative to the CES models. As ACR points out, the simplicity of their formula implies that all the micro-level information put into CES models about the characteristics of firms or countries has no effect on the computed gains from trade in those models, so long as the model matches the import behavior and gravity relationship from the data. The non-homothetic approach breaks this tight link. As our examples demonstrate, the varying trade elasticities we derive can depend fundamentally on country characteristics, as well as the composition of imports and domestic consumption.

Since the CES models are widely used to measure the gains from trade, a natural question is how the gains from trade in the non-homothetic models compare to those in the CES models. We derive sufficient conditions on the trade elasticity to sign this difference. Not surprisingly, the key determinant of how the implied gains from trade differ in the two environments depends on how the trade elasticity changes with trade costs. We prove that

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2In the body of the paper, we also impose additive separability. This is not required for our main results, but it greatly simplifies the analysis. The appendix generalizes our procedure to preferences that are non-separable.
if the trade elasticity is increasing in trade costs, then the gains from trade in the non-homothetic model are higher than in the CES models. The intuition for this is simple. Consider a country that is starting with extremely high trade costs. An increasing trade elasticity means that small reductions in trade costs from their very high levels has a large impact on imports. The effect of reducing trade costs on imports gets lower and lower as trade costs decline and imports increase. At observed levels of trade, trade elasticities are low relative to where they would be with much higher trade costs. Therefore, using the elasticity estimated from observed levels of trade and assuming it is constant underestimates the gains from trade.

Although our results generalize the logic of the ACR formula, this paper cannot be said to generalize the results of ACR. First, ACR had a much richer production structure than is under consideration here. Our production is simple and competitive, while theirs is more general. Second, that paper emphasizes the equivalence of margins of adjustment, many of which are absent here. It may be possible to incorporate additional margins into the framework presented here, and to allow for richer types of production.

The other strong assumption maintained throughout the paper is that there is a representative household whose welfare we are measuring. Given that the model is explicitly non-homothetic, we cannot rely on the familiar aggregation results, except to assume that all households in the country of analysis are identical. The goal of this paper is to compare our results to those generated by the CES trade models, so we save the case of within-country inequality for future work.

2 Model with Two Countries

We begin by considering a two country trade model. We take the perspective that country 0 is the country of analysis, and country 1 is the rest of the world. Our motivating question is: how much higher is the real income of country 0 with the observed level of trade than it would be in autarky? In this section, we show how to answer this question in a very general way.

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3 See Fajgelbaum and Khandelwal (2014) as an example of current work on this topic.

4 Instead of two countries, our results are unchanged if there are $N + 1$ countries, where country 0 is the country of analysis and the other $N$ countries are all identical to one another. In that case, the results here are the same when considering trade cost changes between country 0 and all of the $N$ other countries simultaneously. The case with heterogeneous trading partners is considered in Section 4.
2.1 Household’s problem

The household in country 0 has an additively separable utility function\(^5\) and solves the following problem:

\[
\max \sum_{n=0}^{1} \int_{i \in \Omega_n} u_{ni}(c_n(i)) di
\]

\[
st : \sum_{n=0}^{1} \int_{i \in \Omega_n} \tau_{0n}p_n(i)c_n(i) di = w_0L_0
\]

\[
\forall n, \forall i : c_n(i) \geq 0
\]

We choose to have the wage in country 0 be numeraire. Here \(\tau\) is the iceberg transportation cost\(^6\) on imported goods, and \(L_0\) is the endowment of labor available in country 0. We set \(\tau_{00} = 1\). The set of goods available from country \(n\) that the country 0 household can consume is \(\Omega_n\). Here we require only that \(u_{ni}\) be strictly increasing, strictly concave\(^7\) and twice differentiable. In general, these preferences are non-homothetic.

2.2 Production

All production in the model is done by competitive final goods producers who use technology linear in labor.\(^8\) A firm in country \(n\) selling product \(i\) has productivity \(a_n(i)\). In order to sell \(y_{mn}^m(i)\) units to the household in country \(m\), the firm must produce \(\tau_{mn}y_{mn}^m(i)\) units. The firm solves:

\[
\max \sum_{m} \tau_{mn}p_n(i)y_{mn}^m(i) - w_nl_n(i)
\]

\[
st : \sum_{m} \tau_{mn}y_{mn}^m(i) = a_n(i)l_n(i)
\]

\(^5\) The general, non-separable case is considered in Section 5. Our results are unchanged, but computing the trade elasticity is more complicated.

\(^6\) All trade costs in the model take the form of iceberg trade costs. As in Arkolakis, et al (2012), our results do not go through if it is instead a tariff that is rebated to the household.

\(^7\) Due to strict concavity and the fact that the sets of goods produced by each country are disjoint, this model excludes goods that are perfectly substitutable across countries.

\(^8\) What is crucial for the results in this paper is that the price of goods in country 0 relative to the wage in country 0 (the numeraire) does not change in response to changes in trade costs. This is true with perfect competition, which we assume. It is also true with monopolistic competition and fixed markups, which is if \(\forall n, i : c_n(i)u''_{ni}/u'_{ni}\) is constant, as in Example 2 given later.
The fact that these firms are competitive implies that their price is equal to their marginal cost and their profits are zero:

\[ p_n(i) = \frac{w_n}{a_n(i)} \]

### 2.3 Market Clearing

In each country, labor is inelastically supplied and the equilibrium wage clears the labor market:

\[ \forall m, \int_{i \in \Omega_m} l_m(i) di = L_m \]

### 2.4 Trade Elasticity

As has been shown in Arkolakis, et al (2012), the trade elasticity (that is, the elasticity of imports as a fraction of domestic consumption to trade costs) is the main determinant of gains from trade in a CES model. Following their definition, we first define the import share from country \( n \) as:

\[ \lambda_{0n} = \frac{\int_{i \in \Omega_n} \tau_{0n} p_n(i) c_n(i) di}{\sum_{m=0}^{1} \int_{i \in \Omega_m} \tau_{0m} p_m(i) c_m(i) di} \]

Then the trade elasticity is defined\(^9\) as:

\[ \varepsilon_T(\tau_{01}) \equiv \frac{\partial \log(\lambda_{01}/\lambda_{00})}{\partial \log(\tau_{01})} \frac{\partial \log(w_1)}{\partial \log(\tau_{01})} \frac{1}{1 + \frac{\partial \log(w_1)}{\partial \log(\tau_{01})}} \]

Here we write the elasticity as a function of trade costs to emphasize that in general it is not a constant as it is in CES models. This means that changes in trade costs have a non-constant marginal effect on imports, hence a “nonlinear gravity” relationship.

Also, let us define:

\[ I_n(i) = \begin{cases} 
0 & \text{if } c_n(i) = 0 \\
1 & \text{if } c_n(i) > 0
\end{cases} \]

Now we derive a formula that can be used to solve for the trade elasticity. This result is summarized in Theorem 1.

\(^9\)Notice that this is a modified definition from that in Arkolakis, et al (2012). Here the elasticity of real exchange rates (the ratio of domestic and foreign wages) with respect to trade costs is not constant, so we must adjust by that term.
Theorem 1: Whenever $\lambda_{00} \in (0, 1)$:

$$
\varepsilon_T(\tau_{01}) = \left(1 + \frac{1}{L_0 \lambda_{01}} \int_{i \in \Omega_1} \tau_{01} p_1(i) \frac{u'_0(i)}{u'_0(i)} I_n(i) \, di\right) \frac{1}{L_0 \lambda_{00}} \int_{i \in \Omega_0} p_0(i) \frac{u'_0(i)}{u'_0(i)} I_n(i) \, di \\
\frac{1}{L_0} \sum_{m=0}^1 \int_{i \in \Omega_m} \tau_{0m} p_m(i) \frac{u'_m(i)}{u'_m(i)} I_n(i) \, di
$$

The proof of Theorem 1 is left to the appendix. The formula is rather complicated, but its usefulness can be illustrated by some examples.

2.5 Examples

Here we solve out several examples to demonstrate the procedure for finding the trade elasticity, and for discussion later in the paper.

2.5.1 Example 1: Constant Elasticity of Substitution (CES)

Constant elasticity of substitution is a special case of the preferences given above, and, as in Arkolakis, et al (2012), CES preferences should yield a constant trade elasticity. Therefore, first we should demonstrate that this is true.

CES preferences can be written as:

$$
U(c_{00}, c_{01}) = \frac{\sigma}{\sigma - 1} c_{00}^{1-1/\sigma} + \frac{\sigma}{\sigma - 1} c_{01}^{1-1/\sigma}
$$

Then,

$$
u'_0(i) = -\sigma c_{00}, \quad \frac{u'_1(i)}{u'_0(i)} = -\sigma c_{01}, \quad \lambda_{00} = \frac{p_0 c_{00}}{L_0}, \quad \lambda_{01} = \frac{\tau_{01} p_1 c_{01}}{L_0}
$$

Then applying Theorem 1, we get:

$$
\varepsilon_T(\tau_{01}) = \frac{-\sigma (1 - \sigma)}{-\sigma} = 1 - \sigma
$$

which is a constant.

2.5.2 Example 2: Heterogeneous Income Elasticity

Following Fieler (2011), suppose utility is non-homothetic and has goods with differing income elasticities. Specifically, suppose goods imported from the rest of the world have a different income elasticity than those goods produced in the domestic country. Let utility be given by:

$$
U = \int_{\Omega_0} \omega_0(i) \frac{\alpha}{\alpha - 1} c_{00}(i)^{1-1/\alpha} \, di + \int_{\Omega_1} \omega_1(i) \frac{\gamma}{\gamma - 1} c_{10}(i)^{1-1/\gamma} \, di
$$

Taking ratios of first and second derivatives yields:

$$\forall i, \frac{u'_0(i)}{u''_0(i)} = -\alpha c_0(i), \frac{u'_1(i)}{u''_1(i)} = -\gamma c_1(i)$$

Applying Theorem 1, we get:

$$\varepsilon_T(\tau_{01}) = -\frac{\alpha (\gamma - 1)}{\alpha \lambda_{00} + \gamma \lambda_{01}}$$

Now we see that the trade elasticity is no longer constant. As trade costs decline, \(\lambda_{00}\) shrinks and \(\lambda_{01}\) (which is equal to \(1 - \lambda_{00}\)) grows. If \(\alpha > \gamma > 1\), then the trade elasticity gets larger as trade costs get larger (that is, \(\lambda_{00}\) grows), and the opposite if \(\gamma > \alpha > 1\). Notice also that if \(\alpha = \gamma\), we recover the CES case, as expected.

### 2.5.3 Example 3: Consumption Requirement

Markusen (1986) introduced the idea of consumption requirements into utility functions in trade models to better match important facts about the trade patterns of goods across countries. We first consider a simple case of this, and generalize it in the next example. Suppose there is a good produced in the home country that requires a minimum consumption level, and a manufacturing good imported from abroad that does not. Preferences are given by:

$$U(c_{00}, c_{01}) = (c_{00} - \bar{c})^{1-1/\sigma} + c_{01}^{1-1/\sigma}$$

Then, letting \(L_0 = 1\),

$$\frac{u'_0(i)}{u''_0(i)} = -\sigma (c_{00} - \bar{c}), \frac{u'_1(i)}{u''_1(i)} = -\sigma c_{01}$$

And,

$$\varepsilon_T(\tau_{01}) = -\left(1 - \frac{p_0 \bar{c}}{\lambda_{00}}\right) \frac{\sigma - 1}{1 - p_0 \bar{c}}$$

If \(\sigma > 1\) and the household is capable of satisfying its consumption requirement, then the trade elasticity decreases as \(\lambda_{00}\) increases (that is, when trade costs increase). Note that the opposite is true if \(\bar{c} < 0\), assuming that \(\lambda_{00} > 0\).

### 2.5.4 Example 4: Many Goods with Additive Constants

Consider a case similar to Simonovska (2014), but where we do not require the additive constants to be positive nor that they be the same for all goods. Suppose \(L_0 = 1\) and utility
is given by:

\[
U = \int_{i \in \Omega_0} \alpha_0(i) \log(c_0(i) + \bar{c}_0(i)) + \int_{i \in \Omega_1} \alpha_1(i) \log(c_1(i) + \bar{c}_1(i))
\]

s.t. : \[ 1 \geq \int_{i \in \Omega_0} p_0(i)c_0(i) + \int_{i \in \Omega_1} \tau_01p_1(i)c_1(i) \]

Where \( \sum_n \int_{\Omega_n} \alpha_n(i)di = 1 \) and imposing that \( \int_{\Omega_1} \bar{c}_1(i)/a_1(i)di > 0 \). Then applying Theorem 1 yields:

\[
\varepsilon_T = \left(1 - \frac{1}{\lambda_{00}} \int_{i \in \Omega_1} \tau_01p_1(i)(c_1(i) + \bar{c}_1(i))I_n(i)di \right) \frac{1}{\lambda_{00}} \int_{i \in \Omega_0} -p_0(i)(c_0(i) + \bar{c}_0(i))I_n(i)di
\]

Let \( \mu \) be the Lagrange multiplier on the household’s budget constraint. Then taking first order conditions and using the definition of \( \lambda_{00} \) we can prove two useful equalities:

\[
\frac{1}{\mu} = \frac{\lambda_{00} + \int_{i \in \Omega_0} p_0(i)\bar{c}_0(i)di}{\int_{\Omega_0} \alpha_0(i)} = 1 + \sum_n \int_{\Omega_n} \tau_{0n}p_n(i)\bar{c}_n(i)
\]

Then using the definition of \( \lambda_{01} \) and rearranging:

\[
\frac{\int_{i \in \Omega_1} \tau_01p_1(i)\bar{c}_1(i)di}{\lambda_{01}} = -1 + \frac{\lambda_{00} + \int_{i \in \Omega_0} p_0(i)\bar{c}_0(i)di}{\lambda_{01}} \cdot \frac{\int_{\Omega_1} \alpha_1(i)di}{\int_{\Omega_0} \alpha_0(i)di}
\]

Then we can rewrite the trade elasticity as a function of only country 0 prices (that are independent of trade costs), preference parameters, and \( \lambda_{00} \):

\[
\varepsilon_T = \frac{\int_{\Omega_0} \alpha_0(i)I_n(i)}{\lambda_{00}} \left(1 - \frac{\lambda_{00} + \int_{i \in \Omega_0} p_0(i)\bar{c}_0(i)I_n(i)di}{\lambda_{00}} \cdot \frac{\int_{\Omega_1} \alpha_1(i)I_n(i)di}{\int_{\Omega_0} \alpha_0(i)I_n(i)di} \right)
\]

Interestingly, none of the values of \( c_1(i) \) enter the trade elasticity. That information is already encoded in the import penetration term \( \lambda_{00} \). Not surprisingly, whether the trade elasticity here is increasing or decreasing is ambiguous, and depends on the signs and magnitudes of the \( \bar{c} \) terms.

This demonstrates that there are cases with many goods that differ in complex ways in which the trade elasticity can be solved in closed form. However, it is not true that every possible preference specification has a closed form for the trade elasticity. Even this example cannot be solved in closed form except in the logarithmic case. Our result is useful because

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\[^{10}\text{This second condition is equivalent to the trade elasticity being negative. Let us emphasize that we do not require each value of } \bar{c}_1(i) \text{ to be positive.} \]
it does work in many cases, hence expanding the set of trade models that are analytically tractable, and because the trade elasticity we derive is directly related to the welfare gains from trade, as discussed in the next subsection.

### 2.6 Welfare Elasticity

The trade elasticity is useful because of its connection to the welfare gains from trade. Arkolakis, et al (2012) showed, in models with constant trade elasticities, the welfare elasticity is equal to the reciprocal of the trade elasticity, which makes computing the welfare gains from trade trivial in that class of models. We show that this is also true in our environment, even though the trade elasticity is no longer a constant. This result is summarized in Theorem 2 below. We first define some notation.

In this environment there is not a perfect price index so changes in real income are not a simple function of changes in the price level. Instead we define changes in real income as the compensating variation needed to make the representative household indifferent between the allocation they receive facing current trade costs and current income, and the allocation they would receive with new trade costs. We will denote real income as $W$.

The dual of the household’s problem described above is:

$$ I = \min \sum_{n=0}^{1} \int_{i \in \Omega_n} \tau_{0n} p_n(i) c_n(i) di $$

s.t. $\bar{U} = \sum_{n=0}^{1} \int_{i \in \Omega_n} u_{ni}(c_n(i)) di$

Note that $I$ is not real income. For a given $\bar{U}$, $I$ is increasing in $\tau_{01}$ since the household would need more units of income to afford the same utility level. Instead, changes in real income are defined as the equivalent loss in income associated with an increase in trade costs. Therefore,

$$ \frac{\partial \log(W)}{\partial \log(\tau_{01})} = - \frac{\partial \log(I)}{\partial \log(\tau_{01})} $$

Following Arkolakis, et al (2012), we define the welfare elasticity as the elasticity of real income with respect to changes in expenditure on domestic goods, $\lambda_{00}$. That is,

$$ \varepsilon_W(\tau_{01}) \equiv \frac{\partial \log(W)}{\partial \log(\tau_{01})} = \frac{\partial \log(I)}{\partial \log(\tau_{01})} = \frac{\partial \log(\lambda_{00})}{\partial \log(\tau_{01})} $$

As with the trade elasticity, the welfare elasticity is explicitly written as a function of trade
costs to emphasize that it is not a constant. The relationship between the welfare and trade elasticities is described in Theorem 2.

**Theorem 2** For all \( \tau_{01} \),

\[
\varepsilon_W(\tau_{01}) = \frac{1}{\varepsilon_T(\tau_{01})}
\]

**Proof.** First note that \( \lambda_{00} + \lambda_{01} = 1 \), so that

\[
\frac{\partial \lambda_{00}}{\partial \log(\tau_{01})} = -\frac{\partial \lambda_{01}}{\partial \log(\tau_{01})} \implies \lambda_{00} \frac{\partial \log(\lambda_{00})}{\partial \log(\tau_{01})} = -\lambda_{01} \frac{\partial \log(\lambda_{01})}{\partial \log(\tau_{01})}
\]

Then notice that:

\[
\frac{\partial \log(\lambda_{01}/\lambda_{00})}{\partial \log(\tau_{01})} = \frac{\partial \log(\lambda_{01})}{\partial \log(\tau_{01})} - \frac{\partial \log(\lambda_{00})}{\partial \log(\tau_{01})} = -(\frac{\lambda_{00}}{1 - \lambda_{00}} + 1) \frac{\partial \log(\lambda_{00})}{\partial \log(\tau_{01})} = -\frac{1}{1 - \lambda_{00}} \frac{\partial \log(\lambda_{00})}{\partial \log(\tau_{01})}
\]

Using the envelope theorem on the dual of the consumer’s problem written above, and noting that \( \forall i, p_1(i) = w_1/a_1(i) \), yields:

\[
\frac{\partial \log(I)}{\partial \log(\tau_{01})} = (1 - \lambda_{00}) \left( 1 + \frac{\partial \log(w_1)}{\partial \log(\tau_{01})} \right)
\]

Then,

\[
\frac{1}{\varepsilon_W(\tau_{01})} = -\frac{\partial \log(\lambda_{00})}{\partial \log(\tau_{01})} = -\frac{1}{1 - \lambda_{00}} 1 + \frac{\partial \log(w_1)}{\partial \log(\tau_{01})} = \varepsilon_T(\tau_{01})
\]

This theorem provides an easy means of solving for gains from trade, even in models that are not analytically tractable. Example 2 above is a model that cannot (except in particular parameter cases) be solved in closed form. However, using Theorem 1 we can solve for the trade elasticity. Using Theorem 2 to get the welfare elasticity, we can then solve analytically for gains from trade by simple integration.

### 3 Comparison to CES Gains from Trade

The procedure described in section 2 is useful because it provides a means of measuring gains from trade in a simple way even for complex models. However, this is only interesting if it provides new insights into the determinants of gains from trade not already present in the CES trade models. Therefore, an important question to ask is how the gains from trade implied by the procedure above differ from those models.
To answer this question, we derive sufficient conditions on the trade elasticity, $\varepsilon_T$, for the gains from trade implied by the Arkolakis, et al (2012) formula to be an upper or lower bound on the gains implied by the procedure described in section 2. To do this, first consider the gains from trade implied by a given trade elasticity, $\varepsilon_T$. Rearranging the definition of the welfare elasticity and applying Theorem 2 implies:

$$\int_{\tau_{01}}^{\tau_{AUT}} d\log(W) = \int_{\tau_{01}}^{\tau_{AUT}} \frac{1}{\varepsilon_T(\tau)} d\log(\lambda_{00})$$

Recall that the trade elasticity is only defined when $\lambda_{00}$ and $\lambda_{01}$ are both positive. Therefore here the upper limit of integration is the supremum of trade costs where that is true, which may or may not be finite\(^{11}\).

If the trade elasticity were a constant, the formula from Arkolakis, et al (2012) is immediate. However, here it is a function. Therefore we proceed by integrating by parts:

$$\int_{\tau_{01}}^{\tau_{AUT}} d\log(W) = \log \left( \frac{W^{AUT}}{W^{TRADE}} \right) = -\frac{1}{\varepsilon_T(\tau_{01})} \log(\lambda_{00}) + \int_{\tau_{01}}^{\tau_{AUT}} \frac{\log(\lambda_{00})}{\varepsilon_T(\tau)^2} \frac{\partial \varepsilon_T}{\partial \log(\tau)} d\log(\tau)$$

or,

$$\log \left( \frac{W^{TRADE}}{W^{AUT}} \right) = \frac{1}{\varepsilon_T(\tau_{01})} \log(\lambda_{00}) + \int_{\tau_{01}}^{\tau_{AUT}} -\frac{\log(\lambda_{00})}{\varepsilon_T(\tau)^2} \frac{\partial \varepsilon_T}{\partial \log(\tau)} d\log(\tau)$$

Again, the first term is exactly the formula from Arkolakis, et al (2012), while the second term depends crucially on how the trade elasticity changes. Obviously, if the trade elasticity is a constant the second term is zero. We can compare the gains from trade implied by this formula directly to that implied by the ACR formula as follows:

**Theorem 3** Let the “gains from trade at $\tau_{01}$” be defined as the increase in real income associated with decreasing trade costs from $\tau^{AUT}$ to $\tau_{01}$. Consider two versions of the model described in Section 2 with the same value of $\lambda_{00}$: one with a variable trade elasticity $\varepsilon_T(\tau)$ where $\varepsilon_T$ is bounded away from zero, and one with a constant trade elasticity equal to $\varepsilon = \varepsilon_T(\tau_{01})$. Then the model with a variable trade elasticity has higher (lower) gains from trade at $\tau_{01}$ than the model with a constant elasticity if $\varepsilon_T$ is a monotone increasing (decreasing) function of $\tau$.

**Proof.** Note that $\forall \tau, \lambda_{00} \in (0, 1) \implies \log(\lambda_{00}) < 0$, and clearly $\varepsilon_T(\tau)^2 > 0$. Therefore, for all $\tau$,

$$\text{sign} \left( -\frac{\log(\lambda_{00})}{\varepsilon_T(\tau)^2} \frac{\partial \varepsilon_T}{\partial \log(\tau)} \right) = \text{sign} \left( \frac{\partial \varepsilon_T}{\partial \log(\tau)} \right)$$

\(^{11}\)For example, if $c_0$ is produced in country 0 and $c_1$ is produced in country 1, then if $\bar{c} > 0$ and $U = c_0^{1-1/\sigma} + (\bar{c} + c_1)^{1-1/\sigma}$, $\tau^{AUT}$ is finite, whereas if $\bar{c} = 0$ it is infinite.
Suppose that $\varepsilon_T$ is increasing in $\tau$. Then the term within the integral is positive for all $\tau$, hence:

$$0 < \int_{\tau_0}^{\tau_{AUT}} - \frac{\log(\lambda_{00})}{\varepsilon_T(\tau)^2} \frac{\partial \varepsilon_T}{\partial \log(\tau)} d\log(\tau) = \log\left(\frac{W^{TRADE}}{W^{AUT}}\right) - \frac{1}{\varepsilon_T(\tau_0)} \log(\lambda_{00})$$

The first term on the right hand side is the gains from trade in the model with a variable trade elasticity, and the second term is the gains from trade in the constant elasticity model. Therefore, the gains from trade are higher in the variable elasticity model than in the constant elasticity model. The same argument applies for the case with a decreasing trade elasticity.

The usefulness of this sufficient condition is apparent as an easy way to check how non-homotheticities affect gains from trade. In Example 3 above, we can see immediately that the trade elasticity is decreasing in $\lambda_{00}$, and since $\lambda_{00}$ is increasing in trade costs, also decreasing in $\tau$. Applying Theorem 3 above, we can then immediately say that the gains from trade are therefore smaller in that model than they would be in a model with a constant trade elasticity. This conclusion can be reached quickly without having to solve the full model. Likewise, in Example 2, we can see that whether or not the ACR equation is an upper or lower bound depends on if the income elasticity of domestic or foreign goods is larger. We could intrepret this to say that countries that produce highly income elastic goods have gains from trade higher than implied by the ACR equation, while countries that produce low income elasticity goods have lower gains from trade. This points to a role for country heterogeneity in determining the gains from trade, which is absent in CES models.

This result demonstrates that the ACR result is useful even in this environment with general preferences as a bound on the possible gains from trade implied by classes of non-homothetic models.

4 Extension to Model with N Trading Partners

We now analyze the effects on country 0 of changing trade costs with its N other trading partners. We assume that a parameter $\tau$ governs trade costs between country 0 and all its trading partners.\textsuperscript{12} Then we have two theorems analogous to Theorems 1 and 2 in this

\textsuperscript{12}That is, $\tau_{0j} = \hat{\tau}_{0j} \tau$ and we will be considering changes in the common component $\tau$. 

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environment. First, we define the trade elasticity as:

\[ \varepsilon_T(\tau) = \frac{\partial \log \left( \frac{1 - \lambda_{00}}{\lambda_{00}} \right)}{\partial \log(\tau)} \sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \]

Notice that if all N trading partners are identical, then this trade elasticity is the same as that given in Section 2.

**Theorem 4** Whenever \( \lambda_{00} \in (0, 1) \),

\[ \varepsilon_T(\tau) = \frac{1}{L_0 \lambda_{00}} \int_{i \in \Omega_0} p_0(i) \frac{w'_n(i)}{w_n(i)} di \left( 1 + \frac{\sum_{n=1}^{N} \lambda_{0n} L_0 (1 + \frac{\partial \log(w_n)}{\partial \log(\tau)}) \sum_{m=0}^{N} T_{0m} P_m(i) \frac{u'_m(i)}{u_m(i)} di}{\sum_{n=1}^{N} \lambda_{0n} L_0 (1 + \frac{\partial \log(w_n)}{\partial \log(\tau)})} \right) \]

The proof is left to the appendix. The complication introduced with multiple trading partners is that wages may have differing responses in different countries, so that the effective change in the price of goods may differ across countries. This general equilibrium effect would require one to know how wages change as trade costs change in order to compute the trade elasticity.\(^{13}\) With this definition of the trade elasticity, then the analogue of Theorem 2 is essentially unchanged from before. The welfare elasticity is defined as:

\[ \varepsilon_W(\tau) = \frac{\partial \log(W)}{\partial \log(\tau)} - \frac{\partial \log(I)}{\partial \log(\tau)} - \sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \]

**Theorem 5** \( \forall \tau \),

\[ \varepsilon_W(\tau) = \frac{1}{\varepsilon_T(\tau)} \]

**Proof.** Applying the envelope theorem to the dual of the household’s problem,

\[ \frac{\partial \log(W)}{\partial \log(\tau)} = -\frac{\partial \log(I)}{\partial \log(\tau)} - \sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \]

and

\[ \frac{\partial (1 - \lambda_{00})}{\partial \log(\tau)} = -\frac{\partial \lambda_{00}}{\partial \log(\tau)} \Rightarrow \frac{\partial \log \left( \frac{1 - \lambda_{00}}{\lambda_{00}} \right)}{\partial \log(\tau)} = - \left( 1 + \frac{\lambda_{00}}{1 - \lambda_{00}} \right) \frac{\partial \log \left( \lambda_{00} \right)}{\partial \log(\tau)} = - \frac{1}{1 - \lambda_{00}} \frac{\partial \log \left( \lambda_{00} \right)}{\partial \log(\tau)} \]

\(^{13}\) Notice that if preferences are CES as in Example 1 above, we see that the terms for the changes in wages cancel out. This illustrates how CES models avoid this issue entirely.
Then

\[ \varepsilon_W(\tau) = -\frac{\partial \log(I)}{\partial \log(\lambda_0)} = \frac{1}{1-\lambda_{00}} \sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) = \sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) = \frac{1}{\varepsilon_I(\tau)} \]

The usefulness of this result is limited by the presence of the general equilibrium terms. In principle, one could ignore these terms and only measure the partial equilibrium gains from trade. Alternatively, one could use the labor market clearing conditions to solve for the general equilibrium terms. In the appendix, we show how to solve for those terms in general.

5 Non-Seperable Preferences

For simplicity in previous sections we assumed that the utility function was additively seperable. In this section we dispense with that assumption and show that our results are unchanged. That is, the household’s problem is now:

\[
\max U \left( \{ \{ c_n(i) \} \}_{i \in \Omega_n} \right)_{n=0}^{N} \\
\text{s.t.} : \sum_{n=0}^{N} \tau_{0n} p_n(i) c_n(i) \leq L_0
\]

This specification allows for complementarity or substitutability of goods both within and between countries.

Let \( H \) be the Hessian matrix of \( U \). Because \( U \) is strictly concave and twice continuously differentiable, \( H \) is negative definite and invertible.\(^{14}\) The \( n(i) \) row of \( H \) contains all the second order partial derivatives of good \( i \) in country \( n \). It is useful to define the following terms:

\[
A_n(j) = \sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \int_{\Omega_n} H_{(n(i),m(j))}^{-1} \frac{\partial U}{\partial c_n(i)} I_n(i) di \\
B_n(j) = \sum_{n=0}^{N} \int_{\Omega_n} H_{(n(i),m(j))}^{-1} \frac{\partial U}{\partial c_n(i)} I_n(i) di
\]

\(^{14}\)If \( H \) is an infinite matrix, additional regularly assumptions may be necessary. In that case, by inverse we mean the left-hand reciprocal of \( H \), as described in Cooke (2014). A sufficient condition that we use in our example is that the set of goods can be partitioned into disjoint subsets \( \{ S_j \} \) such that \( i \in S_n, j \in S_m, m \neq n \implies U_{ij} = 0. \)
Then the general case of Theorem 1 is as follows:

**Theorem 6** Whenever \( \lambda_{00} \in (0, 1) \),

\[
\varepsilon_T = -\frac{1}{\lambda_{00}} \int_{\Omega_0} p_0(j) A_0(j) \, \frac{1}{L_0} \sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) + \sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j) B_m(j) \left[ 1 + \frac{\sum_{m=0}^{N} \tau_{0m} p_m(j) A_m(j)}{L_0 \sum_{n=1}^{N} \lambda_{0n}} \right]
\]

The proof of this theorem is available in the appendix. Note that this generalizes the previous results. If \( U \) was additively separable, then \( H \) is a diagonal matrix, and so is \( H^{-1} \). This implies that:

\[
\forall j \forall n : B_n(j) = \frac{u_n'(j)}{u_n''(j)} I_n(j) \\
\forall j : A_0(j) = 0 \\
\forall j \forall n \geq 1 : A_n(j) = \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \frac{u_n'(j)}{u_n''(j)} I_n(j)
\]

Substituting this into the formula in Theorem 6 returns the result from Theorem 4.

Due to the greater flexibility of this preference structure, solving for the trade elasticity is necessarily more complicated. Therefore, we provide an example to show how this result can be used to generate interesting results.

### 5.1 Example: Heterogeneous Elasticities of Substitution

Suppose there are two countries, and for each good produced in country 0, there is a corresponding good in country 1 that is its imperfect substitute. Furthermore, we allow for the elasticity of substitution to vary for each good, so that some goods are more substitutable across countries than others (e.g., iron ore is more substitutable than wine). Preferences are given by:

\[
U = \int_{\Omega} \alpha(i) \left( c_0(i)^{\gamma(i)-1} + c_1(i)^{\gamma(i)-1} \right)^{\gamma(i)-1-\beta} \frac{\gamma(i)-1}{\beta} \, di
\]

Here we require that \( \forall i, \gamma(i) > 1 \) and \( \beta > 1 \). The elasticity of substitution between any good and its corresponding good from the other country is \( -\gamma(i) \). Its elasticity of substitution with any other good is determined by \( \beta \).

This preference structure is useful because it admits a block diagonal Hessian. That is, consider the following matrix:

\[
h(i) = \begin{bmatrix}
\frac{\partial^2 U}{\partial c_0(i)^2} & \frac{\partial^2 U}{\partial c_0(i) \partial c_1(i)} \\
\frac{\partial^2 U}{\partial c_1(i) \partial c_0(i)} & \frac{\partial^2 U}{\partial c_1(i)^2}
\end{bmatrix}
\]
Then the Hessian of $U$ is composed of the block matrices $h(i)$ along the diagonal, and zeros everywhere else. Then the inverse of $H$ is also block diagonal where each block is the inverse of $h(i)$. Because $h(i)$ is 2x2, it is very easy to compute its inverse. We can write it easily with some notation for product-level expenditure:

$$X_{0n}(j) = \tau_{0n} p_n(j) c_n(j)$$
$$X_{0n} = \int_{\Omega} \tau_{0n} p_n(j) c_n(j) \, dj$$

We get:

$$h^{-1}(i) = \begin{bmatrix}
    -\frac{c_0(j) \frac{\gamma(j) X_{00}(j) + \beta X_{01}(j)}{u_0(j) X_{00}(j) + X_{01}(j)}}{u_0(j) X_{00}(j) + X_{01}(j)} & -\frac{c_1(j) \frac{(\gamma(j) - \beta) X_{00}(j)}{u_0(j) X_{00}(j) + X_{01}(j)}}{u_0(j) X_{00}(j) + X_{01}(j)} \\
    \frac{c_1(j) \frac{\gamma(j) - \beta) X_{00}(j)}{u_0(j) X_{00}(j) + X_{01}(j)}}{u_0(j) X_{00}(j) + X_{01}(j)} & -\frac{c_1(j) \frac{\gamma(j) X_{01}(j) + \beta X_{00}(j)}{u_1(j) X_{00}(j) + X_{01}(j)}}{u_1(j) X_{00}(j) + X_{01}(j)}
\end{bmatrix}$$

Applying the definitions of $A$ and $B$ above:

$$A_0(j) = \left(1 + \frac{\partial \log(w_1)}{\partial \log(\tau_{01})}\right) c_0(j) \frac{(\gamma(j) - \beta) X_{01}(j)}{X_{00}(j) + X_{01}(j)}$$
$$A_1(j) = -\left(1 + \frac{\partial \log(w_1)}{\partial \log(\tau_{01})}\right) c_1(j) \frac{\gamma(j) X_{00}(j) + \beta X_{01}(j)}{X_{00}(j) + X_{01}(j)}$$
$$B_0(j) = -\beta c_0(j)$$
$$B_1(j) = -\beta c_1(j)$$

Then applying the formula for the trade elasticity implies:

$$\varepsilon_T = 1 - \beta - \int_{\Omega} (\gamma(j) - \beta) \frac{X_{00}(j) X_{01}(j)}{X_{00}(j) X_{01}(j) + X_{00}(j) + X_{01}(j)} \, dj$$

This result is interesting because it incorporates information about the patterns of trade and consumption into the trade elasticity. The trade elasticity depends on each good’s domestic expenditure relative to total domestic spending, the imports of each good relative to total imports, and the total expenditure on that good relative to the total budget.

6 Conclusion

Here we conclude.
Appendix

7.1 Proof of Theorems 1 and 4

Theorem 4 generalizes Theorem 1, so we first prove Theorem 4 and show that it implies Theorem 1. We begin by noting that the wage in country 0 is numeraire and rearranging the definition of \( \lambda_{00} \):

\[
L_0 \lambda_{00} = \int_{\Omega_0} p_0(i) c_0(i) = \int_{\Omega_0} \frac{c_0(i)}{a_0(i)}
\]

Let the Lagrange multiplier on the household’s non-negativity constraint be \( \eta_n(i) \). We write the first order condition of the country 0 household’s problem as:

\[
u_n'(c_n(i), i) = \mu \tau_{0n} p_n(i) - \eta_n(i) \implies c_n(i) = (u_n'(i))^{-1} (\mu \tau_{0n} p_n(i) - \eta_n(i))
\]

where \( \mu \) is the Lagrange multiplier on the household budget constraint. Because each \( u \) function is strictly increasing, strictly concave and twice differentiable, the inverse of the first derivative exists and is itself differentiable. Notice that \( \eta_n(i) \) is equal to:

\[
\eta_n(i) = \begin{cases} 
0 & \text{if } c_n(i) > 0 \\
\mu \tau_{0n} p_n(i) - u_n'(0, i) & \text{if } c_n(i) = 0
\end{cases}
\]

It is useful to note that it’s derivative is therefore:

\[
\forall n \geq 1 : \frac{\partial \eta_n(i)}{\partial \log(\tau_{0n})} = \begin{cases} 
0 & \text{if } c_n(i) > 0 \\
\mu \tau_{0n} p_n(i) \left[ 1 + \frac{\partial \log(w_n)}{\partial \log(\tau_{0n})} + \frac{\partial \log(\mu)}{\partial \log(\tau_{0n})} \right] & \text{if } c_n(i) = 0
\end{cases}
\]

Then:

\[
L_0 \lambda_{00} = \int_{\Omega_0} \frac{(u_n'(i))^{-1} (\mu p_0(i) - \eta_n(i))}{a_0(i)}
\]

\[
\frac{\partial \log(\lambda_{00})}{\partial \log(\tau_{0n})} = \frac{\frac{\partial \log(\mu)}{\partial \log(\tau_{0n})} \int_{\Omega_0} p_0(i) \frac{\mu p_0(i)}{u_n'(i)} - (1 - L_0(i)) p_0(i) \frac{\mu p_0(i)}{u_n'(i)} I_0(i)}{L_0 \lambda_{00}} = \frac{\frac{\partial \log(\mu)}{\partial \log(\tau_{0n})} \int_{\Omega_0} p_0(i) \frac{w_n'(i)}{w_n'(i)} I_0(i)}{L_0 \lambda_{00}}
\]
Here we use the fact that, since \( u'_n(k) \) is differentiable and strictly decreasing (hence, always strictly negative), then:

\[
\frac{\partial (u'_n(i))}{\partial \log(\tau_{0n})}^{-1} (\mu \tau_{0n} P_n(i) - \eta_n(i)) = \frac{\mu \tau_{0n} P_n(i) \left[ 1 + \frac{\partial \log(w_n)}{\partial \log(\tau_{0n})} + \frac{\partial \log(\mu)}{\partial \log(\tau_{0n})} \right] - \frac{\partial \eta_n(i)}{\partial \log(\tau_{0n})} u''_n(i)}{u''_n(i - 1) (\mu \tau_{0n} P_n(i) - \eta_n(i)), i)}
\]

\[
= \left\{ \begin{array}{ll}
0 & \text{if } c_n(i) = 0 \\
1 + \frac{\partial \log(w_n)}{\partial \log(\tau_{0n})} + \frac{\partial \log(\mu)}{\partial \log(\tau_{0n})} \frac{u'_n(c_n(i), i)}{u''_n(c_n(i), i)} & \text{if } c_n(i) > 0
\end{array} \right.
\]

Next we have to solve for the change in the Lagrange multiplier on the budget constraint. The budget constraint of the household in country 0 is:

\[
L_0 = \sum_{n=0}^{1} \int_{\Omega_n} \tau_{0n} P_n(i) c_n(i) = \sum_{n=0}^{1} \int_{\Omega_n} \tau_{0n} P_n(i) \left( u'_n(i) \right)^{-1} (\mu \tau_{0n} P_n(i))
\]

Then:

\[
\frac{\partial \log(\mu)}{\partial \log(\tau)} = -\frac{\sum_{n=1}^{N} \int_{\Omega_n} I_n(i) \left( \tau_{0n} P_n(i) c_n(i) + \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)} \right) \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right)}{\sum_{n=0}^{N} \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}
\]

\[
= -\frac{\sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \lambda_0 + \sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}{\sum_{n=0}^{N} \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}
\]

Combining this with the equation above and substituting into the definition of \( \varepsilon_T \) yields:

\[
\varepsilon_T = \frac{-\frac{\partial \log(\lambda_{00})}{\partial \log(\tau)} \sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right)}{\sum_{n=1}^{N} \lambda_{0n} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right)} = \frac{\int_{\Omega_0} I_0(i) \frac{u'_0(i)}{u''_0(i)} I_0(i) \left( 1 + \frac{\sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}{\sum_{n=0}^{N} \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}} \right) \lambda_{00}}{\sum_{n=0}^{N} \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}
\]

This is the term that appears in Theorem 4. To get the term in Theorem 1, assume that \( N = 1 \):

\[
\varepsilon_T = \frac{\int_{\Omega_0} I_0(i) \frac{u'_0(i)}{u''_0(i)} I_0(i) \left( 1 + \frac{\sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}{\sum_{n=0}^{N} \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}} \right) \lambda_{00}}{\sum_{n=0}^{N} \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}
\]

\[
= \frac{\int_{\Omega_0} I_0(i) \frac{u'_0(i)}{u''_0(i)} I_0(i) \left( 1 + \frac{1}{\lambda_{01}} \int_{\Omega_1} I_1(i) \tau_{01} P_1(i) \frac{u'_1(i)}{u''_1(i)} \right)}{\sum_{n=0}^{N} \int_{\Omega_n} I_n(i) \tau_{0n} P_n(i) \frac{u'_n(i)}{u''_n(i)}}
\]

which is equivalent to the statement of Theorem 1.
7.2 Solving for Changes in Wages with N Trading Partners

With N trading partners, there are terms that appear in the trade elasticity based on the changes in wages in all countries besides country 0 (where the change is zero as $w_0$ is numeraire). To solve for these terms, we use information from the N labor market clearing conditions of the trading partners. Differentiating the labor market clearing conditions with respect to changes in log-trade costs implies:

$$\forall n \geq 1 : 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} = \sum_{m=1}^{N} \tau_{mn} p_n(i) \frac{u_{ni}^{m'}}{u_{ni}^{m}} - \int_{\Omega_n} \tau_{0n} p_n(i) \left( c_n(i) + \frac{u_{ni}^{0'}}{u_{ni}^{0}} \right) -$$

$$\sum_{m=0}^{N} \frac{\partial \log(\mu^m)}{\partial \log(\tau)} \int_{\Omega_n} \tau_{mn} p_n(i) \frac{u_{ni}^{m'}}{u_{ni}^{m}}$$

where we use m superscripts to denote the consumption and Lagrange multipliers of other countries. Differentiating each of the N budget constraints of the trading partners implies:

$$\forall m \geq 1 : \frac{\partial \log(\mu^m)}{\partial \log(\tau)} = \left( 1 + \frac{\partial \log(w_m)}{\partial \log(\tau)} \right) w_m L_m + \sum_{n=1}^{N} \int_{\Omega_n} \tau_{mn} p_n(i) \frac{u_{ni}^{m'}}{u_{ni}^{m}} - \int_{\Omega_0} \tau_{0m} p_0(i) \frac{u_{0i}^{m'}}{u_{0i}^{m}}$$

$$- \sum_{n=0}^{N} \int_{\Omega_n} \tau_{mn} p_n(i) \frac{u_{ni}^{m'}}{u_{ni}^{m}}$$

and differentiating country 0’s budget constraint yields:

$$\frac{\partial \log(\mu^0)}{\partial \log(\tau)} = - \sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \int_{\Omega_n} \tau_{0n} p_n(i) \left( c_n(i) + \frac{u_{ni}^{0'}}{u_{ni}^{0}} \right)$$

$$\sum_{n=0}^{N} \int_{\Omega_n} \tau_{0n} p_n(i) \frac{u_{0i}^{m'}}{u_{0i}^{m}}$$

Notice that we now have 2N+1 equations and 2N+1 unknowns: the changes in Lagrange multipliers, and the changes in wages. Moreover, they are all linear equations. Therefore, we proceed by substituting all the terms derived from the budget constraints into the terms derived from the labor market clearing conditions, which leaves N linear equations and N
unknowns. To write those terms we define the following terms:

\[
G_n = 1 - \frac{\sum_{k=1}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{0n}(i)} \right) - \sum_{k=1}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{kn}(i)} \right) - \sum_{i=0}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{kn}(i)} \right)}{\sum_{i=0}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{kn}(i)} \right)}
\]

\[
H_{kn} = \frac{w_k \sum_{j=0}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{0n}(i)} \right) - \sum_{i=0}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{kn}(i)} \right) - \sum_{i=0}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{kn}(i)} \right) \frac{\partial \ln}{\partial \tau_{kn}(i)}}{\sum_{i=0}^{N} \int_{\Omega} \tau_{kn} p_n(i) \left( c_n^k(i) + \frac{\partial \ln}{\partial \tau_{kn}(i)} \right)}
\]

\[
W_n = 1 + \frac{\partial \ln(w_n)}{\partial \ln(\tau)}
\]

Then let \( H \) be a matrix where the \((k, n)\) element is \( H_{kn} \), \( G \) be a column vector where the \( n \) column is \( G_n \), and \( W \) be a column vector where the \( n \) column is \( W_n \). Then:

\[
W = G + WH \implies W = (I - H)^{-1} G
\]

Given \( W \), these values can be substituted into the equation in Theorem 4 to give the trade elasticity with multiple trading partners.

### 7.3 Proof of Theorem 6

If preferences are non-separable, then the household’s first order condition is:

\[
\forall n \geq 1: \frac{\partial U}{\partial c_n(i)} = \mu \tau_{0n} p_n(i) - \eta_n(i)
\]

\[
n = 0: \frac{\partial U}{\partial c_0(i)} = \mu p_0(i) - \eta_0(i)
\]

Each term on the left hand side can depend on any number of the other goods available to the household. Next we differentiate each such first order condition with respect to log-trade costs:

\[
c_n(i) > 0 \implies \frac{\partial}{\partial \ln(\tau)} \left[ \frac{\partial U}{\partial c_n(i)} \right] = \sum_{m=0}^{N} \int_{\Omega_m} I_m(j) \frac{\partial c_m(j)}{\partial \ln(\tau)} \frac{\partial^2 U}{\partial \ln(\tau) \partial c_n(i) \partial c_m(j)} dj = \mu \tau_{0n} p_n(i) \left( 1 + \frac{\partial \ln(w_n)}{\partial \ln(\tau)} + \frac{\partial \ln(\mu)}{\partial \ln(\tau)} \right)
\]
Let $\hat{H}$ be the Hessian of $U$. Let $H$ be a matrix constructed by deleting every row and column whose good has zero consumption. In Section 6, there is some discussion of sufficient conditions for $H$ and its inverse to be well-defined, which we assume here. Let $n(k)$ be the index of good $k$ in country $n$. Then:

$$\frac{\partial c_m(j)}{\partial \log(\tau)} = \frac{\partial \log(\mu)}{\partial \log(\tau)} \sum_{n=0}^{N} \int_{\Omega_n} H_{n(i),m(j)}^{-1} \frac{\partial U}{\partial c_n(i)} I_n(i) di + \sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \int_{\Omega_n} H_{n(i),m(j)}^{-1} \frac{\partial U}{\partial c_n(i)} I_n(i) di$$

Differentiating the budget constraint of the household yields:

$$0 = \int_{\Omega_0} p_0(j) \frac{\partial c_0(j)}{\partial \log(\tau)} dj + \sum_{n=1}^{N} \int_{\Omega_n} \tau p_n(j) c_n(j) \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \int_{\Omega_n} H_{n(i),m(j)}^{-1} \frac{\partial U}{\partial c_n(i)} I_n(i) di + \sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \lambda_0$$

which implies:

$$\frac{\partial \log(\mu)}{\partial \log(\tau)} = \frac{\sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \left[ \lambda_0 + \sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j) \sum_{n=0}^{N} \int_{\Omega_n} H_{n(i),m(j)}^{-1} \frac{\partial U}{\partial c_n(i)} I_n(i) di dj \right] - \sum_{n=0}^{N} \int_{\Omega_n} \tau_{0m} p_m(j) \sum_{n=0}^{N} \int_{\Omega_n} H_{n(i),m(j)}^{-1} \frac{\partial U}{\partial c_n(i)} I_n(i) di dj}{\sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \lambda_0}$$

Recall the definition of $\lambda_{00}$:

$$L_0 \lambda_{00} = \int_{\Omega_0} p_0(j) c_0(j) \Rightarrow \frac{\partial \log(\lambda_{00})}{\partial \log(\tau)} = \frac{\int_{\Omega_0} p_0(j) \frac{\partial c_0(j)}{\partial \log(\tau)} \lambda_{00}}{\int_{\Omega_0} p_0(j) \frac{\partial c_0(j)}{\partial \log(\tau)} \lambda_{00}}$$

Using the definitions of the $A_n(i)$ and $B_n(i)$ terms from Section 5, we can simplify these terms as:

$$\frac{\partial \log(\mu)}{\partial \log(\tau)} = -\frac{\sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \lambda_0 + \sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j) A_m(j)}{\sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j) B_m(j)}$$

$$\frac{\partial c_0(j)}{\partial \log(\tau)} = A_0(j) - B_0(j) \frac{\sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \lambda_0 + \sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j) A_m(j)}{\sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j) B_m(j)}$$
This implies:

$$\frac{\partial \log(\lambda_{00})}{\partial \log(\tau)} = \frac{\int_{\Omega_0} p_0(j)A_0(j)}{L_0\lambda_{00}} - \frac{\int_{\Omega_0} p_0(j)B_0(j)}{L_0\lambda_{00}} \sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \lambda_n + \sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j)A_m(j) \sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j)B_m(j)$$

Applying the definition of the trade elasticity yields:

$$\varepsilon_T = -\frac{\int_{\Omega_0} p_0(j)A_0(j)}{\sum_{n=1}^{N} \lambda_n \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right)} + \frac{\int_{\Omega_0} p_0(j)B_0(j)}{\sum_{n=1}^{N} \int_{\Omega_m} \tau_{0m} p_m(j)B_m(j)} \left[ 1 + \frac{\sum_{m=0}^{N} \int_{\Omega_m} \tau_{0m} p_m(j)A_m(j)}{L_0 \sum_{n=1}^{N} \left( 1 + \frac{\partial \log(w_n)}{\partial \log(\tau)} \right) \lambda_n} \right]$$

This completes the proof.

8 References

References


