Cash-in-the-Market Asset Pricing in a Monetary Model with Over-the-Counter Asset Markets

Fabrizio Mattesini
University of Rome, “Tor Vergata”

Ed Nosal
Federal Reserve Bank of Chicago

August 2011
This version: June 2014

Abstract

We study how asset prices are affected by the amount of “liquidity” or cash that is available in asset markets. We find that higher levels of liquidity lead to higher asset price and lower bid-ask spreads. An increase in inflation increases asset returns and decreases asset prices. The amount of liquidity available in asset markets depends on the fraction of agents who do not have immediate consumption needs, which itself is a random variable. This implies that asset prices will fluctuate over time even though asset fundamentals are unchanged.

1 Introduction

Assets that are subject to search and bargaining frictions trade in over-the-counter (OTC) markets. In many of these markets, trade is intermediated by middlemen, such as brokers or dealers. Dealers negotiate bid and ask prices and match sellers—who have assets and want to acquire money—with buyers—who have money and want to acquire assets—respectively. Notice an important component for both sides of the dealer’s trade is money: assets trade for money and money trades for assets. In frictionless Arrow-Debreu markets, where agents trade against their budget constraints and money serves no purpose (and does not exist), asset prices depend only the value of their future dividends. In frictional markets, where agents trade with agents and money is needed to support trade, one might conjecture that, in addition to future dividends, asset prices also depend on the amount of money or liquidity that is available in the OTC market. This paper examines this conjecture. Specifically, we study how aggregate liquidity in OTC asset markets affects asset prices.
At its heart, this paper is about the interaction of OTC markets and money. In the analysis, the existence of OTC markets is justified along fundamental lines; by search and bargaining frictions. It would seem rather odd, then, to justify the existence of money along non-fundamental lines, e.g., by a cash-in-advance constraint or money in the utility function. Hence, similar to the treatment of OTC markets, money emerges in the analysis because of frictions: frictions related to commitment and record keeping. These are standard frictions in monetary economics, so our basic model appeals to a standard one in monetary economics, Lagos and Wright (2005). Money is essential and the framework is designed to address policy issues, such as the impact that a change in inflation has on economic activity. To this model, we add an OTC financial market. In the spirit of Duffie, Garleanu and Pederson (2005) and Lagos and Rocheteau (2009), the hallmark of the OTC asset market consists of: (1) a middleman—the dealer—who intermediates asset trade between buyers and sellers and (2) asset prices determined by bargaining. Since the dealer has bargaining power, bid and ask prices naturally arise. The sellers of assets are a fraction $\sigma$ of investors that have successfully searched for a dealer and have imminent consumption opportunities that require cash. The buyers are the fraction $1 - \sigma$ that have successfully searched for a dealer and don’t have these opportunities (and, therefore, don’t have an immediate need for money). In addition to bargaining powers of the various agents and OTC matching probabilities, we find that asset prices depend on the aggregate of amount of liquidity or money that is available the OTC market, where aggregate liquidity equals the money holdings of the $1 - \sigma$ investors that do not have imminent consumption opportunities and are matched with the dealer. Similar to Allen and Gale (2005, 2007), there is “cash-in-the-market” pricing for assets in the sense that higher levels of cash/liquidity are associated with higher asset prices. In contrast to Allen and Gale, however, our model has something that actuality resembles cash—fiat money—as opposed to a short term real asset that is simply labelled as ‘cash.’

Each period the fraction of agents that get a consumption opportunity, $\sigma$, is drawn from a probability distribution, i.e., the economic environment is characterized by aggregate risk. Hence, the amount of cash available in the OTC market varies over time. This means that asset prices will fluctuate over time even though asset fundamentals are unchanging. The aggregate amount of cash-in-the-market has implications for the bid-ask spread: Higher levels of liquidity in the OTC market are associated with lower bid-ask spreads. Hence, changes in liquidity in the OTC market brought about by changes in $\sigma$, the OTC matching probability or changes in inflation will affect asset prices, asset returns and bid-ask spreads. Our model
predicts a positive relationship between asset returns and bid-ask spreads, and this relationship is well documented in the literature, (e.g., Amihud and Mendelson (1986) and Amihud, Mendelson, Pedersen (2005)). Our model also predicts a positive relationship between inflation and asset returns or, equivalently, a negative relationship between inflation and asset prices. This prediction is consistent with the observation that periods of low inflation are usually associated with periods of high asset prices (Christiano, Ilut, Motto and Rostagno (2010)).

This paper contributes to the literature on asset pricing in OTC markets that starts with Duffie, Garleanu and Pederson 2005. Lagos and Rocheteau (2009) relax their indivisible asset assumption. But both papers assume that assets can be purchased by “transferable utility,” or that buyers purchase the asset by producing a numeraire good that enters the seller’s utility function in an additive and linear manner. This means that liquidity—the object that buys assets—can be interpreted as being potentially in infinite supply. This is unappealing. To limit the amount of liquidity, Geromichalos and Herrenbrueck (2012) and Lagos and Zhang (2013), like this paper, embed an OTC financial market in a monetary model. Geromichalos and Herrenbrueck (2012) have their buyers and sellers directly meet (with some probability) in the OTC financial market—there is no dealer—which implies there is no bid-ask spread. In Lagos and Zhang (2013), buyers and sellers can either meet one another or a dealer in the OTC financial market. Dealers have access to a competitive interdealer market where they can off load their positions. In both of these papers, like this one, asset prices deviate from their fundamental values. However, neither of these papers features a phenomenon that resembles cash-in-the-market pricing as an equilibrium outcome, which is a focus of this paper. As well, a novel prediction of our model is that asset prices and inflation are negatively correlated. In Geromichalos and Herrenbrueck (2012) and Lagos and Zhang (2013), the effect of inflation on OTC asset prices is ambiguous. We attribute this to the differences between the OTC bargaining environments in the three papers.

The paper is organized as follows. The next section presents the model environment. In Section 3, the bargaining model is presented and analyzed. Section 4 derives agents’ value functions and characterizes equilibrium. The relationship between liquidity and assets prices, returns and bid-ask spreads is explored in Section 5. Section 6 analyzes the effect that inflation has on assets prices, returns and bid-ask spreads. Section 7 concludes.
2 Environment

Time is discrete and continues forever. Each time period has 3 subperiods: a financial market, FM, followed by a decentralized goods market, DM, and finally by a competitive rebalancing market, CM. There is a unit measure of infinitely-lived agents called investors that participate in all subperiods/markets; a unit measure of infinitely-lived agents called decentralized market good producers, or simply producers, that participate in the DM and CM; and an infinitely-lived agent called a dealer that participates only in the FM. Financial services are produced in the FM; decentralized market goods are produced in the DM; and a numeraire consumption good is produced in the CM.

There are 2 assets, money and a real asset. The quantity of money, $M_t$, grows at constant gross rate $\mu$ so that $M_{t+1} = \mu M_t$. New money is injected—$\mu > 1$—or withdrawn—$\mu < 1$—by lump-sum transfers $T_t = M_t(\mu - 1)$ in the CM. Let $\phi_t$ represent the amount of the numeraire good that one unit of money buys in the CM of period $t$ and $z_t = \phi_t m_t$ is real cash balances (measured in terms of the date $t$ numeraire good). Each real asset provides a single dividend payout of $\delta$ numeraire goods and then “dies.”

Investors need money to purchase consumption goods in the DM. To this end, they accumulate money balances in the CM. At the beginning of the FM, investors learn if they get a consumption opportunity in the subsequent DM. Those who get an opportunity would like to hold more money balances and less real assets, and those who do not, would like to hold less money balances and more real assets. A dealer operates an over-the-counter market that allows investors to reallocate their money and real asset holdings. Investors who have consumption opportunities enter the DM and search for them. All investors and producers re-enter the CM to consume and rebalance their money holdings, and so on.

More formally, in period $t - 1$, investors exit the CM with $m_t$ units of money and ownership of $a_{t-1}$ units of the real asset. At the beginning of period $t$, investors enter the FM, and the following sequence of events occur:

- each unit of real assets pays a dividend of $\delta$, which the asset owner consumes;

- all investors receive ownership of $\bar{a}$ new assets, where each asset pays a single dividend payment $\delta$ at the beginning of period $t + 1$;

- investors learn if they get consumption opportunities in the subsequent DM. A fraction
\(\sigma \in [0, 1]\) of investors get a consumption opportunity and a fraction \(1 - \sigma\) do not.

- all investors contact the dealer;
- the dealer intermediates asset trade between investors who have consumption opportunities and those who don’t, where asset ownership is exchanged for money.

As in Lagos and Rocheteau (2009), we assume that investors can only buy and sell assets in the FM through a dealer.\(^1\) In the FM and DM we must distinguish between the measure \(\sigma\) of investors who get consumption opportunities (and want to sell assets) and the measure \(1 - \sigma\) of investors who do not (and want to buy assets). Call the former consumers and the latter liquidity providers. The dealer bargains with consumers and liquidity providers to determine the prices at which assets are bought and sold, as well as the quantities of assets that are bought and sold. The bargaining environment is described in Section 3. The dealer buys \(\tau^b_t\) assets from a consumer at a unit price of \(\phi^b_t\), where \(\phi^b_t\) is the amount of assets that the dealer purchases per unit of money.\(^2\) One can interpret \(\phi_t / \phi^b_t \equiv p_t^b\) as the real bid price—the amount of real balances the dealer needs to buy one unit of the asset. The dealer sells \(\tau^a_t\) assets to a liquidity provider at a unit price of \(\phi^a_t\), where \(\phi^a_t\) is the amount of assets that a liquidity provider receives per unit of money. One can interpret \(\phi_t / \phi^a_t \equiv p_t^a\) as the real ask price—the amount of real balances obtained by the dealer for one unit of the asset.

Consumers enter the DM of period \(t\) and are randomly matched with producers.\(^3\) The probability that a consumer is matched with a producer is \(\sigma_D\). Matched consumers and producers bargain over the amount of the DM consumption good, \(q_t\), the producer exchanges for the consumer’s real balances, \(\tau^m_t\). The cost of producing \(q_t\) units of the investment good is \(c(q_t)\). For simplicity we assume that \(c(q_t) = q_t\). The consumer values \(q_t\) as \(u(q_t)\), where \(u' > 0, u'' < 0, u'(0) = \infty\) and \(u'(\infty) = 0\). The DM consumption good is perishable. We assume the consumer makes a take-it-or-leave-it offer to the producer, which means the consumer can extract all of the match surplus, where the match surplus is given by \(u(q_t) - q_t\). Match surplus is maximized at \(q_t = q^*\), where \(q^*\) solves \(u'(q^*) = 1\).

---

\(^1\)Duft, Garleanu and Pedersen (2005) assume that entrepreneurs can also directly meet one another and trade. We exclude this possibility to simplify the analysis. The main results hold if we allow for direct FM meetings between entrepreneurs.

\(^2\)Both Lagos and Rocheteau (2009) and Duffie, Garleanu and Pedersen (2005) are non-monetary models, where non-investors purchase assets with with transferable utility.

\(^3\)Liquidity providers do not enter the DM since they do not have any consumption opportunities.
When investors enter the CM of period $t$ they do not know if they will get consumption opportunities in the DM of period $t + 1$. They do know, however, that in period $t + 1$ a fraction $\sigma$ of them will get consumption opportunities in the subsequent DM, where $\sigma$ is independently and identically distributed over time with probability distribution function $f(\sigma)$ and $\int_0^1 f(\sigma) = 1$. The $\sigma$ realization is public information in the FM. The ex post probability that any particular investor gets a consumption opportunity in the next DM is $\sigma$. Investors can produce and consume in the CM. Let $x_t$ represent the CM numeraire good and $\ell_t$ the amount labor used to produce it. One unit of labor produces one unit of the numeraire good. An investor’s utility of consuming $x_t$ units of the numeraire good is linear and equal to $x_t$; his disutility of labor is linear and equal to $\ell_t$. Hence, the investor’s preferences over the consumption and production of the numeraire good is $x_t - \ell_t$. The numeraire good is perishable.

The producer is only active in the DM and CM. In the CM the producer can consume but cannot produce. The utility associated with consuming $x_t$ units of the CM good is linear and equal to $x_t$. Hence, the producer’s preferences in period $t$ are described by $x_t - q_t$, where $q_t$ is the amount of the consumption good produced in the DM. Given the DM bargaining protocol—the consumer makes a take-it-or-leave-it offer to the producer—all producers, whether they are matched or unmatched, get utility equal to zero in each period $t$. (A matched producer receives money in the DM which he uses to buy the numeraire good in the CM.)

The dealer is only active in the FM. The dealer’s preferences are linear in dividends and, hence, his objective is to maximize real asset holdings in each period.

Real assets “reside” in the FM in the sense that investors do not carry the real assets into the DM and CM. All agents are anonymous in the DM and CM. In any subperiod, investors cannot commit to promises made in previous subperiods. In the FM there exists a “book entry system” that verifies agents’ identities and documents asset ownership—who owns how many assets. The book entry system can issue a physical claim regarding asset ownership to investors, but these claims can be costlessly counterfeited. This, coupled with an absence of record keeping for agents in the DM and CM subperiods and investors’ inability to commit, implies that only money will be used as a medium of exchange in those subperiods. Money cannot be counterfeited. This is why investors value and accumulate money balances in the CM, and why producers are willing to accept it in the DM. Since the book entry system can
only record asset ownership, money is also needed as a medium of exchange to buy assets in the FM.

3 Bargaining in the FM

The dealer acts as a middleman, matching consumers with liquidity providers. One can interpret the dealer as running a “matched book” in the sense that other than obtaining a payoff for his services from a real asset, he does not take a directional position in the assets he trades. Bargaining between the dealer, consumers and liquidity providers occurs in two distinct stages. In the first stage, the dealer and consumers are matched and bargain over the bid price and quantity, \((p^b, \tau^b)\); in the second stage, the dealer and liquidity providers are matched and bargain over the ask price and ask quantity, \((p^a, \tau^a)\). The dealer has a “family” in \([0, 1]\), where family members do all the bargaining and act to maximize the real asset holdings of the dealer. The family interpretation allows dealer family members to be matched one-on-one with the consumers in the first stage and the remaining family members to be matched one-on-one with the liquidity providers in the second stage. When it leads to no confusion, we will refer to a dealer family member as simply a dealer (since, after all, the family member represents the interests of the dealer).

The bargaining outcome in each match—\((p^b, \tau^b)\) in the first stage consumer-dealer match and \((p^a, \tau^a)\) in the second stage liquidity provider-dealer match—is determined by the (Kalai) proportional bargaining solution. In the first stage, a consumer receives a \(\theta\) share of the consumer-dealer match surplus and the dealer receives a \(1 - \theta\) share. Similarly, in the second stage, the liquidity provider receives \(\theta\) of the liquidity provider-dealer match surplus, while the dealer gets \(1 - \theta\).

The dealer or his family members cannot observe the money holdings of consumers or liquidity providers prior to being matched. Once a dealer family member is matched, he can observe the real balances of the agent in his match—either a consumer in the first stage or an liquidity provider in the second stage—but cannot observe real balances of consumers

---

4It cannot not, for example, document owners and issuers of IOU’s.

5Because the dealer is able to match consumers and liquidity provides, the dealer has no need for a competitive interdealer market to offload undesirable positions, as in Duffie, Garleanu and Pederson (2005), and Lagos and Rocheteau (2009).

6When it does not cause any confusion, we will suppress time subscripts on variables.
or liquidity providers in other matches. Similarly, a consumer or liquidity provider cannot observe the money holdings of other consumers or liquidity providers.

In the bargaining game, it will be helpful to distinguish between real balances held by a consumer, \( z \), and real balances held by a liquidity provider, \( \tilde{z} \). We assume that \( z < q^* \). If \( z \geq q^* \), then the bargaining problem is irrelevant: the consumer has sufficient real balances to extract the maximum surplus in the DM if he is matched and, as a result, has no (strict) incentive to sell his assets in the FM.\(^7\)

In a first-stage match, the consumer and family member dealer must form expectations regarding the amount of real balances that will be supplied to their match by liquidity providers (at the second stage). The dealer allocates the same (expected) level of real balances from liquidity providers to each consumer-family dealer match. Consumers and family member dealers expect that each liquidity provider has \( \tilde{z} \) real balances. Hence, the dealer allocates \( \tilde{z}(1-\sigma)/\sigma \) real balances to each match. Suppose that in a first stage match the negotiated bid price and bid quantity is \((p^b, \tau^b)\). Notice that the family member dealer cannot pay for the \( \tau^b \) assets in the first stage, as this is not feasible—the family member dealer does not have any real balances at this time. Instead, in the first stage the family member dealer buys the assets from the consumer on consignment, as in Rubinstein and Wolinsky (1986, 589-591). The “contract” \((p^b, \tau^b)\) is financed in the second stage by the family member dealers that bargain with liquidity providers. Specifically, in the second stage the family member dealers will, in equilibrium, obtain \( p^b \tau^b \) real balances from liquidity providers in exchange for assets. Upon receiving these real balances, the dealer transfers them to the consumer and thus fulfills the first stage contract \((p^b, \tau^b)\). We assume that the dealer can commit to pay back consumers and that the first stage bargain \((p^b, \tau^b)\) is not subject to renegotiation. If the dealer does not receive \( p^b \tau^b \) real balances from liquidity providers—because, for example, liquidity providers that fund the bid contract \((p^b, \tau^b)\) accumulated less real balances than was expected—then the bid contract \((p^b, \tau^b)\) specifies that the dealer buy as many of the consumers assets that he can at the unit price \( p^b \) and return those assets that cannot be financed.\(^8\)

In each of the \( \sigma \) first stage matches a bid price and bid quantity \((p^b, \tau^b)\) is negotiated. At the beginning of the second stage, before liquidity providers and family member dealers are matched, these contracts, and assets associated with them, are divided among the \( 1-\sigma \)

\(^7\)If the growth rate of money is greater than that given by the Friedman Rule, i.e., \( \mu > \beta \), then \( z < q^* \).

\(^8\)The dealer never has an incentive to “hold back” real balances for himself and return assets since he values the asset’s dividends and does not value real balances.
family members who will bargain with the liquidity providers. The contracts and assets are
distributed among the second stage family dealers so that each bid contract will be allocated
to \( \varepsilon (1 - \sigma)/\sigma \) real balances.\(^9\) The surplus to be split in the second stage per unit of a bid
contract, i.e., for contract \((p^b, 1)\), is \( \beta \delta - p^b \). Since the dealer must pay \( p^b \) for each asset that
he buys from the consumer and the value of the asset is \( \beta \delta \), the surplus per asset purchased
by the dealer is \( \beta \delta - p^b \). (Recall that an asset pays a dividend \( \delta \) at the beginning of the next
FM.) The equilibrium match surplus that a liquidity provider and dealer generate per unit
of a bid contract can be expressed as

\[
\theta \left( \beta \delta - p^b \right) = (\beta \delta - p^a) \tau_1^a
\]

and

\[
(1 - \theta) \left( \beta \delta - p^b \right) = \beta \delta \left( 1 - \frac{p^b}{p^a} \right),
\]

respectively, where \( \tau_1^a \) represents the amount of asset that the liquidity provider purchases
per unit of a bid contract. The left side of (1) is a liquidity provider’s share of the match
surplus and the right side expresses this surplus in terms of an ask price and ask quantity.
The left side of (2) is the dealer’s share of the match surplus. The dealer keeps \( 1 - \tau_1^a \) of the
assets per unit that a consumer sells, resulting in a payoff of \( \beta \delta (1 - \tau_1^a) \) to the dealer. Since
real balances received by consumers equals real balances given up by liquidity providers, i.e.,
\( p^b = p^a \tau_1^a \), the dealer’s payoff per asset purchased can also be expressed as the right side of (2).

The equilibrium ask price is obtained by rearranging (2),

\[
p^a = \frac{p^b \beta \delta}{\theta \beta \delta + (1 - \theta) p^b},
\]

and the equilibrium ask quantity per unit that the dealer buys is obtained by rearranging
(1) so that we get

\[
\tau_1^a = \frac{\theta \left( \beta \delta - p^b \right)}{\beta \delta - p^a},
\]

which, using (3), can be rewritten as

\[
\tau_1^a = \frac{\theta \beta \delta + (1 - \theta) p^b}{\beta \delta}.
\]

\(^9\)For example, if \( \sigma = 0.5 \), then each second stage family dealer is given one bid contract; if \( \sigma = 0.75 \), then
each second stage family dealer is given 3 bid contracts; and if \( \sigma = 0.25 \), each second stage family dealer is
given 1/3 of a bid contract.
Notice that the ask price, $p^a$, is invariant to the amount of assets that the consumer sells in the first stage bargaining. That is, if the member dealer brings in $\tau^b$ assets to the stage 2 bargain, then his surplus is given by the left (or right) side of (2) times $\tau^b$, which implies that the ask price is given by (3).

Now let’s move to the first stage bargain between the consumer and family member dealer. The amount of real balances that a consumer receives in the FM from selling assets, $p^b\tau^b$, equals $\min \{ \bar{z}, \min \{ p^b\bar{a}, q^* - z \} \}$, where $\bar{z} = \bar{z}(1 - \sigma)/\sigma$ is the total amount of liquidity providers’ real balances per consumer or the maximum amount of real balances that the dealer can transfer to the consumer in exchange for his assets. If

$$\min \{ \bar{z}, \min \{ p^b\bar{a}, q^* - z \} \} = q^* - z,$$

then we say that the consumer is *unconstrained* in the FM because he is able to purchase $q^*$ in the DM if he is matched. If the consumer is unconstrained, then he has sufficient assets to obtain real balances equal to $q^* - z$, i.e., $p^b\bar{a} \geq q^* - z$, and the dealer has sufficient real balances—provided by $(1 - \sigma)/\sigma$ liquidity providers—to purchase the assets, $\bar{z} \geq q^* - z$. If

$$\min \{ \bar{z}, \min \{ p^b\bar{a}, q^* - z \} \} = p^b\bar{a},$$

then we say the consumer is *asset constrained* in the FM because his asset holdings, along with his own real balance holdings, are insufficient to purchase output $q^*$ in the DM if he is matched. If the consumer is asset constrained, then he sells all of his assets in the FM, which implies that $p^b\bar{a} < \bar{z}$, but is unable to purchase $q^*$ in the DM if he is matched because $p^b\bar{a} + z < q^*$. Finally, if

$$\min \{ \bar{z}, \min \{ p^b\bar{a}, q^* - z \} \} = \bar{z},$$

then we say the consumer is *liquidity constrained* in the FM because he would like to sell more assets so that he is able to purchase more DM output $q < q^*$ if he is matched. If the consumer is liquidity constrained, then he sells assets and gets all of the real balances available from the dealer, $\bar{z} < \min \{ p^b\bar{a}, q^* - z \}$, but is unable to purchase $q^*$ in the DM if he is matched because $\bar{z} + z < q^*$. For convenience we will say that $\bar{a}$ is “large” if $p^b\bar{a} \geq q^* - z$ and that $\bar{a}$ is “small” if $p^b\bar{a} < q^* - z$. Note that if $\bar{a}$ is large, then a consumer can never be asset constrained and if $\bar{a}$ is small, then he can never be unconstrained.

The surplus the consumer receives from selling $\tau^b$ assets at price $p^b$, where $p^b\tau^b =$
\[ \min \{ \tilde{z}, \min \{ p^b \tilde{a}, q^* - z \} \},^{10} \]

\[ \sigma_D u \left( z + \min \{ \tilde{z}, \min \{ p^b \tilde{a}, q^* - z \} \} \right) + (1 - \sigma_D) \left( z + \min \{ \tilde{z}, \min \{ p^b \tilde{a}, q^* - z \} \} \right) - \beta \delta \tau^b \]

\[ - \left[ \sigma_D u (z) + (1 - \sigma_D) z \right]. \]

The second line is (minus) the expected DM payoff that the consumer receives if he does not sell any assets in the FM; the first line is the expected DM payoff if he sells assets in the FM minus the loss in the value of dividends associated with selling \( \tau^b \) in the FM. The consumer’s surplus is, by construction, equal to \( \theta \) times the match surplus. Recall that the surplus that the dealer obtains from this match is equal to \( p^b b \), which represents \( 1 - \theta \) times the match surplus, i.e., match surplus is equal to \( p^b b = (1 - \theta) \). Therefore, we have

\[ \sigma_D \left[ u \left( z + \min \{ \tilde{z}, \min \{ p^b \tilde{a}, q^* - z \} \} \right) - u(z) \right] + (1 - \sigma_D) \min \{ \tilde{z}, \min \{ p^b \tilde{a}, q^* - z \} \} - \beta \delta \tau^b \]

\[ = \frac{\theta}{1 - \theta} \left( \beta \delta - p^b \right) \tau^b. \quad (5) \]

Suppose first that \( \tilde{a} \) is large, i.e., \( p^b \tilde{a} \geq q^* - z \). In this case, (5) is one equation in one unknown, \( \tau^b \), and, as a result, the first-stage bargaining solution \( (p^b, \tau^b) \) is given by

\[ p^b = (\tau^b)^{-1} \min \{ q^* - z, \tilde{z} \} \]

and

\[ \tau^b = \begin{cases} 
\{ \theta (q^* - z) + (1 - \theta) \left[ \sigma_D \left[ u (q^*) - u (z) \right] \right] + (1 - \sigma_D) \left( p^b b \right) \}/\beta \delta & \text{if } q^* - z \leq \tilde{z} \\
\{ \theta \tilde{z} + (1 - \theta) \left[ \sigma_D \left[ u (z + \tilde{z}) - u (z) \right] \right] + (1 - \sigma_D) \tilde{z} \}/\beta \delta & \text{if } q^* - z > \tilde{z} 
\end{cases}. \quad (7) \]

Now suppose that \( \tilde{a} \) is small, i.e., \( p^b \tilde{a} < q^* - z \). If the consumer turns out to be liquidity constrained, \( p^b \tilde{a} > \tilde{z} \), then the stage 1 bargaining solution, \( (p^b, \tau^b) \), is \( p^b = \tilde{z} / \tau^b \) and \( \tau^b \) is given by the the lower branch of (7). If the consumer turns out to be asset constrained, i.e., \( \tilde{z} > p^b \tilde{a} \), then, clearly, \( \tau^b = \tilde{a} \). To determine the bid price, we must describe the surplus the consumer receives in the first stage bargain with the dealer, which is

\[ \sigma_D u \left( z + p^b \tilde{a} \right) + (1 - \sigma_D) \left( z + p^b \tilde{a} \right) - \beta \delta \tilde{a} \]

\[ - \left[ \sigma_D u (z) + (1 - \sigma_D) z \right]. \]

\(^{10}\)It should be clear that if \( p^b \tau^b < \min \{ \tilde{z}, \min \{ p^b \tilde{a}, q^* - z \} \} \), then agents are leaving surplus on the table and this is not consistent with equilibrium.
The equilibrium bid price, which we denote by $\tilde{p}^b$, solves
\begin{align}
\sigma_Du(z + \tilde{p}^ba) + (1 - \sigma_D)(z + \tilde{p}^ba) - \beta\delta \tilde{a} - [\sigma_Du(z) + (1 - \sigma_D)\tilde{z}] = \frac{\theta}{1 - \theta}(\beta\delta - \tilde{p}^b)a. \tag{8}
\end{align}

The left side of (8) is the consumer’s surplus and the right side is the consumer’s share of the match surplus. Therefore, the bargaining solution to the first stage bargaining problem when the consumer is asset constrained is given by $p^b = \tilde{p}^b$ and $\tau^b = \tilde{a}$, where $\tilde{p}^b$ is given (implicitly) by (8).

In summary, assuming that a consumer holds $z$ real balances, each liquidity provider holds $\tilde{z}$ real balances and the fraction of investors that are consumers in the FM is $\sigma$, the first-stage bargaining solution $(p^b(z, \tilde{z}, \sigma), \tau^b(z, \tilde{z}, \sigma))$ is given by
\begin{align}
p^b(z, \tilde{z}, \sigma) = \begin{cases} 
(q^* - z)/\tau^b & \text{if } p^b\tilde{a} \geq q^* - z \leq \tilde{z} \\
\tilde{z}/\tau^b & \text{if } \min\{p^b\tilde{a}, q^* - z\} > \tilde{z} \\
p^b(z, \sigma) & \text{if } p^b\tilde{a} < q^* - z \text{ and } p^b\tilde{a} < \tilde{z}
\end{cases}, \tag{9}
\end{align}
where $\tilde{p}^b(z, \sigma)$ solves (8) and
\begin{align}
\tau^b(z, \tilde{z}, \sigma) = \begin{cases} 
\left\{\theta(q^* - z) + (1 - \theta)[\sigma_D[u(q^*) - u(z)] + (1 - \sigma_D)(q^* - z)]\right\}/\beta\delta & \text{if } p^b\tilde{a} \geq q^* - z \leq \tilde{z} \\
\left\{\theta\tilde{z} + (1 - \theta)[\sigma_D[u(z + \tilde{z}) - u(z)] + (1 - \sigma_D)\tilde{z}]\right\}/\beta\delta & \text{if } \min\{p^b\tilde{a}, q^* - z\} > \tilde{z} \\
\tilde{a} & \text{if } p^b\tilde{a} < q^* - z \text{ and } p^b\tilde{a} < \tilde{z}
\end{cases}.
\tag{10}
\end{align}

In addition, assuming that each consumer holds real balances equal to $z$, using (3) and (4), the second-stage bargaining solution $(p^a(z, \tilde{z}, \sigma), \tau^a(z, \tilde{z}, \sigma))$ is given by
\begin{align}
p^a(z, \tilde{z}, \sigma) = \frac{p^b(z, \tilde{z}, \sigma)\beta\delta}{\beta\delta + (1 - \theta)p^b(z, \tilde{z}, \sigma)}, \tag{11}
\end{align}
and
\begin{align}
\tau^a(z, \tilde{z}, \sigma) = \frac{\theta\beta\delta + (1 - \theta)p^b(z, \tilde{z}, \sigma)\tau^b(z, \tilde{z}, \sigma)\sigma}{\beta\delta}(1 - \sigma). \tag{12}
\end{align}

Notice that (4) describes the amount of assets that liquidity providers purchase per unit of asset that the dealer buys. Since family member dealers bring in $\tau^b(z, \tilde{z}, \sigma)\sigma/(1 - \sigma)$ assets into the second stage match, the amount of assets purchased by a liquidity provider is given by (12).\footnote{When the consumer and dealer bargain they both believe that each liquidity provider is holding $\tilde{z}$ real}
4 Equilibrium

The analysis so far assumes that the investor’s real balance holdings are $z$ if he is a consumer $z$ and $\hat{z}$ if he is a liquidity provider. Real balances, however, are a choice variable for the investor in the CM. In this section, we first derive the investor’s CM real money demand function. Then we characterize the symmetric steady-state equilibrium for the economy.

In the CM of period $t$, an investor decides on the amount of real balances that he will hold in period $t+1$, $z_{t+1}$. This decision depends on the level of real balances that the other investors are accumulating. In his decision problem, the investor believes that all other investors will accumulate real balances equal to $\hat{z}_{t+1}$. The investor’s real money demand function can be deduced from his value function in the CM. The value function for an investor in the CM, $W$, of period $t$ is,

$$W(z_t, a_t) = \max_{z_{t+1}} \left\{ x_t - \ell_t + \beta \int_\sigma [\sigma Z_1(z_{t+1}, a_t) + (1 - \sigma) Z_0(z_{t+1}, a_t)] f(\sigma) d\sigma \right\}$$

s.t. $x_t + (\phi_t/\phi_{t+1}) z_{t+1} = \ell_t + z_t + T_t$

or

$$W(z_t, a_t) = z_t + T_t + \max_{z_{t+1}} \left\{ -(\phi_t/\phi_{t+1}) z_{t+1} + \beta \int_\sigma [\sigma Z_1(z_{t+1}, a_t) + (1 - \sigma) Z_0(z_{t+1}, a_t)] f(\sigma) d\sigma \right\}$$  \hspace{1cm} (13)

where $Z_1$ is the value function of an investor that becomes a consumer in the DM and $Z_0$ is FM value function of an investor that becomes a liquidity provider.

Note that the ask price, $p^a$, is independent of the actual real balances that the dealer secures in the second stage bargain. This is because the ask price, which is determined in the second bargaining stage, is a function of the bid price—see (3)—and the bid price is determined in the first bargaining stage. Hence, the bid price is a “predetermined” variable in the second stage, and by (3), so is the ask price. Of course, the bid price is not independent of (some measure of) the real balances; it depends on the liquidity providers’ equilibrium real balance holdings per consumer, $\tilde{z} = \tilde{z}(1 - \sigma)/\sigma$. That is, (to reemphasize) the dealer and consumer bargain under the belief that liquidity providers are holding the equilibrium level of real balances, $\hat{z}$. If a liquidity provider accumulates more than the equilibrium real balances, then in the second stage liquidity provider-dealer bargaining match, the liquidity provider will purchase $\tau^a$ assets, given by (4), at the price $p^a$. That is, the liquidity provider does not purchase any additional assets with his “extra” real balances because the ask price is essentially predetermined and the dealer brings in $\tau^b\sigma/(1 - \sigma)$ real assets to the match. If a liquidity provider accumulates less than the equilibrium real balances, say $\tilde{z} < \hat{z}$, then in the stage 2 dealer-liquidity provider match the liquidity provider purchases $\tilde{z}/p^a$ assets, which is less than the equilibrium amount.

\footnote{Notice the slight change in notation. In the previous section $\tilde{z}$ represented the real balances of liquidity providers. Here $\tilde{z}$ represents the real balances held by all other investors. When an investor makes his real balance decision, he must form expectations over what other investors are doing. When we focus our analysis on symmetric steady state equilibrium, $\tilde{z}$ will represent the proposed equilibrium real balances for investors.}
The value functions in the FM for a given period, \( t + 1 \), and realization, \( \sigma \), are,

\[
Z_1(z_{t+1}, a_t) = \delta a_t + V_1(z_{t+1} + p_{t+1}^b(z_{t+1}, \hat{z}_{t+1}, \sigma) \tau_{t+1}^b(z_{t+1}, \hat{z}_{t+1}, \sigma), \bar{a} - \tau_{t+1}^b(z_{t+1}, \hat{z}_{t+1}, \sigma)) \tag{14}
\]

and

\[
Z_0(z_{t+1}, a_t) = \delta a_t + V_0 \left( z_{t+1} - p_{t+1}^a(\hat{z}_{t+1}, z_{t+1}, \sigma) \tau_{t+1}^a(\hat{z}_{t+1}, z_{t+1}, \sigma), \bar{a} + \tau_{t+1}^a(\hat{z}_{t+1}, z_{t+1}, \sigma) \right), \tag{15}
\]

where \( V_1 \) is the value function of the consumer in the DM and \( V_0 \) is the liquidity provider’s DM value function. The first argument in the bid or ask variables represents the real balances of the consumer and the second argument represents the real balances of the liquidity provider. Since the investor is a consumer for bid variables and a liquidity provider for ask variables, the “ordering” of \( z_{t+1} \) and \( \hat{z}_{t+1} \) will differ for bid and ask variables, see above. To reduce notational burden, we will define \( p_{t+1}^{b, \sigma} \equiv p_{t+1}^b(z_{t+1}, \hat{z}_{t+1}, \sigma), \tau_{t+1}^{b, \sigma} \equiv \tau_{t+1}^b(z_{t+1}, \hat{z}_{t+1}, \sigma), p_{t+1}^{a, \sigma} \equiv p_{t+1}^a(\hat{z}_{t+1}, z_{t+1}, \sigma) \) and \( \tau_{t+1}^{a, \sigma} \equiv \tau_{t+1}^a(\hat{z}_{t+1}, z_{t+1}, \sigma) \). In the FM, the consumer sells \( \tau_{t+1}^{b, \sigma} \) assets and receives \( p_{t+1}^{b, \sigma} \bar{a}^{b, \sigma} \) real balances; the liquidity provider buys \( \tau_{t+1}^{a, \sigma} \) assets in exchange for \( p_{t+1}^{a, \sigma} \bar{a}^{a, \sigma} \) real balances. Plugging (14) and (15) into (13) we get,

\[
W_t(z_t, a_t) = z_t + \beta \delta a_t + T_t
\]

\[
\max_{z_t} \left\{ -(\phi_t / \phi_{t+1}) z_{t+1} + \beta \int \sigma V_1(z_{t+1} + p_{t+1}^{b, \sigma} \tau_{t+1}^{b, \sigma}, \bar{a} - \tau_{t+1}^{b, \sigma}) + (1 - \sigma) V_0(z_{t+1} - p_{t+1}^{a, \sigma} \tau_{t+1}^{a, \sigma}, \bar{a} + \tau_{t+1}^{a, \sigma}) \right\} d\sigma \right\}
\]

If a consumer is matched in the DM, then he purchases the DM good from a producer. The consumer transfers \( \tau_{t+1}^{m} \) units of real balances for \( q_{t+1} \) units of the consumer good in the DM of period \( t \). The value functions in the DM of period \( t + 1 \) are

\[
V_1(z_{t+1}, a_{t+1}) = \sigma_D \left[ u(q_{t+1}) + W(z_{t+1} - \tau_{t+1}^{m}, a_{t+1}) \right] + (1 - \sigma_D) W(z_{t+1}, a_{t+1})
\]

\[
= \sigma_D \left[ u(q_{t+1}) - q_{t+1} \right] + W(z_{t+1}, a_{t+1})
\]

and

\[
V_0(z_{t+1}, a_{t+1}) = W(z_{t+1}, a_{t+1}) \tag{18}
\]

Recall that \( \tau_{t+1}^{m} = q_{t+1} \) since the consumer makes a take-it-or-leave-it offer to the producer.
Plugging (17) and (18) into (16), we get

\[ W_t(z_t, a_t) = z_t + \beta \alpha a_t + T_t + \max_{z_{t+1}} \{- (\phi_t / \phi_{t+1}) z_{t+1} + \beta \int_{\sigma} [\sigma \sigma_D(u(q^\sigma_{t+1}) - q^\sigma_{t+1})] + \sigma W(z_{t+1} + p_{t+1}^b z_{t+1}, \bar{a} - \tau_{t+1}^a) + (1 - \sigma) W(z_{t+1} - p_{t+1}^a z_{t+1}, \bar{a} + \tau_{t+1}^a)] f(\sigma) d\sigma \}, \]

where

\[ q^\sigma_{t+1} = \min\{z_{t+1} + p_t^b \tau_{t+1}^b, q^* \}, \]

and \( q^* \) solves \( u'(q^*) = 1 \). Linearity of \( W(z, a) \) implies that (19) can be simplified to,

\[ W_t(z_t, a_t) = z_t + T + \beta \alpha a_t + \beta^2 \delta \bar{a} + \beta W(0, 0) + \max_{z_{t+1}} \{- (\phi_t / \phi_{t+1}) z_{t+1} + \beta \int_{\sigma} [(\sigma p_{t+1}^b \tau_{t+1}^b - (1 - \sigma) p_{t+1}^a \tau_{t+1}^a) + \beta \delta [(1 - \sigma) \tau_{t+1}^a - \sigma \tau_{t+1}^b] + \sigma \sigma_D[u(q^\sigma_{t+1}) - q^\sigma_{t+1}] f(\sigma) d\sigma \}. \]

Notice that the investor’s CM period \( t \) decision problem—the “\( \max \)” term in (21)—does not depend on the amount of real balances, \( z_t \), he brings into the CM. Hence, all investors, independent of their history, face the identical CM decision problem.

The investor’s CM real balance decision is given by the solution to

\[ \max_{z_{t+1}} \{- (\phi_t / \phi_{t+1}) z_{t+1} + \beta z_{t+1} + \beta \int_{\sigma} [(\sigma p_{t+1}^b \tau_{t+1}^b - (1 - \sigma) p_{t+1}^a \tau_{t+1}^a) + \beta \delta [(1 - \sigma) \tau_{t+1}^a - \sigma \tau_{t+1}^b] + \sigma \sigma_D[u(q^\sigma_{t+1}) - q^\sigma_{t+1}] f(\sigma) d\sigma \}. \]

The first two terms are standard and represent the date \( t \) cost and the date \( t + 1 \) benefit of accumulating \( z_{t+1} \) real balances in the CM of period \( t \). The first term of the integral expression of (22),

\[ \int_{\sigma} [\sigma p_{t+1}^b \tau_{t+1}^b - (1 - \sigma) p_{t+1}^a \tau_{t+1}^a] f(\sigma) d\sigma. \]

represents the expected transfer of real balances to the investor (when he is a consumer) and from the investor (when he is a liquidity provider) in the FM. The investor is interested in knowing how a change in his real balances affects the amount of real balances he can obtain in the FM, \( p_t^b \tau_t^b \), at the margin when he is a consumer and the amount of that he pays for assets in the FM, \( p_t^a \tau_t^a \), at the margin when he is a liquidity provider. Although, in equilibrium, \( \sigma p_t^b \tau_t^b = (1 - \sigma) p_t^a \tau_t^a \), a change in the right side, brought about by a change in real balance holdings, need not equal the change in the left side.
Suppose the investor marginally increases his holdings of real balances in the CM of period \( t \) from the equilibrium level. If the investor turns out to be a consumer, then he has more real balances of his own to spend in the DM. The higher real balances changes the solution to the bargaining problem since the bargaining surplus at the margin decreases and this will affect the cash, \( p_{t+1}^b r_{t+1}^b \), that he receives and assets that he sells, \( r_{t+1}^b \), at the margin.

Similarly, if the consumer turns out to be a liquidity provider, a change in real balances may affect the amount of assets he buys, \( r_{t+1}^a \), and what he pays for them, \( p_{t+1}^a r_{t+1}^a \), at the margin.

The second term on the second line of (22), i.e.,
\[
\beta \delta \int \sigma \left[(1 - \sigma) \tau_{t+1}^a - \sigma \tau_{t+1}^b \right] f(\sigma) d\sigma
\]
represents the expected value of future dividends transferred to the dealer. Finally, the third line of (22), i.e.,
\[
\int \sigma \sigma M \left[u \left(q_{t+1}^\sigma - q_{t+1}^\sigma\right) - q_{t+1}^\sigma\right] f(\sigma) d\sigma
\]
represents the consumer’s surplus in the DM. The quantity \( q_{t+1}^\sigma \) is a function of \( \tau_{t+1}^b \) and \( p_{t+1}^b \), i.e.,
\[
q_{t+1}^\sigma = z_{t+1} + \min\{q^* - \hat{z}_{t+1}(1 - \sigma)/\sigma, p_{t+1}^b \tau_{t+1}^b\}.
\]

In the investor’s money demand problem (22), the bargaining variables present themselves as \( \tau_{t+1}^a, p_{t+1}^a \tau_{t+1}^a, \tau_{t+1}^b, \) and \( p_{t+1}^b \tau_{t+1}^b \). We first examine how these variables are affected by a change in the investor’s real balances in the CM, and then we can characterize the solution to the investor’s money demand problem (22). The precise solution to the investor’s CM money demand depends on whether \( \bar{a} \) is large or small. We consider each case separately.

**Asset endowment \( \bar{a} \) is large**

The asset endowment \( \bar{a} \) is large when \( p_{t+1}^b \bar{a} > q^* - z_{t+1} \). The effect that a change in real balances has on bid and ask variables depends on whether or not the investor is liquidity constrained in his FM match (if he turns out to be a consumer).

Suppose first the investor turns out to be a consumer who is unconstrained in his FM match; this implies that \( p_{t+1}^b \tau_{t+1}^b = q^* - z_{t+1} \). Clearly,
\[
\partial \left(p_{t+1}^b \tau_{t+1}^b\right) / \partial z_{t+1} = -1.
\]

The effect that an increase in the consumer’s real balances, \( z_{t+1} \), has on the bid quantity can
be determined by using the upper expression in (10); that is,

\[
\frac{\partial \tau^b_{t+1}}{\partial z_{t+1}} = -\frac{\theta + (1 - \theta)[\sigma D u'(q^*) + 1 - \sigma D]}{\beta \delta} < 0.
\]  
\[ (24) \]

Now suppose that the consumer is liquidity constrained; this implies that

\[
p^b b \tau^b t+1 = z_{t+1},
\]

where \( z_{t+1} = \hat{z}_{t+1}(1 - \sigma)/\sigma \). Equation (25) immediately implies that

\[
\partial (p^b b \tau^b t+1) / \partial z_{t+1} = 0;
\]

an increase in the consumer’s real balances has no effect on the total value of asset that he sells. The effect that an increase in the consumer’s real balances has on the bid quantity can be determined by using the middle expression in (10), that is

\[
\frac{\partial \tau^b_{t+1}}{\partial z_{t+1}} = (1 - \theta)\sigma D \frac{u'(z_{t+1} + \hat{z}_{t+1}) - u'(\hat{z}_{t+1})}{\beta \delta} < 0.
\]

\[ (26) \]

This result is somewhat interesting: If the consumer brings in more real balances, he will sell less assets in the FM but at a higher price.

If the investor turns out to be a liquidity provider, then a marginal increase in the investor’s real money holdings does not affect \( \tau^b_{t+1} \) and \( p^b b \tau^b t+1 \). This is because \( \tau^b_{t+1} \) and \( p^b b \tau^b t+1 \) are determined in the first stage when the dealer and consumer bargain under the belief that liquidity providers are holding the equilibrium level of real balances, \( \hat{z} \)—see discussion in footnote 11. Therefore, if \( z_{t+1} \geq \hat{z}_{t+1} \), then an increase in investor’s real money holdings will not affect \( \tau^a_{t+1} \) and \( p^a t+1 \tau^a t+1 \) when he turns out to be a liquidity provider—i.e.,

\[
\partial (p^a t+1(\hat{z}_{t+1}, z_{t+1}, \sigma) \tau^a t+1(\hat{z}_{t+1}, z_{t+1}, \sigma)) / \partial z_{t+1} = 0
\]

and

\[
\partial \tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma) / \partial z_{t+1} = 0.
\]

However, if \( z_{t+1} < \hat{z}_{t+1} \), then

\[
\partial (p^a t+1(\hat{z}_{t+1}, z_{t+1}, \sigma) \tau^a t+1(\hat{z}_{t+1}, z_{t+1}, \sigma)) / \partial z_{t+1} = 1
\]

and

\[
\partial \tau^a_{t+1}(\hat{z}_{t+1}, z_{t+1}, \sigma) / \partial z_{t+1} = 1/p^a t+1.
\]
(when $z_{t+1} < \hat{z}_{t+1}$, an additional unit of real balances can purchase $1/p_{t+1}^a$ units of the asset).

We can now solve the investor’s CM maximization problem (22). The first-order condition to this problem is,

$$\frac{\phi_t}{\phi_{t+1}} - 1 = \int_0^{\sigma^*(z_{t+1})} \{ -1 \cdot \sigma - \delta \cdot (1 - \sigma) + 0 \cdot \beta \delta (1 - \sigma)$$

$$- \beta \delta \frac{\partial r_{t+1}^h}{\partial z_{t+1}} + \sigma \sigma_D [u'(q^*) - 1]} \} f(\sigma) d\sigma$$

$$+ \int_0^{\sigma^*(z_{t+1})} \{ 0 \cdot \sigma + \delta \cdot \frac{(1 - \sigma)}{p_{t+1}^a} + 0 \cdot \beta \delta (1 - \sigma)$$

$$- \beta \delta \frac{\partial r_{t+1}^b}{\partial z_{t+1}} + \sigma \sigma_D [u'(z_{t+1} + \hat{z}_{t+1}) - 1]} \} f(\sigma) d\sigma,$$

where

$$\delta = \begin{cases} 0 & \text{if } z_{t+1} \geq \hat{z}_{t+1} \\ 1 & \text{if } z_{t+1} < \hat{z}_{t+1} \end{cases},$$

and $\sigma^*(z_{t+1}) = \hat{z}_{t+1}/(q^* + \hat{z}_{t+1} - z_{t+1})$. Using the Fisher equation $(1 + i_t) = (\phi_t/\phi_{t+1})(1 + r)$ and $\beta = 1/(1 + r)$, the nominal interest rate, $i_t$, can be expressed as

$$i_t = \frac{\phi_t/\phi_{t+1}}{\beta} - 1.$$  (27)

The above first-order condition can be simplified to

$$i_t = \int_0^{\sigma^*(z_{t+1})} \{ -\delta \cdot (1 - \sigma) - 1 + \theta + (1 - \theta) [\sigma_D u'(q^*) + 1 - \sigma_D] \} f(\sigma) d\sigma$$

$$+ \int_0^{\sigma^*(z_{t+1})} \{ \delta \cdot \frac{(1 - \sigma)}{p_{t+1}^a} + \sigma \cdot (1 - \theta) [u'(z_{t+1} + \hat{z}_{t+1}) - u'(z_{t+1})] \} f(\sigma) d\sigma$$

$$+ \int_0^{\sigma^*(z_{t+1})} \sigma \sigma_D [u'(z_{t+1} + \hat{z}_{t+1}) - 1] f(\sigma) d\sigma,$$

or

$$i_t = -\int_0^{\sigma^*(z_{t+1})} \delta \cdot (1 - \sigma) f(\sigma) d\sigma + \int_0^{\sigma^*(z_{t+1})} \delta \cdot \frac{1 - \sigma}{p_{t+1}^a} f(\sigma) d\sigma +$$

$$\sigma_D \int_0^{\sigma^*(z_{t+1})} \sigma [u'(z_{t+1}) - 1] + \theta [u'(z_{t+1} + \hat{z}_{t+1}) (1 - \sigma) / \sigma - 1]} f(\sigma) d\sigma.$$  (28)
Asset Endowment $\bar{a}$ is small

The asset endowment $\bar{a}$ is small when $p_{t+1}^b \bar{a} < q^* - z_{t+1}$. If the investor turns out to be a consumer and is liquidity constrained, $p_{t+1}^b \bar{a} > \tilde{z}_{t+1}$, then the effect of an increase in real balances on $p_{t+1}^b$ and $\tau_{t+1}^b$ is exactly the same as above, i.e., $\partial (p_{t+1}^b \tau_{t+1}^b) / \partial z_{t+1} = 0$ and $\partial \tau_{t+1}^b / \partial z_{t+1}$ is given by (26). If the consumer is asset constrained, $p_{t+1}^b \bar{a} \leq \tilde{z}_{t+1}$, then, from (8) we get

$$\frac{\partial p_{t+1}^b}{\partial z_{t+1}} = -\sigma_D \frac{u'(z_{t+1} + p_{t+1}^b \bar{a}) - u'(z_{t+1})}{\sigma_D u'(z_{t+1} + p_{t+1}^b \bar{a}) \bar{a} + (1 - \sigma_D) \bar{a} + \theta / (1 - \theta) \bar{a}} > 0,$$

(29)

which implies that $\partial (p_{t+1}^b \tau_{t+1}^b) / \partial z_{t+1} = \bar{a} \partial p_{t+1}^b / \partial z_{t+1} > 0$. Since $\tau_{t+1}^b = \bar{a}$ when the consumer is asset constrained, we get $\partial \tau_{t+1}^b / \partial z_{t+1} = 0$. If the investor turns out to be a liquidity provider, then the effect of an increase is his real balance holdings on $\tau_{t+1}^b$, $p_{t+1}^b \tau_{t+1}^b$ and $p_{t+1}^a \tau_{t+1}^a$ are the same as described above when $\bar{a}$ is large.

The first order condition for the investor’s CM maximization problem (22) is

$$\frac{\phi_t}{\phi_{t+1}} - 1 = \int_{\tilde{\sigma}(z_{t+1})}^{\tilde{\sigma}(z_{t+1})} \{ \frac{\partial \tilde{p}_{t+1}^b}{\partial z_{t+1}} \bar{a} \cdot \sigma - \delta \cdot (1 - \sigma) + 0 \cdot \beta \delta (1 - \sigma) - 0 \cdot \beta \delta \sigma \} d\sigma$$

$$+ \sigma_D [u'(z_{t+1} + \tilde{p}_{t+1}^b \bar{a}) - 1] f(\sigma) d\sigma$$

$$+ \int_{\tilde{\sigma}(z_{t+1})}^{1} \{ 0 \cdot \sigma - \delta \cdot \frac{(1 - \sigma) \tilde{p}_{t+1}^a}{\tilde{p}_{t+1}^a} + 0 \cdot \beta \delta (1 - \sigma) \} + \beta \delta \frac{\partial \tau_{t+1}^b}{\partial z_{t+1}} \sigma + \sigma_D [u'(z_{t+1} + \tilde{z}_{t+1}) - 1] f(\sigma) d\sigma$$

or

$$\frac{\phi_t}{\phi_{t+1}} - 1 = - \int_{0}^{\tilde{\sigma}(z_{t+1})} \delta \cdot (1 - \sigma) f(\sigma) d\sigma + \int_{0}^{\tilde{\sigma}(z_{t+1})} \sigma \{ \frac{\partial \tilde{p}_{t+1}^b}{\partial z_{t+1}} \bar{a} + \sigma_D [u' \left( z_{t+1} + \tilde{p}_{t+1}^b \bar{a} \right) - 1] \} f(\sigma) d\sigma$$

$$+ \int_{\tilde{\sigma}(z_{t+1})}^{1} \delta \cdot \frac{(1 - \sigma) \tilde{p}_{t+1}^a}{\tilde{p}_{t+1}^a} f(\sigma) d\sigma + \int_{\tilde{\sigma}(z_{t+1})}^{1} \sigma \{ - \beta \delta \frac{\partial \tau_{t+1}^b}{\partial z_{t+1}} + \sigma_D [u' \left( z_{t+1} + \tilde{z}_{t+1} \right) - 1] \} f(\sigma) d\sigma,$$

where limit of integration is $\tilde{\sigma}(z_{t+1}) = \tilde{z}_{t+1} / (\tilde{p}_{t+1}^b \bar{a} + \tilde{z}_{t+1})$ and $\tilde{p}^b$ is given implicitly by (8).
Substituting for the derivatives, this equation can be rewritten as

\[ i_t = - \int_{0}^{\delta(z_{t+1})} \delta \cdot (1 - \sigma) f(\sigma)d\sigma + \sigma_D \int_{0}^{\delta(z_{t+1})} \sigma \Theta(z_{t+1}, \sigma) f(\sigma)d\sigma \]
\[ + \int_{\delta(z_{t+1})}^{1} \frac{\delta \cdot (1 - \sigma)}{\bar{p}_t^\sigma} f(\sigma)d\sigma + \sigma_D \int_{\delta(z_{t+1})}^{1} \sigma \Phi(z_{t+1}, \sigma) f(\sigma)d\sigma, \]

where

\[ \Theta(z_{t+1}, \sigma) = - \frac{(1 - \theta)[u'(z_{t+1} + \hat{z}_t^\sigma \bar{\alpha}) - u'(z_{t+1})]}{(1 - \theta)\sigma_D[u'(z_{t+1} + \hat{z}_t^\sigma \bar{\alpha}) - 1] + 1} + u'(z_{t+1} + \hat{z}_t^\sigma \bar{\alpha}) - 1 > 0 \]

and

\[ \Phi(z_{t+1}, \sigma) = (1 - \theta)[u'(z_{t+1}) - 1] + \theta\{u'[z_{t+1} + \hat{z}_t(1 - \sigma)/\sigma] - 1\} > 0. \]

**Equilibrium**

We focus on our analysis on a symmetric steady-state equilibrium. In a steady state equilibrium, the real money supply is constant over time, \( M_{t+1}\phi_t = M_t\phi_t \), which implies that

\[ \frac{M_{t+1}}{M_t} = \frac{\phi_t}{\phi_{t+1}} = \mu. \]  

(31)

As well, all investors accumulate the same level of real balances in the CM of period \( t \), \( z_{t+1} = \hat{z}_{t+1} \). Since, the real money supply is constant over time, we have that, in a symmetric steady state equilibrium, \( z_{t+1} = \hat{z}_{s+1} = \hat{z}_{t+1} = \hat{z}_{s+1} = z \) for all \( s, t \). An implication of constant and equal real balance holdings over investors and time is that, for any given \( \sigma \), the values of bid and ask variables are time invariant, see (9), (10), (11) and (12).

For a given the money growth rate \( \mu \) and, hence, nominal interest rate \( i = \mu/\beta - 1 \geq 0 \), when \( \bar{\alpha} \) is large, investors choose real balances, \( z \), to satisfy condition (28) where \( z_{t+1} = \hat{z}_{t+1} = z \), i.e., they choose \( z \) to satisfy

\[ i = \sigma_D \int_{z/q^\sigma}^{1} \sigma\{(1 - \theta)[u'(z) - 1] + \theta[u'(z/\sigma) - 1]\} f(\sigma)d\sigma. \]

(32)

Notice that when \( z_{t+1} = \hat{z}_{t+1} = z \), the indicator function \( \delta \) is equal to zero. Denote the solution to (32) as \( z(\mu) \). When \( \bar{\alpha} \) is small, investors choose real balances, \( z \), to satisfy condition (30) where \( z_{t+1} = \hat{z}_{t+1} = z \), i.e., they choose \( z \) to satisfy

\[ i = \sigma_D \int_{z/(\bar{\rho}^\bar{\alpha} + z)}^{1} \sigma \Theta(z, \sigma) f(\sigma)d\sigma \]
\[ + \sigma_D \int_{z/(\bar{\rho}^\bar{\alpha} + z)}^{1} \sigma \Phi(z, \sigma) f(\sigma)d\sigma. \]

(33)
Denote the solution to (33) as \( \bar{z}(\mu) \).

Up to this point we have characterized “\( \bar{a} \) being large” as \( p^b \bar{a} \geq q^* - z \) and “\( \bar{a} \) being small” by \( p^b \bar{a} < q^* - z \). This characterization is problematic because it depends on an endogenous variable, \( z \). We now provide a characterization for “\( \bar{a} \) being large” or “\( \bar{a} \) being small” that relies only on fundamental (exogenous) parameters. Assume that \( \bar{a} \) is large. The investor’s real balance holdings in the CM is \( \bar{z}(\mu) \). Since (by assumption), \( \bar{a} \) is large, the consumer will be unconstrained for all \( \sigma \leq z^*(\mu)/q^* \). If \( \sigma \in (0, z^*(\mu)/q^*) \), then the amount of assets that the investor sells in the FM is invariant to \( \sigma \) and given by the upper branch of (7), where we substitute \( z = \bar{z}(\mu) \). Denote the amount of assets that the consumer sells when he is unconstrained as \( \bar{\tau}^b(\mu) \). Hence, \( \bar{a} \) is large if \( \bar{a} \geq \bar{\tau}^b(\mu) \) and small if \( \bar{a} < \bar{\tau}^b(\mu) \). If \( \bar{a} \) is large, then the investor’s real CM balance holdings will be given by the solution to (32), which is \( \bar{z}(\mu) \); and if it is small, then his real CM balance holdings will be given by the solution to (33), which is \( \bar{z}(\mu) \).

Since \( z_t = \hat{z}_t = z \), we can compactly write the bid price and bid quantity as \( p^b(z, \sigma) \) and \( \tau^b(z, \sigma) \), respectively. Using (9) and (10), if \( \bar{a} \geq \bar{\tau}^b(\mu) \), then the bid price and bid quantity are given by, respectively,

\[
p^b(z, \sigma) = (\tau^b)^{-1} \min \{ z(1 - \sigma)/\sigma, q^* - z \}
\]

and

\[
\tau^b(z, \sigma) = \begin{cases} 
\{ \theta (q^* - z) + (1 - \theta) [\sigma_D [u (q^*) - u (z)] + (1 - \sigma_D) (q^* - z)] \}/\beta \delta & \text{if } q^* \leq z/\sigma \\
\{ \theta z(1 - \sigma)/\sigma + (1 - \theta) [\sigma_D [u (z/\sigma) - u (z)] + (1 - \sigma_D) z(1 - \sigma)/\sigma] \}/\beta \delta & \text{if } q^* > z/\sigma 
\end{cases}
\]

where \( z = \bar{z}(\mu) \). If \( \bar{a} < \bar{\tau}^b(\mu) \), then the bid price and bid quantity are given by, respectively,

\[
p^b(z, \sigma) = \begin{cases} 
z(1 - \sigma)/[\sigma \tau^b(z, \sigma)] & \text{if } \bar{\tau}^b(z, \sigma) \bar{a} \geq z(1 - \sigma)/\sigma \\
\bar{p}^b(z, \sigma) \bar{a} & \text{if } \bar{\tau}^b(z, \sigma) \bar{a} < z(1 - \sigma)/\sigma 
\end{cases}
\]

and

\[
\tau^b(z, \sigma) = \begin{cases} 
\{ \theta z(1 - \sigma)/\sigma + (1 - \theta) [\sigma_D [u (z/\sigma) - u (z)] + (1 - \sigma_D) z(1 - \sigma)/\sigma] \}/\beta \delta & \text{if } \bar{\tau}^b(z, \sigma) \bar{a} \geq z(1 - \sigma)/\sigma \\
\bar{a} & \text{if } \bar{\tau}^b(z, \sigma) \bar{a} < z(1 - \sigma)/\sigma 
\end{cases}
\]

where \( z = \bar{z}(\mu) \) and \( \bar{p}^b(z, \sigma) \) solves (8). From (11) and (12), the ask price, \( p^a(z, \sigma) \), and ask quantity, \( \tau^a(z, \sigma) \), are given by

\[
p^a(z, \sigma) = \frac{p^b(z, \sigma) \beta \delta}{\theta \beta \delta + (1 - \theta) p^b(z, \sigma)},
\]
and
\[ \tau^a(z, \sigma) = \frac{\theta \beta \delta + (1 - \theta) p^b(z, \sigma)}{\beta \delta} \tau^b(z, \sigma) \sigma / (1 - \sigma), \] (39)
respectively.

The equilibrium DM output, (20), can be written as
\[ q(z, \sigma) = \min\{z + p^b(z, \sigma) \tau^b(z, \sigma), q^*\}. \] (40)

In period \( t \), the aggregate demand for real balances evaluated in period \( t \) prices is \( \mu z_{t+1} = \mu z \) and the total supply of real balances is \( \phi_t M_t \). Since \( z_t = z \) for \( t \), the market clearing price of money is
\[ \phi_t = \frac{\mu z}{M_t}. \] (41)

We can now provide a definition of equilibrium for our economy.

**Definition 1** Given a money growth rate \( \mu \), an initial money supply, \( M_0 \), and a per period asset endowment \( a \), a steady-state equilibrium is characterized by: (i) real balances \( z \), given by (32) if \( a \geq \bar{z}(\mu) \) or (33) if \( a < \bar{z}(\mu) \); (ii) a set of CM prices \( \{\phi_t\}_{t=0}^{\infty} \), given by (41); and for each realization of \( \sigma \in [0,1] \), (iii) a bid price \( p^b(z, \sigma) \), given by (34) if \( a \geq \bar{z}(\mu) \) or (36) if \( a < \bar{z}(\mu) \); (iv) a bid quantity \( \tau^b(z, \sigma) \), given by (35) if \( a \geq \bar{z}(\mu) \) or (37) if \( a < \bar{z}(\mu) \); (v) an ask price \( p^a(z, \sigma) \), given by (38); (vi) an ask quantity \( \tau^a(z, \sigma) \), given by (39); and (vii) a DM output level \( q(z, \sigma) \), given by (40).

We provide a proof for existence and uniqueness of equilibrium in the Appendix. The proof requires the properties of the comparative static \( dz/d\mu \), which are derived in Section 6.

### 5 Returns, Spreads and Cash-in-the-Market Pricing of Assets

We are interested in understanding how supply and demand for liquidity in the FM affects asset prices, returns and bid-ask spreads. Although liquidity is a nebulous concept, there are a number of ways one can attempt to measure or characterize it. One measure of liquidity is the fraction of investors that turn out to be liquidity providers in the FM, \( 1 - \sigma \). Once investors exit the CM and enter the FM, their liquidity needs may be revised: An investor who
turns out to be a consumer in the subsequent DM may desire additional real balances, and an investor who turns out to be a liquidity provider is willing to supply them. The total amount of real balances that liquidity providers can supply to consumers in the FM is \((1 - \sigma)z\). When \(\sigma\) is “large,” there may be a shortage of FM liquidity since the measure of liquidity providers in the FM is “small.” And, when \(\sigma\) is “small,” liquidity may be plentiful in the FM.

Another, rather obvious, measure of the supply of liquidity is the amount of real balances, \(z\), that investors accumulate in the CM. A change in various model parameters may cause consumers to change their real balance holdings. For example, a change in the money growth rate changes the cost of holding real balances and, thus the amount of real balances that investors accumulate in the CM. We defer this discussion to the subsequent section.

The equilibrium ask and bid prices are given by (3) and (34) or (36), respectively. There are a couple of ways to measure asset returns, e.g., a gross bid return, \(r^b\), and a gross ask return, \(r^a\). These returns can be measured as

\[
r^b = \frac{\delta}{p^b}
\]

and \(r^a = \frac{\delta}{p^a}\). Using (3), we can define the equilibrium bid-ask spread as

\[
\frac{p^a}{p^b} = \frac{\beta \delta}{\theta \beta \delta + (1 - \theta) p^b}.
\]

Notice that the wedge between the bid and ask price is created by the bargaining friction. When the bargaining friction disappears, i.e., \(\theta = 1\), the dealer receives no surplus and \(p^a = p^b\).

The effect that a change in the liquidity measure \(\sigma\) has on the bid price is simply \(\partial p^b / \partial \sigma\), and on the (bid) asset return is,

\[
\frac{\partial r^b}{\partial \sigma} = -\frac{\delta}{(p^b)^2} \frac{\partial p^b}{\partial \sigma}.
\]

Note that, not surprisingly, bid prices and asset returns are negatively correlated.\(^\text{13}\) The

\(^\text{13}\)If one is interested in the “ask return,” then, using (43), we have

\[
r^a = \theta r^b + (1 - \theta),
\]

and

\[
\frac{\partial r^a}{\partial \sigma} = \theta \frac{dr^b}{d\sigma}.
\]
effect that a change in the liquidity measure \( \sigma \) has on the bid-ask spread is

\[
\frac{\partial (p^a/p^b)}{\partial \sigma} = -\frac{\beta \delta (1 - \theta)}{[\theta \beta \delta + (1 - \theta) p^b]^2} \frac{\partial p^b}{\partial \sigma}.
\] (45)

The finance literature, e.g., Amihud and Mendelson (1986) and Amihud, Mendelson, Pedersen (2005), has consistently documented a positive correlation between changes in asset returns and bid-ask spreads. Our bargaining environment predicts that such a relationship will prevail (at least for the liquidity measure \( \sigma \)), i.e., simply compare (44) and (45),

\[
\frac{\partial (p^a/p^b)}{\partial \sigma} = \frac{\beta (1 - \theta) (p^b)^2}{[\theta \beta \delta + (1 - \theta) p^b]^2} \frac{\partial r_t^b}{\partial \sigma}.
\]

Let’s examine the effect that a change in the composition of consumers and liquidity providers in the FM has on assets prices. In particular, what is \( \partial p^b/\partial \sigma \)? Recall that the value of \( \sigma \) is drawn from the probability distribution \( f(\sigma) \), where \( \sigma \in [0, 1] \). Hence, in a steady-state equilibrium, \( \sigma \) will take on different values over time. If the consumer is not liquidity constrained in the FM, i.e., either \( p^b \tilde{a} < \tilde{z} = z(1 - \sigma)/\sigma \) or \( q^* - z < \tilde{z} \), then it is obvious from the equilibrium conditions—(34) and (35) if \( \tilde{a} \) is large or (36) and (37) if \( \tilde{a} \) is small—that \( \partial p^b/\partial \sigma = 0 \). Intuitively, since the amount of assets that consumers sell in the FM is unaffected by an increase in FM liquidity when the consumer is not liquidity constrained, then so too is the (bid) price.

Now suppose that the consumer is liquidity constrained in the FM, i.e., \( \tilde{z} < \min \{p^b \tilde{a}, q^* - z\} \). Then \( p^b \tau^b = \tilde{z} \). If \( p^b \tau^b = \tilde{z} \), then conditions (34) and (35) when \( \tilde{a} \) is big or (36) and (37) when \( \tilde{a} \) is small both imply that

\[
\frac{\partial p^b}{\partial \tilde{z}} \frac{\partial \tilde{z}}{\partial \sigma} = -\frac{p^b (1 - \theta) \sigma_D}{\beta \delta \tilde{z}^2} \left[ u(z + \tilde{z}) - u(z) - \tilde{z} u'(z + \tilde{z}) \right] \frac{\tilde{z}}{\sigma^2} < 0.
\] (46)

If the fraction of liquidity providers in the FM decreases, then the amount of real balances available in a bargaining match also falls. As a result, the marginal value of the DM expected surplus increases, which makes real balances more valuable to the consumer. Hence, the consumer is willing to sell the real asset at a lower bid price, i.e., \( \partial p^b/\partial \sigma < 0 \). This result is reminiscent of cash-in-the-market pricing, see Allen and Gale (2005, 2007). Cash-in-the-market pricing has the basic flavor that asset prices fall below their fundamental values.

\[\text{In fact, as long as the change in asset prices are brought about by a change in a variable or parameter that is not equal to } \beta, \delta, \text{ or } \theta, \text{ we can conclude there is a positive correlation between changes in asset returns and the bid-ask spread.}\]
because, relative to the amount of assets that are liquidated, cash is in scarce supply. If, however, cash is “plentiful,” then asset prices will be at their fundamental values. In our environment, asset prices in the FM are always below their fundamental values because asset markets are not competitive. But, the amount of cash (real balances) available in the (FM) market, \((1 - \sigma)z\), can directly influence asset prices: Higher cash holdings can lead to higher asset prices. This result may seem anomalous from a classic asset pricing theory perspective, where the value of an asset is equal to the discounted stream of its dividends.

Over time, asset prices will “fluctuate” in the steady state equilibrium. More specifically, when \(\bar{a}\) is large consumers are liquidity constrained for all \(\sigma > q^*/\bar{z}(\mu)\) and when it is small, they are liquidity constrained for all \(\sigma > \bar{z}(\mu)/(p^b\bar{a} + \bar{z}(\mu))\). As a result, when \(\bar{a}\) is large asset prices are “constant” over \(\sigma \in (0, q^*/\bar{z}(\mu))\) and declining over \(\sigma \in (q^*/\bar{z}(\mu), 1)\); when \(\bar{a}\) is small, asset prices are constant over \(\sigma \in (0, \bar{z}(\mu)/(p^b\bar{a} + \bar{z}(\mu)))\) and declining over \(\sigma \in (\bar{z}(\mu)/(p^b\bar{a} + \bar{z}(\mu)), 1)\). Therefore, over time, asset prices will “move around” since the asset price will reflect, among other things, the amount of liquidity available in the FM, \((1 - \sigma)z\), and the amount of liquidity varies over time because \(\sigma\) is a random variable. Once again, from a classic asset pricing theory perspective, this result appears to be anomalous because asset price changes are not accompanied by new information regarding asset fundamentals.

## 6 Inflation, Asset Prices and Returns

We now examine the effect that changes in liquidity measure \(z\) has on asset prices and returns. Since real balances, \(z\), is an endogenous choice variable, something (exogenous) must change in order to induce a change in \(z\). We focus our attention on a change in the money growth rate, \(\mu\), which also changes inflation one-to-one, and the nominal interest rate, \(i\), by the factor \(1/\beta\). We first determine how a change in money growth or inflation affects real balances; we are then in a position to examine how inflation, by changing real balances, affects asset prices and returns.

We first suppose that the asset endowment, \(\bar{a}\), is large. From the equilibrium condition (32), the relationship between equilibrium real balances and the (gross) inflation rate is given
by
\[
\frac{dz}{d\mu} = \{\beta \sigma_D \int_{z/q^*}^{1} \sigma[(1 - \theta)u''(z) + \theta u''(z/\sigma)]f(\sigma)d\sigma \\
- \beta/q^*[(1 - \theta)[u'(z) - 1]f(z/q^*)]^{-1} < 0, \tag{47}
\]
which means that an increase in inflation decreases investors’ real balance holdings.

Now suppose that the asset endowment, \(\bar{a}\), is small. From the equilibrium condition (33), the relationship between equilibrium real balances and the (gross) inflation rate in the steady state is given by
\[
\frac{dz}{d\mu} = \beta \sigma_D \{ \int_{\tilde{\sigma}(z)}^{\sigma(z)} \frac{d\Theta(z, \sigma)}{dz} f(\sigma)d\sigma \\
+ \int_{\tilde{\sigma}(z)}^{1} \frac{d\Phi(z, \sigma)}{dz} f(\sigma)d\sigma \\
+ \frac{d\tilde{\sigma}(z)}{dz}[\Theta(z, \tilde{\sigma}) - \Phi(z, \tilde{\sigma})]f(\tilde{\sigma})\}^{-1}, \tag{48}
\]
where
\[
\tilde{\sigma}(z) = \frac{z}{\bar{p}^b \bar{a} + z},
\]
\[
\frac{d\Theta(z, \sigma)}{dz} = \frac{u''(z + \bar{p}^b_{t+1} \bar{a})}{(1 - \theta)\sigma_D[u'(z + \bar{p}^b_{t+1} \bar{a}) - 1] + 1} \left[ \theta + (1 - \theta)\sigma_D(u'(z + \bar{p}^b_{t+1} \bar{a}) - 1) (u'(z) - 1) \right] \\
+ \frac{(1 - \theta)u''(z)}{(1 - \theta)\sigma_D[u'(z + \bar{p}^b_{t+1} \bar{a}) - 1] + 1} + \frac{(1 - \theta)[u'(z + \bar{p}^b_{t+1} \bar{a}) - u'(z)]}{(1 - \theta)u''(z + \bar{p}^b_{t+1} \bar{a})\{\sigma_D[u'(z + \bar{p}^b_{t+1} \bar{a}) - 1] + 1\}^2} < 0,
\]
and
\[
\frac{d\Phi(z, \sigma)}{dz} = (1 - \theta)u''(z) + \theta u''(z/\sigma)/\sigma < 0,
\]
and
\[
\frac{d\tilde{\sigma}}{dz} = \bar{a}[\bar{P}^b - z \frac{dp^b}{dz}] . \tag{49}
\]
By construction, the consumer is both asset constrained and money constrained at \(\tilde{\sigma}\), i.e., \(\bar{p}^b \bar{a} = z(1 - \tilde{\sigma})/\tilde{\sigma}\). It is straightforward to show that \(\Theta(z, \tilde{\sigma}) - \Phi(z, \tilde{\sigma}) < 0\). Intuitively, \(\Theta(z, \sigma)\) represents the marginal benefit to an investor of having an additional unit of real balances when consumers turn out to be asset constrained and \(\Phi(z, \sigma)\) represents the marginal benefit when consumers turn out to be money constrained. Since this difference is negative, the marginal benefit is higher when consumers are money constrained as opposed to asset constrained.
If the asset price elasticity, \( \left( \frac{d\tilde{p}^b}{p^b} \right)(dz/z) \), is less than one or equal to one, then, from (49), \( d\tilde{\sigma}/dz > 0 \) and, therefore, \( dz/d\mu < 0 \). Unfortunately, the size of the asset price price elasticity depends on preferences. For example, if the utility function \( u \) is described by

\[
u(q) = \frac{q^{1-\alpha}}{1-\alpha}, \tag{50}\]

then for \( 0 < \alpha < 1 \), we have \( d\tilde{\sigma}/dz > 0 \). However, when \( \alpha > 1 \), it is not possible to analytically sign \( d\tilde{\sigma}/dz \) or, more importantly, \( dz/d\mu \). We will deal with the small \( \tilde{a} \) case by numerically evaluating the derivative (48). Common to all of our examples are the utility function, given by (50), a uniform distribution for \( f(\sigma) \), and parameter values \( \sigma_D = 0.8 \), \( \beta = 0.98 \), and \( \delta = 0.5 \). We consider various values for the preference parameter \( \alpha \), the nominal interest rate, \( i \), asset endowment, \( \tilde{a} \), and bargaining parameter, \( \theta \); in particular \( \alpha \in \{2.5, 3, 4, 5, 6, 7, 10, 15, 20, 40\} \), \( i \in \{0.01, 0.02, 0.03, \ldots, 0.24, 0.25\} \), \( \tilde{a} \in \{0.01, 0.05, 0.1\} \), and \( \theta = \{0.1, 0.5, 0.9\} \). For the sets of parameters that imply that \( \tilde{a} \) is small, we find that every numerical example is characterized by \( dz/d\mu < 0 \). Hence, we can reasonably assume that when asset holdings, \( \tilde{a} \), are small, an increase in the money growth rate decreases equilibrium real balance holdings. This, of course, is a standard and robust result in monetary economics.

We can now investigate the effect that inflation—or, equivalently, a change in real balances—has on asset prices, asset real returns and the bid-ask spread. The effect that a change in real balances (for all investors) has on the bid asset price depends on whether the consumer is unconstrained, asset constrained or liquidity constrained in the FM. If consumers are not liquidity constrained in their bargaining matches\(^{16} \), i.e., \( p^b r^b = q^* - z \), then using (35), (34) can be written as

\[
\beta \delta \frac{q^* - z}{p^b} = \theta (q^* - z) + (1 - \theta) \left\{ \sigma_D [u'(q^*) - u(z)] + (1 - \sigma_D)(q^* - z) \right\}
\]

or

\[
\frac{\beta \delta}{p^b} = \theta + (1 - \theta) \frac{\sigma_D u'(q^*) - u(z)}{q^* - z} + (1 - \theta)(1 - \sigma_D).
\tag{51}\]

Therefore,

\[
\frac{dp^b}{dz} = \frac{p^b(1 - \theta)\sigma_D}{\beta \delta (q^* - z)^2} \left[ u'(z)(q^* - z) - u(q^*) + u(z) \right] > 0.
\]

\(^{15}\)Some parameter configurations do not have \( \tilde{a} \) small. For example, when \( \tilde{a} = 0.1 \) and \( \alpha \in \{20, 40\} \), \( \tilde{a} \) is never “small” for any \( 0.01 \leq i \leq 0.25 \).

\(^{16}\)This necessarily implies that the asset endowment, \( \tilde{a} \), is large.
If consumers are asset constrained,\textsuperscript{17} i.e., $p^b \tilde{a} < \tilde{z}$, where $z(1 - \sigma)/\sigma$ and $p^b \tilde{a} + z < q^*$, then using (8), we get

$$\frac{dp^b}{dz} = \frac{(1 - \theta)[\sigma_D u(z + p^b \tilde{a}) - 1]}{(1 - \theta)[\sigma_D u(z) - 1] + 1} > 0.$$ 

Finally, if consumers are liquidity constrained,\textsuperscript{18} i.e., $p^b \tau^b = \tilde{z}$, where $\tilde{z} = z(1 - \sigma)/\sigma$, then using (35), (34) can be written as

$$\beta \delta \frac{z(1 - \sigma)/\sigma}{p^b} = \theta z(1 - \sigma)/\sigma + (1 - \theta) [\sigma_D [u(z/\sigma) - u(z)] + (1 - \sigma_D) z(1 - \sigma)/\sigma$$

or

$$\frac{\beta \delta}{p^b} = \theta + (1 - \theta) \sigma_D \frac{u(z/\sigma) - u(z)}{z(1 - \sigma)/\sigma} + (1 - \theta) (1 - \sigma_D).$$

Therefore,

$$\frac{dp^b}{dz} = \frac{p^b}{\beta \delta z^2(1 - \sigma)/\sigma} \{u'(z) z - u'(z/\sigma)z/\sigma + u(z/\sigma) - u(z)\} > 0.$$

An increase in the real balances of all investors—and hence, an increase in the total real money supply—increases asset prices, and this is independent of whether the consumer is liquidity constrained, asset constrained or unconstrained. The intuition behind this result is straightforward. There are two effects associated with an increase in real balances that work in the same direction. If a consumer’s real balances increase (holding the liquidity provider’s balances constant), then the marginal value of the bargaining match surplus falls. If a liquidity provider’s real balances increase (holding the consumer’s balances constant), then although the match surplus increases, the marginal value of the surplus falls. (This latter effect is not operative if the consumer is not liquidity constrained.) In both cases, the consumer’s value for an additional unit of real balances falls, which implies that bid asset price increases, i.e., the consumer gives up a smaller quantity of the asset for an additional unit of real balances. Interestingly, higher levels of liquidity, as measured by $z$, are associated with higher asset prices: this is the cash-in-the-market effect at work. Since an increase in inflation decreases real balance holdings, an increase in inflation actually decreases asset prices. This is consistent with the observation that periods of low inflation are usually associated with periods of high asset prices. This is in contrast to the standard so-called Mundell-Tobin effect, which is present in many models of money, that posits a positive relationship between inflation and asset prices.

\textsuperscript{17}This necessarily implies that the asset endowment, $\tilde{a}$, is small.

\textsuperscript{18}Investors can be liquidity constrained when $\tilde{a}$ is large and when it is small.
The effect that a change in real balances has on asset returns and the bid-ask spread is similar to the effect of a change in \( \sigma \), (44) and (45); in particular,

\[
\frac{dr^b}{dz} = -\frac{\delta}{(p^b)^2} \frac{dp^b}{dz},
\]

and

\[
\frac{d\left(p^a/p^b\right)}{dz} = -\frac{\beta \delta (1 - \theta)}{[\theta \beta \delta + (1 - \theta) p^b]^2} \frac{dp^b}{dz}.
\]

Hence, asset returns and the bid-ask spreads are positively correlated. An increase in inflation increases asset returns, as well as the bid-ask spread. The latter result reinforces a standard view of liquidity and bid-ask spreads: An increase in inflation reduces real balances (liquidity) in the economy which results in higher bid-ask spreads.

We conclude this section by pointing out that there is a subtle but important difference in interpreting the results for liquidity measure \( z \) and liquidity measure \( \sigma \). In the steady-state equilibrium, investors accumulate \( \mu z \) real balances in the CM in period \( t \) measured in period \( t \) prices, and then enter the FM. The amount of liquidity available to consumers, \( z(1 - \sigma) \), will fluctuate over time, as will asset prices, asset returns and bid-ask spreads. So we can sensibly ask, in the equilibrium, how does an increase or decrease in liquidity affect asset prices, asset returns and bid-ask spreads? In contrast, in equilibrium, there is a unique value of real balances, \( z \), given by the solution to (32) or (33). We can compare differences in asset prices, asset returns and bid-ask spreads associated with different values of real balances—that result from different money growth rates—but it should be recognized that this is a comparison across equilibria associated with different economies.

7 Conclusions

We examined a world where money is needed to facilitate investments. Since money is costly to hold, investors tend to an amount that is insufficient to purchase the efficient of (DM) output. A financial market that allows consumers to trade their less liquid assets for more liquid ones can improve matters. When financial markets are over-the-counter, we find that asset prices depend on the amount of liquidity that is available in the market. Interestingly, the amount of liquidity that is available depends on the ex post distribution of (DM) consumption opportunities: individuals who do not find opportunities are willing to provide liquidity to those that do. If few consumption opportunities are available, then the supply
of liquidity will be high and so will asset prices. If money becomes more costly to hold because inflation increases, agents will hold less real balances. This implies higher inflation is associated with in lower levels of liquidity available in the OTC financial market, and will result in lower asset prices. This is in contrast to the standard Mundell-Tobin effect, where an increase in inflation implies that asset prices will increase. Since our financial markets are over-the-counter, a bid-ask spread emerges as long as the agent who operates the market, the dealer, has some bargaining power. If liquidity in financial market changes because of changes in inflation or investors’ investment opportunities, bid-ask spreads are negatively correlated with asset returns.

8 References


Appendix

In this appendix we provide a proof for the existence and uniqueness of equilibrium.

**Proposition 2** (i) In the large asset case, \( \tilde{a} \geq \tau^b(\mu) \), an equilibrium exists and is unique.
(ii) In the small asset case, \( \tilde{a} < \tau^b(\mu) \), an equilibrium exists. The equilibrium is unique if \( dz/d\mu < 0 \); a sufficient condition for uniqueness is \((d\tilde{p}^b/p^b)(dz/z) \leq 1\).

**Proof.** (i) Denote right side of (32) by \( \Lambda(z) \). The Inada conditions imply that \( \lim_{z \to 0} \Lambda(z) = \infty \) and \( \lim_{z \to \infty} \Lambda(z) = -\sigma_D \int_0^1 \sigma f(\sigma) d\sigma < 0 \). From (47) we have \( d\Lambda/dz < 0 \). Since \( \Lambda(z) \) is continuous, there is a unique \( z > 0 \) that solves \( \Lambda(z) = i = \mu/\beta - 1 \geq 0 \). Given such \( z \), (34) and (35) uniquely determine \( \tau^b \) and \( p^b \), respectively, (3) and (4) uniquely determine \( \rho^a \) and \( \tau^a \), respectively, (40) uniquely determines the level of DM output \( q \), and (41) uniquely determines the market clearing price of money, \( \phi_i \).

(ii) Denote the right side of (33) by \( \Gamma(z) \). The Inada conditions imply that \( \lim_{z \to 0} \Gamma(z) = \infty \) and \( \lim_{z \to \infty} \Gamma(z) = -\sigma_D \int_0^1 \sigma f(\sigma) d\sigma - \sigma_D \int_1^\infty \sigma f(\sigma) d\sigma = -\sigma_D \int_0^1 \sigma f(\sigma) d\sigma < 0 \). Since \( \Gamma(z) \) is continuous, there exists at least one \( z > 0 \) that solves \( \Gamma(z) = i = \mu/\beta - 1 \). Given such a \( z \), (36) and (37) uniquely determine \( \tau^b \) and \( p^b \), respectively, (3) and (4) uniquely determine \( \rho^a \) and \( \tau^a \), respectively, (40) uniquely determines the level of DM output \( q \), and (41) uniquely determines the market clearing price of money, \( \phi \). If \( dz/d\mu < 0 \), then \( \Gamma'(z) < 0 \), and there
exists a unique $z$ that solves $\Gamma(z) = i = \mu/\beta - 1 \geq 0$. A sufficient condition for uniqueness is that $(d\bar{p}/\bar{p})(dz/z) \leq 1$, i.e., if $(d\bar{p}/\bar{p})(dz/z) \leq 1$, then $dz/d\mu < 0$. ■

Remark 3 For utility function (50), we were unable to find a set of parameters for which $dz/d\mu > 0$. From this, we conclude that there exists a large set of parameters for which the equilibrium is unique for the small asset case.