On the Robustness of the Competitive Equilibrium: Utility-Improvements and Equilibrium points

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Current Version: March 7, 2014

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Abstract: We show that the set of competitive equilibrium points of a pure exchange economy are the equilibrium points of a broader class of better-response demands than the usual utility-maximizing demand functions. The better-response demands are derived from assigning weights to all commodity bundles with higher utility than the current commodity bundle, with the greatest weights being placed on the commodity bundles with the highest utility gain. The usual utility-maximizing demand functions are then those in which the weight on the utility-maximizing bundle is one. We also show that these better-response demands belong to a large class of response maps that are generated by monotonic transformations of the utility functions and/or monotonic transformations of the weights assigned to the commodity bundles.

Keywords: Competitive equilibrium, Nash map, bounded rationality, general equilibrium, pure exchange economy, Abstract economy

JEL Classification Numbers: Primary C71, C62, D51. Secondary D01.

1Earlier versions of this paper were presented at the Midwest Theory and International Trade Conference at Ohio State University, October 2008, the Microeconomics Workshop of the Economics Department, Purdue University, November 2008, the Seminar Series at the Center for Economic Studies and Planning, Jawaharlal Nehru University, July 2009 and the NSF/NBER Conference on General Equilibrium and Mathematical Economics, University of Iowa, October 2011. The author thanks the participants at these conferences for their comments. The author would also like to thank Marcus Berliant, Bernard Cornet, Will Geller, Aditya Goenka, Anjan Mukherji, William Novshek, Cheng Zhong Qin, Herakles Polemarchakis, Iryna Topolyan, Myrna Wooders and Nicholas Yannelis for their comments and suggestions but is especially indebted to an anonymous referee for very detailed comments. The usual disclaimer applies.

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1 Introduction

As is well known, the competitive equilibrium or the Walrasian equilibrium prices are prices at which the market clears (that is, prices at which the excess demand is zero) when economic agents respond to market prices by choosing the utility maximizing bundle of goods and services. The inference that can then be made is that the competitive equilibrium allocation of goods and services can prevail only if all agents trade their optimal or utility maximizing bundle at the market prices. Little has been said about what would happen if some of the economic agents respond to market prices by demanding bundles different from the utility-maximizing bundles. One is then led to conclude that in such cases the market would be in disequilibrium, or, if the deviations from utility maximizing behavior are not too large then trade would take place at approximate equilibrium market prices.

What we show here is that if the economic agents use a demand function that belongs to a class of better-response demand functions, then the market clearing prices are exactly the same as the competitive equilibrium prices. More precisely we show that if consumer $i$ offers to trade commodity bundles like

$$d_i(x^i, p) = \frac{m^i(x^i) + \int_{\gamma_i(p, p, \omega^i)} \phi^i(x^i, a) da}{1 + \int_{\gamma_i(p, p, \omega^i)} \phi^i(x^i, a) da}$$

when the market price is $p$ and the preceding commodity bundle offered for trade is $x^i$, then the resulting market clearing prices are the competitive equilibrium prices. Here $\omega^i$ is the endowment vector of agent $i$ and $\gamma_i(p, p, \omega^i)$ is the tradable boundary of the budget set of consumer $i$ given the endowment $\omega^i$ and the price vector $p$. $m^i(x^i)$ is the bundle in $\gamma_i(p, p, \omega^i)$ that is closest to the bundle $x^i$, and the function

$$\phi^i(x^i, a) = \max\{0, [u^i(a) - u^i(x^i)]\}$$

is the gain in utility from bundle $a$ over bundle $x^i$. Note that this demand function can be written as

$$d_i(x^i, p) = \beta(m^i(x^i))m^i(x^i) + \int_{\gamma_i(p, p, \omega^i)} \beta(x^i, a) da,$$

that is, as a weighted average of the bundle $m^i(x^i)$ and the bundles that have higher utilities than $x^i$, where the weight on bundle $a$ is $\beta(x^i, a) = \frac{\phi^i(x^i, a)}{1 + \int_{\gamma_i(p, p, \omega^i)} \phi^i(x^i, a) da}$ and the weight on the bundle $m^i(x^i)$ is $\beta(m^i(x^i)) = \frac{1}{1 + \int_{\gamma_i(p, p, \omega^i)} \phi^i(x^i, a) da}$. The first thing to note about this better-response demand function $d_i$ is that it is not the consumer’s utility maximizing bundle, but only a weighted average of all the bundles that are in the boundary of the budget set $\gamma_i(p, p, \omega^i)$ of the consumer and which have higher utilities.
than the bundle $m^i(x^i)$. The other significant feature of this demand function is that the weights $\beta(x^i, a)$ of the commodity bundles $a$ that are preferred to $x^i$ increase with the utility gain from the bundles $a$, the largest weight being on the utility-maximizing bundle.

One interpretation of these weights could be that the weights are the probabilities with which an agent chooses a particular bundle. As the weights are larger for commodity bundles with higher utility gains, a bundle with a higher utility gain has a greater probability of being chosen. That an agent does not immediately choose a utility maximizing bundle may then be due to errors that the agent makes in choosing the commodity bundle. Since the probability of choosing a commodity bundle with a larger utility gain is higher, an agent then makes a less costly error (that is, a smaller loss in utility from the maximum possible) with a higher probability than a more costly error (that is, a larger loss in utility from the maximum possible). Thus, although economic agents may be prone to making errors, they make more costly errors with a smaller probability and less costly errors with a higher probability$^3$.

It is useful to note here that the result presented here is quite different from the observation that perturbations of the utility maximizing choices give us approximate equilibrium points, and that as the perturbations go to zero, the approximate equilibrium points converge to exact equilibrium points; where such convergence is guaranteed by the upper semi-continuity of the equilibrium correspondence. The equilibrium points resulting from consumers using the better-response demand functions $d_i(x^i, p)$ are exact equilibrium points. This is thus distinct from studies, mostly in game theory, that look at the outcomes of games when players do not fully maximize utility because of bounded rationality. In [9] and [10], players observe utilities with some errors which can then affect their utility maximizing choice; the error in observing the true payoff can lead to non-utility maximizing choices. Such choices can lead to payoffs that are actually less than the current payoff of the player. In [5] players retain a level of bounded rationality and never quote a utility maximizing bundle unless the bounded rationality goes to zero. An alternative approach to bounded rationality is to use the concept of control costs as in [8]. It is another explanation as to why a decision maker may not necessarily quote his or her utility maximizing bundle. In all these alternative approaches the “equilibrium” could be different from the equilibrium obtained when players or agents maximize utility. In our case, the equilibrium is the utility-maximizing equilibrium. This is so because in our case players or agents get to make better choices until an equilibrium is reached as

$^3$This interpretation of how agents respond echoes some of the observations made in game theory. For example, trembling hand perfect equilibrium[22] considers equilibrium points that are robust to “trembles” or errors made by the agents. The concept of “proper” equilibrium [15] is even more closely connected with our interpretation here, as it requires equilibrium points to be robust to trembles in which players make more costly errors with lower probabilities and less costly errors with higher probabilities.
consumers (or the agents or players), while choosing a non-utility maximizing bundle, never make a choice that is “less” preferred to a choice that gives them the same utility as their current choice.

A point worth noting about the better-response demand functions $d_i$ is that these demand functions seem to describe behavior that is distinct from just satisficing. An example of a demand function that would indicate that agents were just satisficing is one in which an agent takes a simple average of all the bundles that are better than the one currently held by the agent. Demand functions generated by such satisficing behavior do not, in general, give equilibrium points that are competitive equilibrium points. The difference between behavior that is represented by the better-response demand functions $d_i$ and general satisficing behavior is that the demand functions $d_i$ result from agents placing increasingly larger weights on bundles with higher utility gains. In fact, as we show, any system of weighting of the commodity bundles that are monotonically increasing in the weights $\beta(x^i, a)$ will generate demand functions that result in equilibrium points that are competitive equilibrium points.

Better-response functions like the better-response demand functions being discussed here have been used in the game theory literature. In proving the existence of a Nash equilibrium in [16] Nash used a better-response function distinct from the best response map that appears in [17]. Nash [18] offered a refined version of the existence result in [16] that used the Brouwer Fixed Point Theorem and showed that the better response map’s fixed points are Nash equilibria, and vice versa. More recently, Becker and Chakrabarti [3] showed that Nash’s better-response function belongs to a class of better-response maps whose fixed points are Nash equilibrium points and extend the results in [16] and [18] to games with a continuum of actions and to games with some forms of non-expected utilities. The interesting feature of the map that is used by Nash in [16] and [18], and which is used in [3], is that this map is a weighted average of the responses that are better than the current strategy; the weights being increasing functions of the gains in payoffs. This better-response function, adapted to the environment of a market and to that of the Abstract Economy associated with that market, gives the better-response demand functions that we discuss here.

As we know, in his seminal papers on the existence of a competitive equilibrium, Debreu [6] uses the concept of an abstract economy and social equilibrium to establish the existence of a competitive equilibrium. The abstract economy is a pseudo-game in which the choice sets of the participants depends on the choices made by all the

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4Such demand functions can be discontinuous and may not generate any equilibrium points at all.

5Nash proves the existence of an equilibrium for games with finite actions and expected utilities.

6Debreu [6] uses a generalized version of Nash’s existence theorem to prove the existence of a social equilibrium and shows that a social equilibrium of a properly defined Abstract Economy is a competitive equilibrium.
participants, unlike that in a game in which the choice sets are independent of the actions of the players. Results for an Abstract Economy can thus be used to establish results for games and conversely.

We use these observations and the fact that there is a close connection between the social equilibrium of the Abstract Economy and the competitive equilibrium to show that the competitive equilibria are the rest points of a market process, in which, the demand functions are not necessarily the utility-maximizing bundles, but better-response demand functions that are the weighted average of all those commodity bundles that have higher utilities than the current bundle. Thus if buyers and sellers respond to the prevailing price so as to buy or sell a basket of goods that is “better” than the one they have, but not necessarily the best, primarily because they may have made errors in choosing the utility maximizing bundle, then not only is there some kind of an equilibrium, it happens to be the competitive equilibrium. This would then imply that the competitive equilibrium is the rest point of market processes in which the economic agents use response maps that belong to a much wider class of response maps than the usual classical demand function.

We work with a pure exchange economy and show that if buyers and sellers as well as the market participant use better-response functions as in [16] and [18] and in [3], then the fixed point of these maps is a social equilibrium of the Abstract economy and, hence, a competitive equilibrium. We also show that if the preference orderings, in addition to being convex, are also monotonic, then the set of competitive equilibrium points and the fixed points of the better-response functions are identical; that is, every fixed point of the better-response function is a competitive equilibrium and every competitive equilibrium is a fixed point of the better-response function. We also show that although the better-response demand functions can change with the utility function, the set of fixed points do not. This is also the case if we take monotonic transformations of the weights assigned by the consumers to commodity bundles that are better than the one they have. Thus an equilibrium is not just the fixed points of the better-response functions of the kind used by Nash in [16] and [18], but of an entire class of them. The competitive equilibrium can thus be viewed not just as an equilibrium that results from consumers behaving as perfect utility-maximizers, but also more generally as the rest point of a market process in which consumers and agents choose consumption bundles that are better than the bundles they have, but not necessarily the best. A similar theme is also pursued in a recent paper by Toda (2013) in [24]. In his case agents do not maximize utility but choose randomly from a set of commodity bundles where the choice set or the “offer” set is determined by the price vector and the average demand. This offer set is updated on the basis of the observed price vector and the updated average demand. One of the results show that if the set of Bayesian General equilibria of the economy coincides with the set of Walrasian equilibria if certain regularity conditions hold. One of these regularity conditions is that the offer set be always a subset of the set of commodity
bundles with higher utility than the current bundle\textsuperscript{7}. This regularity condition thus has the same flavor as that of the better response sets, albeit with the difference that the weights on the commodity bundles are random in [24]. Such results are useful as they provide a much broader foundation for the use of the General Equilibrium Framework and the Market Equilibrium as the most plausible outcome of the market process.

The results here also have some similarity to the work of, say, Becker [2] in which it is argued that a wide range of non-utility maximizing behavior is consistent with a downward-sloping demand curve. In fact a relatively informal discussion shows that as long as consumers seek gains in utility the demand curve is likely to be downward sloping. Experimental results on market equilibrium also seem to be consistent with the theoretical results here. In Smith’s [25] experimental study of markets, a buyer and/or a seller calls out a price at which the trader is willing to trade but the prices called out are not necessarily utility-maximizing trades (indeed it would be hard to verify this), but presumably trades that lead to gains in surplus, much like the trades here that lead to gains in utility or the surplus of the agent. While the results here have interesting connection to partial equilibrium analysis it should be pointed out that the results here go well beyond observations in partial equilibrium analysis and indicate that broad forms of rational behavior are consistent with the competitive equilibrium. The analytical framework used is of general equilibrium theory and the results are focused on the competitive equilibrium.

2 The better-response function and the Social Equilibrium of an Abstract Economy

Consider an abstract economy given by a set of \( N \) players with \( X^j \) as the strategy set of player \( j \) where \( X^j \) is a subset of an Euclidean space\textsuperscript{8}. Let \( X := \prod_{j \in N} X^j \) and \( F^j : X \to X^j \) denote the feasible strategy correspondence of player \( j \). For each \( x \in X \), \( F^j(x) \) gives the set of strategies that are feasible for player \( j \). Each player \( j \in N \) has a payoff function or utility function \( u^j : \text{gr}F^j \to \mathbb{R} \). We note that since \( u^j \) is a function of \((x, x^j) \in X \times X^j\), the payoff of player \( j \) depends on the actions of all the players. An Abstract Economy is thus given by \( \{X^j, F^j, u^j\}_{j \in N} \).

**Definition 1** A Social Equilibrium of an abstract economy \( \{X^j, F^j, u^j\}_{j \in N} \) is an \( n-\)

\textsuperscript{7}While it could be argued that allowing agents to choose in such a wide variety of manner is ad-hoc, the fact that even under such broad behavioral norms the Walrasian equilibrium remains the central equilibrium concept, should be viewed as a remarkable observation about its relevance.

\textsuperscript{8}Much of the subsequent analysis can be done by assuming a more general space for \( X^i \). This certainly seems to be true if, for instance, \( X^i \) is a Hilbert space. We have focused here on establishing the results for the Euclidean space leaving the generalizations for future work.
tuple of strategies $x^* := (x_1^*, \ldots, x_n^*) \in X$ such that for every $j \in N$,
1) $x_j^* \in F^j(x^*)$, and
2) $u^j(x^*, x_j^*) \geq u^j(x^*, \xi_j)$ for all $\xi_j \in F^j(x^*)$. 

Thus, a social equilibrium is the exact analog of the Nash equilibrium for a pseudo game.

It should be noted here that the definition of the social equilibrium is a little different than the original definition in Debreu [6]. There, the feasible correspondence depends on $X_{-i} = \prod_{j \neq i} X^j$ so that the feasible correspondence depends only on the “current” action of the other players. In our case the feasible correspondence depends on $x \in X$ that is on the “current” action of the other players as well as the “current’ action of agent $i$ and maps the current actions into the set of feasible actions of agent $i$. The utility thus depends on the current action of all the agents and the new action of agent $i$. If the new action $y_i \in F^i(x)$ of agent $i$ is the same as the current action $x_i$ then the utility of agent $i$ remains unchanged.

Let

$$\phi^i(x, a) = \max \{0, [u^i(x, a) - u^i(x, x^i)]\}$$

for $x \in X$ and $a \in F^i(x)$. Let $\hat{x}^i$ be defined as

$$d(\hat{x}^i, x^i) = \min_{y \in F^i(x)} d(y, x^i)$$

when $x^i \not\in F^i(x)$. Define the map $m^i : X \to X^i$ as

$$m^i(x^i) = \begin{cases} 
  x^i & \text{if } x^i \in F^i(x) \\
  \hat{x}^i & \text{if } x^i \not\in F^i(x) 
\end{cases}$$

We note that when the correspondence $F^i : X \to X$ is a continuous correspondence that is compact-valued and convex-valued, the map $m^i : X^i \to X^i$ is a continuous function that is the identity map when $x^i \in F^i(x)$. The convexity and compactness of $F^i(x)$ gives the uniqueness of $\hat{x}^i$ and the continuity of $F^i$ then implies that the function $m^i : X^i \to X^i$ is continuous$. We now define the function $b_i : X \to X^i$ as

$$b_i(x) := \begin{cases} 
  \frac{x^i + \int_{F^i(x)} \phi^i(x, a) da}{1 + \int_{F^i(x)} \phi^i(x, a) da} & \text{if } x^i \in F^i(x) \\
  \frac{\hat{x}^i + \int_{F^i(x)} \phi^i(x, a) da}{1 + \int_{F^i(x)} \phi^i(x, a) da} & \text{if } x^i \not\in F^i(x) 
\end{cases}$$

Or simply as

$$b_i(x) = \frac{m^i(x^i) + \int_{F^i(x)} \phi^i(x, a) da}{1 + \int_{F^i(x)} \phi^i(x, a) da}.$$ 

$9$These observations depend quite critically on the fact that $X^i$ is a subset of an Euclidean space. If $X^i$ was a subset of a more general space then these assertions may not always hold.
It is worth noting that $X^i$ is usually a multidimensional space and the element $a \in F^i(x) \subset X^i$ is a vector. Thus when $X^i \subset \mathbb{R}^n$ then $F^i(x)$ is a subset of a Euclidean space and the element $a \in F^i(x)$ is an $n$-dimensional vector $(a_1, \ldots, a_n)$. In such a case the integral in the numerator of the function $b_i$ is

$$\int_{F^i(x)} \phi^i(x, a) da = (\int_{F^i(x)} \phi^i(x, a) da_1, \ldots, \int_{F^i(x)} \phi^i(x, a) da_n)$$

which gives us an $n$-dimensional vector and so the numerator of the map $b_i(x)$ is the sum of two $n$-dimensional vectors. The denominator of the map $b_i$ is a scalar in every case and is an indicator of how much utility can be gained on average by improving on the point $x \in X^i$.

It is important to observe here that in the definition of $b_i(x)$, the integrals $\int_{F^i(x)} \phi^i(x, a) da$ and $\int_{F^i(x)} \phi^i(x, a) da$ are with respect to the Lebesgue measure on $\mathbb{R}^k$, and it is assumed that $F^i(x)$ has the same dimension as $\mathbb{R}^k$. We show later that we can relax this condition and can define $b_i(x)$ more generally even when the dimension of $F^i(x)$ is smaller than the dimension of $\mathbb{R}^k$ or smaller than the dimension of $X^i$. In that case the integral is taken with respect to the Lebesgue measure of the subspace of $\mathbb{R}^k$ in which $F^i(x)$ has full dimension.

The function $b_i : X \rightarrow X^i$ gives a strategy of player $i$ that is a weighted average of the strategies in $F^i(x)$ which have higher utility than the strategy $x^i$ of player $i$, when the strategy combination is $x \in X$. More specifically the map $b_i$ gives a weighted average of the strategies that lead to gains in utility, with larger weights being assigned to strategies that lead to larger gains in utility. This map is a generalization of the map that Nash used in [16] and [18] to prove the existence of the Nash equilibrium. In Becker and Chakrabarti [3] this map is referred to as the better response map. Notice that if $x^i$ is the best response to $x \in X$, then $b_i(x) = x^i$ as $\phi^i(x, a) = 0$ for all $a \in F^i(x)$. Further note that if $x^i \not\in F^i(x)$ then the map picks up the strategy that is closest to $x^i$ namely $\hat{x^i}$ and then takes a weighted average of those strategies that give higher utilities than $x^i$. If none of the strategies in $F^i(x)$ give higher utility than $x^i$ then the map $b_i : X \rightarrow X$ simply gives the strategy $\hat{x^i}$.

**Lemma 1** If $u^i : grF^i \rightarrow \mathbb{R}$ is continuous and $u^i(x, .)$ is quasi-concave on $F^i(x)$ for every $x \in X$, then $\phi^i : grF^i \mapsto \mathbb{R}_+$ is continuous and $\phi^i(x, .)$ is quasi-concave on $F^i(x)$ for every $x \in X$.

**Proof:** This follows in exactly the same way as in lemma 3 of Becker and Chakrabarti [3].

The next lemma proves some useful continuity properties of the better-response function.
Lemma 2 If for each \( i = 1, \ldots, n \), \( X^i \) is a compact and convex subset of a Euclidean space and the correspondences \( F^i(.) \) are upper semi-continuous, lower semi-continuous and nonempty-valued, compact-valued and convex-valued, then the better-response function \( b_i \) is continuous.

The proof of the lemma is given in the appendix.

In proving theorem 1 we had assumed that the interior of \( F^i(x) \) which we will denote by \( \text{int} F^i(x) \) is non-empty as otherwise \( \int_{F^i(x)} \phi^i(x, a)ada \) would be zero even if \( F^i(x) \neq \emptyset \) and \( \phi^i(x, a) > 0 \) for all \( a \in F^i(x) \). In case the dimension of \( F^i(x) \), denoted by \( \text{dim} F^i(x) \), is less than \( n \), the arguments in theorem 1 are no longer valid as \( B(x) \) will not have a nonempty interior. In these cases, if \( F^i(x) \) is a manifold of dimension \( n - 1 \) or less, then the integral

\[
\int_{F^i(x)} \phi^i(x, a)ada 
\]

can then be defined as an integral over the manifold \( F^i(x) \), see for example Spivak[23] and Munkres[14]. We will restrict attention to manifolds \( M \) such that \( M \) has a parametric representation \( G : A \to \mathbb{R}^n \) that belongs to class \( C^1 \), where \( A \subset \mathbb{R}^m \) is a contented subset of \( \mathbb{R}^m \), \( m \leq n \) and \( x = G(t_1, \ldots, t_m) \) for \( x \in M \). We then recall that if \( f : M \to \mathbb{R} \) is a function over the manifold \( M \), then the integral of the real-valued function \( f \) over the manifold \( M \) is given by

\[
\int_M f dvol_m = \int_M f(G(t)) \sqrt{\text{Gram}(\frac{\partial G}{\partial t_1}, \ldots, \frac{\partial G}{\partial t_m})} dt \quad (3)
\]

where

\[
\text{Gram}(\frac{\partial G}{\partial t_1}, \ldots, \frac{\partial G}{\partial t_m}) = \det(DG)^t DG
\]

with \( DG \) being the Jacobian matrix of \( G \) and is given by

\[
DG = \begin{bmatrix}
\frac{\partial G_1}{\partial t_1} & \cdots & \frac{\partial G_1}{\partial t_m} \\
\vdots & \ddots & \vdots \\
\frac{\partial G_n}{\partial t_1} & \cdots & \frac{\partial G_n}{\partial t_m}
\end{bmatrix}
\]

Therefore, if \( F^i(x) \) is the \( m \)-dimensional manifold \( M^i(x) \), and \( G(x, t) \) is the parametric function over the manifold \( M^i(x) \), then

\[
\int_{F^i(x)} \phi^i(x, a)ada = \int_{M^i(x)} \phi(x, G(x, t)) . G(x, t) \sqrt{\det(DG(x, \cdot)^t)(DG(x, \cdot))} dt \quad (4)
\]

The next result shows that if the parametric representation \( G^i(x, t) \) of the manifold has sufficient continuity then the the better response function \( b_i \) is continuous.
Lemma 3 For each $i$ let the feasible sets $F^i(x)$ of agent $i$ be given by manifolds $M^i(x)$. Let $A_i : X \to \mathbb{R}^m$ be a continuous and compact-valued correspondence and let $G^i : \text{Gr} A_i \to \mathbb{R}^n$ give the parametric representation of $M^i(x)$ so that $G^i(x, A_i(x)) = M^i(x)$ for each $x \in X$ and $G^i(x, \cdot)$ is of class $C^1$. If $G^i(x, t)$ is jointly continuous on $\text{Gr} A_i$ then the better response function $b_i$ is continuous.

Proof: We note that if $G^i : \text{Gr} A_i \to \mathbb{R}^n$ is continuous then $\det(DG^i(x, \cdot)^t)(DG^i(x, \cdot))$ is continuous in $x$. Further, the correspondence that maps $x \mapsto M_i(x)$ is a continuous correspondence. Hence,

$$\int_{M^i(x)} \phi(x, G^i(x, t)) G^i(x, t)\sqrt{\det(DG^i(x, \cdot)^t)(DG^i(x, \cdot))} dt$$

is a continuous function of $x$. Therefore, $b_i : X \to X_i$ as defined in (2) is a continuous function. \qed

In cases in which $F^i(x)$ is a manifold of dimension less than $n$, one has to consider $F^i(x)$ as a subset of the affine hull of $F^i(x)$. The relative interior of $F^i(x)$, denoted by $ri\{F^i(x)\}$, is then no longer empty even when $\dim F^i(x) < n$ (see for example Part II, section 6 of [20])\(^{10}\). One can now show that the result in theorem 1 will also hold for the case when $F^i(x)$ is a manifold. The result follows in almost the same manner as in theorem 1.

Corollary 1 For each $i$ let the feasible sets $F^i(x)$ of agent $i$ be given by manifolds $M^i(x)$. Let $A_i : X \to \mathbb{R}^m$ be a continuous and compact-valued correspondence and let $G^i : \text{Gr} A_i \to \mathbb{R}^n$ give the parametric representation of $M^i(x)$ so that $G^i(x, A_i(x)) = M^i(x)$ for each $x \in X$ and $G^i(x, \cdot)$ is of class $C^1$. If $G^i(x, t)$ is jointly continuous on $\text{Gr} A_i$ then the better response map has a fixed point and this fixed point is an equilibrium of the Abstract Economy.

Proof: From lemma 3 it follows that $b$ defined as $b = (b_1, \cdots, b_n)$ has a fixed point. One can now apply the arguments in theorem 1 to complete the proof after noting that the separation theorem is applied to $x^*$ and $ri\{B_i(x^*)\}$. \qed

3 The Competitive Equilibrium and the better-response demand

In [1], the existence of a competitive equilibrium of a private ownership economy is proved by showing that the social equilibrium of the Abstract Economy generated by the private

\(^{10}\)This idea is also used in [3] in section 2.1. There the feasible set is the set of probabilities on a finite set and has Lebesgue measure zero in $\mathbb{R}^{m+1}$. The feasible set is then redefined as a subset of $\mathbb{R}^m$.\]
ownership economy is a competitive equilibrium. In theorem 1 we showed that the fixed points of the better-response function is a social equilibrium of the abstract economy. This observation leads to the next result. We start by first describing a pure exchange economy with $L$ commodities and $m$ consumers. A pure exchange economy is given by $\{X^i, \preceq_i, \omega^i\}_{i=1}^m$, where for each consumer $i = 1, \cdots, m$, the set $X^i$ is the consumption set of consumer $i$; it is a closed and convex subset of the Euclidean space $\mathbb{R}^L$. $\preceq_i$ is the preference ordering of consumer $i$ on $X^i$. $\omega^i$ is the endowment vector of consumer $i$. For each consumer $i$ let $\gamma_i : \Delta^L \times \mathbb{R}^L \to X^i$ defined as $\gamma_i(p, \omega^i) := \{x^i \in X^i | p.x^i \leq p.\omega^i\}$ denote the consumer’s budget correspondence, where $\Delta^L$ is the unit simplex in $\mathbb{R}_+^L$. A competitive equilibrium of a pure exchange economy is then defined as follows.

**Definition 2** The competitive equilibrium of a pure exchange economy is given by $((x^i)^{m}_{i=1}, p^*)$ where $(x^i)^{m}_{i=1} \in X$ and $p^* \in \Delta^L$ such that
(i) $x^*$ is a maximal element of $\{x_i \in X_i | p^*x_i \leq p^*\omega_i\}$ with respect to the preference ordering $\preceq_i$ of consumer $i$ for every $i$, and
(ii) $\sum_{i=1}^m x^i \leq \sum_{i=1}^M \omega_i$.

In Debreu [6] and [11] the existence of a competitive equilibrium is proved by showing that a social equilibrium exists for the Abstract economy generated by the pure exchange economy, and that a social equilibrium of the Abstract economy is a competitive equilibrium of the pure exchange economy. In the Abstract economy in Debreu [6] and in Ichishi [11] the feasible set $F^i : X_{-i} \to X_i$ maps the actions of the other agents into the action set of player $i$, and the feasible set of an agent $i$ in the Abstract economy of the pure exchange economy is given by

$$F^i(x_{-i}, p) = \gamma_i(p, p\omega_i) = \{x_i \in \mathbb{R}^L | px_i \leq p\omega_i\}.$$ 

**Existence of a Competitive Equilibrium** Let $\{X^i, \preceq_i, \omega^i\}_{i=1}^m$ be a pure exchange economy with $L$ commodities and $m$ consumers. For each consumer $i$, let $X^i$ be a nonempty, closed and convex subset of $\mathbb{R}^L$, the preference ordering $\preceq_i$ be complete, transitive, closed and convex, and $\min\{p.x^i | x^i \in X^i\} < p.\omega^i$ for all $p \in \Delta^L$. Then there exists a social equilibrium for the Abstract economy of the pure exchange economy and it is a competitive equilibrium of the pure exchange economy.

It should be noted that the conditions of the existence theorem as stated here are different from the conditions in Arrow and Debreu [1], chapter 4. There Arrow and Debreu prove the existence of a competitive equilibrium for a private ownership economy under the assumption that the preference orderings of the consumers are convex and that there are no saturation points. This version of the result on the existence of a competitive
equilibrium does not assume the that there are no saturation points. In Debreu [6] as well as in [11], the existence of a social equilibrium is proved using the best reply correspondence. As a result, in [6] and [11] the existence of a competitive equilibrium is ultimately proved by using the utility-maximizing demand correspondence of the consumers. The competitive equilibrium according to that approach is thus the rest point of a market process in which the consumers respond to market prices by choosing the utility-maximizing commodity bundles. If one then assumes as in [6] that there are no saturation points then one gets the condition that the competitive equilibrium allocation \((x \ast i\ast)_{i=1}^{m}\) satisfies the condition that \(p\ast x\ast i = p\ast \omega_i\) rather than just \(p\ast x\ast i \leq p\ast \omega_i\).

Given a pure exchange economy, the abstract economy is constructed as follows. Consider the \(m\) consumers as \(m\) players and the market as the \((m+1)^{th}\) player. For \(i = 1, \cdots, m\), the consumption sets \(X^i\) of the consumers are the strategy sets of the \(m\) players. The strategy set of player \(n\) is \(X^n := \Delta^L\), which is the unit simplex in \(\mathbb{R}_+^L\). The idea is that the market participant chooses the price vector in the market and all the price vectors are in \(\Delta^L\). If the preference ordering \(\preceq_i\) is complete, transitive, closed and convex then the preference ordering can be represented by a continuous and quasi-concave utility function \(v^i : X^i \to \mathbb{R}\) for all the players \(i = 1, \cdots, m\). The abstract economy generated by the pure exchange economy is \(\{X^i, u^i, F^i\}_{i=1}^{n}\) where the utility functions are \(u^j((x^j)_i, p), y) = v^j(y)\) for players \(j = 1, \cdots, m\) and \(u^n(((x^n)_i, p), q) = q \cdot \sum_{i=1}^{m}(x^i - \omega^i)\) for player \(n\).

Let \(\bar{\gamma}_i(p, p, \omega^i) = \{z_i \in X^i | p, z_i = p, \omega^i\}\). Thus the correspondence \(\bar{\gamma}_i\) maps prices and commodity bundles into the set of commodity bundles that can be purchased at the price vector \(p\), given the endowment \(\omega^i\)\(^{11}\). Thus, for players \(i = 1, \cdots, m\), the feasible strategy correspondences \(F^i(.)\) of the abstract economy are given by the tradable boundary of the budget set along the hyperplane defined by the prevailing price vector along which goods can be traded given the prevailing price vector. Thus \(F^i((x^i)_i, p) = \bar{\gamma}_i(p, p, \omega^i)\)\(^{12}\), and for the market participant, that is player \(n\), it is given by \(F^n((x^n)_i, p) = \Delta^L\)\(^{13}\).

**Theorem 1** Consider a pure exchange economy \(\{X^i, \preceq_i, \omega^i\}_{i=1}^{m}\) with \(L\) commodities and

---

\(^{11}\)It is important to note at this point the implication of this construction. The commodity bundles that are traded are always on the boundary of the budget set and never in the interior of the budget set. This is consistent with the idea that consumers will trade at the prevailing prices and trade commodity bundles that are on the boundary of the budget set.

\(^{12}\)Note that the feasible set of consumer \(i\) is determined by the value of the endowment of consumer \(i\) at the prevailing price. The better response function \(b_i(.)\) is then interpreted as the “offer” made at the current price. This interpretation is consistent with the usual interpretation of an equilibrium price—it is the price at which trade finally takes place after “out of equilibrium offers” have led to a revision of prices until the equilibrium price has been reached.

\(^{13}\)The Abstract economy here is thus a little different from the Abstract economy described in [6] and [11]. The difference is that the feasible correspondence is restricted to the tradable boundary of the budget set and not the entire budget set \(\{x_i \in \mathbb{R}_+^L | px_i \leq p\omega_i\}\).

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m consumers. Let \( X^i \) be a nonempty, closed and convex subset of \( \mathbb{R}^L \), the preference ordering \( \preceq_i \) be complete, transitive, closed and convex, and \( \min \{ p.x^i | x^i \in X^i \} < p.\omega^i \) for all \( p \in \Delta^L \). In addition let the preference ordering \( \preceq_i \) satisfy local non-satiation. Then the better-response function of the Abstract economy has a fixed point, and each fixed point of the better-response function of the Abstract Economy economy is a competitive equilibrium of the pure exchange economy.

**Proof:** Let \( c > 0 \) be such that \( \sum_{i=1}^{m} \omega_i < (c, c, \ldots, c) = c \). Then define \( X^c_i = X^i \cap [0, c]^L \) for each \( X_i \). Then \( X^c_i \) is a nonempty and compact subset of \( R^L_+ \). Define \( X^c = \prod_{i=1}^{m} X^c_i \) and the map \( b_i^c : X^c \rightarrow X^c_i \) of player \( i \) for \( i = 1, \ldots, m \) in the abstract economy is

\[
b_i^c((x^i)_i, p) = \frac{m^i(x^i) + \int_{\gamma_i(p.p.\omega)} \phi^i(x^i, a)da}{1 + \int_{\gamma_i(p.p.\omega)} \phi^i(x^i, a)da},
\]

where

\[
\phi^i(x^i, a) = \max\{0, [u^i((x^i)_i, p), a) - u^i((x^i)_i, p), x^i)] = \max\{0, [v^i(a) - v^i(x^i)]\}.
\]

The map of player \( n \), the market participant, is

\[
b_n((x^i)_i, p) = \frac{p + \int_{\Delta^L} \phi^n((x^i)_i, p), q)dq}{1 + \int_{\Delta^L} \phi^n((x^i)_i, p), q)dq},
\]

where

\[
\phi^n((x^i, p)_i, q) = \max\{0, [u^n((x^i)_i, p), q) - u^n((x^i)_i, p), p)]\} = \max\{0, [v^i(a) - v^i(x^i)]\}.
\]

We now observe that the feasible sets \( F^i((x^i)_i, p) = \gamma_i(p, p.\omega) \) are \( L-1 \) dimensional manifolds and are given by a continuous and compact-valued correspondence on \( \Delta^L \). We further observe that for any \( p \in \Delta^L \) such that \( p >> 0 \), parametric representations are given by \( G^i(p, t) = \{ x \in \mathbb{R}^L_+ | x_\ell = t_\ell, \ell \neq k, x_k = \frac{p.\omega_k - \sum_{\ell \neq k} t_\ell}{p_k} \} \), where \( \{(t_\ell)_{\ell \neq k} \in \mathbb{R}^{L-1} | 0 \leq \sum_{\ell \neq k} t_\ell \leq 1, t_\ell \geq 0 \} \). The integral \( F^i(x) \) is then given by (14) and it can be seen that it is continuous for \( p >> 0 \). If \( p \) is in the Boundary of \( \Delta^L \), then there is some \( s \) for which \( p_s > 0 \). If \( \{p^n\} \) is a sequence in \( \Delta^L \) such that \( p^n \rightarrow p \), then for all \( n \) sufficiently large \( p^n_s > 0 \). One can then choose a parametrization \( G^i(p^n, t) = \{ x \in \mathbb{R}^L_+ | x_\ell = t_\ell, \ell \neq s, x_s = \frac{p^n.\omega_s - \sum_{\ell \neq s} p^n_\ell t_\ell}{p^n_s} \} \) for all \( n \) sufficiently large. The integrals \( \int_{\gamma_i(p^n, p^n.\omega)} \phi^i(x^i, a)da \) and \( \int_{\gamma_i(p^n, p^n.\omega)} \phi^i(x^i, a)da \) in (15) can then be written as in (14). These integrals converge.
Theorem 2

that we use in the next result are exactly the same as the conditions in [1] as well as [6].

additional assumptions this is indeed the case. Note that these additional conditions

a fixed point of a better-response function. The next result shows that under some mild

a market process in which the consumers respond to market prices by using the market

Proof:

petitive equilibrium of this pure exchange economy is a fixed point of the better-response

orderings

Finally, we show that these fixed points are the competitive equilibrium of the pure

Theorem 2 shows that the competitive equilibrium can be viewed as the rest point of

a market process in which the consumers respond to market prices by using the market

by

The better response maps \( b^i(\cdot) \) for \( i = 1, \ldots, n \) and \( b_n(\cdot) \) as well as the feasible correspondences \( F^i((x^i)_i, p) = \gamma_i(p, p, \omega^i) \) and \( F^n((x^i)_i, p) = \Delta^L \) satisfy all the conditions of Lemma 3 and corollary 1. Therefore, the better-response function \( b^c(\cdot) = (b^1_1(\cdot), \ldots, b^m_n(\cdot), b_n(\cdot)) \) has a fixed point \(( (x^*_i), p^* ) = (x^*, p^*) \). As \((x^*, p^*) = b^c(x^*, p^*) \) we have

for all \( i = 1, \ldots, m \).

We now observe that if \((x^*, p^*)\) is a fixed point of \( b^c(\cdot) \) then it is also a fixed point of

the original best response function \( b(\cdot) \). Note that if \( b(\cdot) \) has a fixed point then it must

satisfy the condition given in (17) so that every fixed point of \( b(\cdot) \) is in \( X^c \). But that

implies that every fixed point of \( b(\cdot) \) is a fixed point of \( b^c(\cdot) \). As it has been shown that

\( b^c(\cdot) \) has a fixed point therefore \( b(\cdot) \) has a fixed point.

Theorem 2 shows that the competitive equilibrium can be viewed as the rest point of

a market process in which the consumers respond to market prices by using the market

variant of the better-response function instead of the demand correspondence. This

result, however, does not tell us whether every competitive equilibrium can be found as

a fixed point of a better-response function. The next result shows that under some mild

additional assumptions this is indeed the case. Note that these additional conditions

that we use in the next result are exactly the same as the conditions in [1] as well as [6].

Theorem 2 Consider a pure exchange economy \( \{X^i, \preceq_i, \omega_i\}_{i=1}^m \) such that the preference

orderings \( \preceq_i \) of the consumers are convex and satisfy non-satiation. Then each competitive

equilibrium of this pure exchange economy is a fixed point of the better-response function of the Abstract

Economy generated by the pure exchange economy.

Proof: Let \(( (x^*_i), p^* )\) be a competitive equilibrium of the pure exchange economy given

by \( \{X^i, \preceq_i, \omega_i\}_{i=1}^m \). For any commodity bundles \( x, y \) let

\[ [v^i(y) - v^i(x)]^+ = \max\{0, (v^i(y) - v^i(x)) \} \]
For the market participant, player $n$, the better-response function, or the better-response price-adjustment function, $((x^i), p) \mapsto \hat{p}$ is given by

$$
\hat{p}((x^i), p) := \frac{p + \int_{\Delta L} [(q - p) \cdot \sum_{i=1}^{m} (d_i((x^i), p) - x^i)]^+ dq}{1 + \int_{\Delta L} [(q - p) \cdot \sum_{i=1}^{m} (d_i((x^i), p) - x^i)]^+ dq}.
$$

(9)

where

$$
[(q - p) \cdot \sum_{i=1}^{m} (d_i((x^i), p) - x^i)]^+ = \max \{0, (q \cdot \sum_{i=1}^{m} (d_i((x^i), p) - x^i) - p \cdot \sum_{i=1}^{m} (d_i((x^i), p) - x^i))\}.
$$

We need to show that $((x^{i^*}, p^*))$ is a fixed point of the maps described in (15) and (16). We first note that as the preference orderings of the consumers are convex and satisfies non-satiation, we have

$$
p^* \cdot x^{i^*} = p^* \cdot \omega^i
$$

for every consumer $i$ so that

$$
\tilde{\gamma}_i(p^*, p^*, x^{i^*}) = \tilde{\gamma}_i(p^*, p^*, \omega^i).
$$

(10)

From (17) it follows that

$$
d_i((x^{i^*}, p^*)) := \frac{m^i(x^{i^*}) + \int_{\tilde{\gamma}_i(p^*, p^*, \omega^i)} [v^i(y) - v^i(x^*)]^+ dy}{1 + \int_{\tilde{\gamma}_i(p^*, p^*, \omega^i)} [v^i(y) - v^i(x^*)]^+ dy}.
$$

(11)

and that $x^{i^*} \in \tilde{\gamma}_i(p^*, p^*, \omega^i)$. Since for every consumer $i$, $x^{i^*}$ maximizes $v^i(y)$ over the budget set $\gamma_i(p^*, p^*, \omega^i)$ (as $((x^{i^*}, p^*))$ is a competitive equilibrium), therefore, $[v^i(y) - v^i(x^{i^*})]^+ = 0$ for every $y \in \tilde{\gamma}_i(p^*, p^*, \omega^i)$. Therefore, from (18) it follows that the better-response function of every consumer $i$ will map $x^{i^*} \rightarrow x^{i^*}$.

From this and from (16) we now have

$$
\hat{p}((x^{i^*}, p^*)) := \frac{p^* + \int_{\Delta L} [(q - p^*) \cdot \sum_{i=1}^{m} (d_i((x^{i^*}), p^*) - x^{i^*})]^+ dq}{1 + \int_{\Delta L} [(q - p^*) \cdot \sum_{i=1}^{m} (d_i((x^{i^*}), p^*) - x^{i^*})]^+ dq}.
$$

(12)

$$
= \frac{p^* + 0}{1 + 0} = p^*.
$$
This completes the proof of the theorem. \qed

The following example illustrates these results in a simple two-person economy.

**Example 1** The better-response demand and the competitive equilibrium of a pure exchange economy with two goods and two consumers.

Consider a pure exchange economy in which there are two consumers 1 and 2 with utility functions given by \( u_1(x_1, y_1) = \sqrt{x_1y_1} \) and \( u_2(x_2, y_2) = \sqrt{x_2y_2} \). Their endowments are given by \( \omega_1(0, 1) \) and \( \omega_2 = (1, 0) \). For any two commodity bundles \((x_1, y_1)\) and \((a, b)\) of consumer 1 let

\[
[u_1(x_1, y_1) - u_1(a, b)]^+ = \max\{0, [u_1(x_1, y_1) - u_1(a, b)]\}.
\]

If we denote the price vector \((p_x, 1 - p_x)\) by \(p\) then the better-response demand function \(d_1\) that takes \((x_1, y_1, p)\) \(\mapsto\) \((x_1', y_1')\) is

\[
d_1(x_1, y_1, p) = \frac{m^1(x_1, y_1) + \int_{\gamma_1(p, p, \omega_1)}[u_1(a, b) - u_1(x_1, y_1)]^+(a, b)d(a, b)}{1 + \int_{\gamma_1(p, p, \omega_1)}[u_1(a, b) - u_1(x_1, y_1)]^+d(a, b)}.
\] (13)

As \(u_1(x, y) = \sqrt{xy}\), the better-response demand function \(d_1(x_1, y_1, p) = (x_1', y_1')\) can be written as

\[
(x_1', y_1') = \frac{m^1(x_1, y_1) + \int_{\{(a, b): a_{p_x} + b(1-p_x) = 1-p_x\}}[\sqrt{ab} - \sqrt{x_1y_1}]^+(a, b)d(a, b)}{1 + \int_{\{(a, b): a_{p_x} + b(1-p_x) = 1-p_x\}}[\sqrt{ab} - \sqrt{x_1y_1}]^+d(a, b)},
\]

for consumer 1. The better-response demand function \(d_2\) for consumer 2 is similar. For the market, if \(x' = x_1' + x_2'\) and \(y' = y_1' + y_2'\) then the better-response price-adjustment function is given by

\[
\hat{p}_x(p, x, y) = \frac{p_x + \int_{\{(q_1, q_2): q_1 + q_2 = 1\}}[(q - p), ((x' - 1), (y' - 1))]^+q_1d(q_1, q_2)}{1 + \int_{\{(q_1, q_2): q_1 + q_2 = 1\}}[(q - p), ((x - 1), (y - 1))]^+d(q_1, q_2)},
\] (14)

and

\[
\hat{p}_y = 1 - \hat{p}_x.
\]

Note that in the definition of the better-response demand functions, the integrals are taken over the tradable boundary of the budget set which is a one-dimensional manifold of \(\mathbb{R}^2\). If \(p_x < 1\) then a parametric representation of the set \(\tilde{\gamma}(p, p, \omega_1)\) is given by

\[
G(t) = (t, \frac{1 - p_x - tp_x}{1 - p_x})
\]
so that
\[ DG = \left[ \frac{1}{1-p_x} \right]. \]
Therefore, the integral
\[
\int_{\{(a,b):ap_x+b(1-p_x)=1-p_x\}} \left[ \sqrt{ab} - \sqrt{x_1 y_1} \right]^+ (a, b) d(a, b)
\]
reduces to
\[
\int_{0}^{1-p_x/p_x} \left[ \sqrt{\frac{t(1-p_x-tp_x)}{1-p_x}} - \sqrt{x_1 y_1} \right]^+ (t, \frac{1-p_x-tp_x}{1-p_x}) \sqrt{DG^t DG} dt
\]
\[
= \int_{0}^{1-p_x/p_x} \left[ \sqrt{\frac{t(1-p_x-tp_x)}{1-p_x}} - \sqrt{x_1 y_1} \right]^+(t, \frac{1-p_x-tp_x}{1-p_x}) \sqrt{1 + \frac{p_x^2}{(1-p_x)^2}} dt.
\]
Now let \((x_1, y_1) = \left( \frac{3}{4}, \frac{1}{4} \right)\) and the price-vector \((\frac{1}{2}, \frac{1}{2})\). Then
\[ u_1(x_1, y_1) = \sqrt{\frac{3}{4} \times \frac{1}{4}} = \sqrt{\frac{3}{16}}. \]
Therefore, the set of commodity bundles that give higher utility than the bundle \((\frac{3}{4}, \frac{1}{4})\) is
\[ \{(a, b) | \sqrt{a \times b} > \sqrt{\frac{3}{16}} \}. \]
Hence, in this case
\[
\int_{\gamma_1(p,p,\omega_1)} [u_1(a, b) - u_1(x_1, y_1)]^+ d(a, b) = \int_{\frac{3}{4}}^{\frac{3}{4}} \sqrt{t(1-t)} t \sqrt{2} dt
\]
and
\[
\int_{\gamma_1(x_1,y_1,p)} [u_1(a, b) - u_1(x_1, y_1)]^+ d(a, b) = \int_{\frac{3}{4}}^{\frac{3}{4}} \sqrt{t(1-t)} \sqrt{2} dt.
\]
Therefore, the better-response demand \(x_1'\) of consumer 1 for the commodity \(x\) is
\[
d_1^x(x_1, y_1, p) = \frac{\frac{3}{4} + \sqrt{2} \int_{\frac{3}{4}}^{\frac{3}{4}} \sqrt{t(1-t)} t dt}{1 + \sqrt{2} \int_{\frac{3}{4}}^{\frac{3}{4}} \sqrt{t(1-t)} dt}.
\]
This shows that the better-response demand of consumer 1 for commodity $x$ will lie between $\frac{3}{4}$ and $\frac{1}{4}$. In fact a direct computation shows that

$$\int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{t(1-t)}dt = 0.119576$$

and

$$\int_{\frac{1}{4}}^{\frac{3}{4}} \sqrt{t(1-t)}dt = 0.239153.$$ 

Therefore,

$$d_1^x(x_1, y_1, p) = \frac{\frac{3}{4} + 1.4142 \times 0.119576}{1 + 1.4142 \times 0.239153} = 0.68681 \approx 0.69.$$ 

This means that

$$d_1^y(x_1, y_1, p) = 1 - 0.69 = 0.31.$$ 

Hence, the better-response commodity bundle is given by \((0.69, 0.31)\).

This is illustrated in figure 1, in which the striped area shows the set of commodity bundles that have higher utility than the bundle \((\frac{3}{4}, \frac{1}{4})\), and are affordable at the price vector \((\frac{1}{2}, \frac{1}{2})\). The utility maximizing bundle in this case is \((\frac{1}{2}, \frac{1}{2})\) which is different from the better-response demand bundle of \((0.69, 0.31)\). Note however that the thick line along the budget line connecting the bundles \((0.31, 0.69)\) and \((0.69, 0.31)\) shows the commodity bundles that can be traded and give higher utilities than the better response demand bundle \((0.69, 0.31)\).
In fact an examination of the maps given in (13) and (14) show that the only fixed point of these better-response demand functions and the better-response price-adjustment function of this two-person, two-good economy is

\[ x_1 = \frac{1}{2}, \quad x_2 = \frac{1}{2}, \quad y_1 = \frac{1}{2}, \quad y_2 = \frac{1}{2}, \quad \text{and} \quad p_1 = \frac{1}{2}. \]

It is also easy to check that the competitive equilibrium of this pure exchange economy is

\[ x_1^* = x_2^* = \frac{1}{2}, \quad y_1^* = y_2^* = \frac{1}{2}, \quad \text{and} \quad p_1^* = \frac{1}{2} \]

the fixed point of the better-response demand functions.

4 Monotonic Transformations and the Invariance of the Equilibrium Points

The fact that the better-response demand functions of a pure exchange economy give us the competitive equilibrium points immediately raises a couple of important questions. One is whether there are other response maps whose fixed points are also the competitive equilibrium points. Another is whether monotonic transformations of the utility functions lead to a different equilibrium point. The first issue is important because if there are other better-response functions that lead to the competitive equilibrium of the economy, then this would imply that the better-response demand functions is one among an entire class of better-response demand functions that lead to the competitive equilibrium. The second issue is important because it addresses the point as to whether the fixed points of the better-response demand functions are sensitive to transformations of the utility functions. Since utility functions are unique only up to a monotonic transformation, both the usual demand functions as well as the equilibrium points are invariant to monotonic transformations of the utility functions. Thus if the response maps give us the competitive equilibrium points then these ought to be invariant to monotonic transformations of the utility functions. We show here that the fixed points of the response maps, and hence the social equilibrium of the Abstract Economy, are invariant to both monotonic transformations of the response maps as well as monotonic transformations of the utility functions.

**Theorem 3** For \( i = 1, \ldots, n \), let \( g_i : \mathbb{R} \to \mathbb{R} \) and \( F_i : \mathbb{R}_+ \to \mathbb{R}_+ \) be increasing real-valued functions. Then each of the fixed points of the response maps given by

\[
\hat{x}_i^i(p, x^i) = \frac{m^i(x^i) + \int_{\gamma_i(p, p, \omega^i)}[g_i(v^i(y)) - g_i(v^i(x^i))]^+ ydy}{1 + \int_{\gamma_i(p, p, \omega^i)}[g_i(v^i(y)) - g_i(v^i(x^i))]^+ dy}.
\]
for the consumers $i = 1, \cdots, n$ and each of the fixed points of the response maps given by

$$\hat{x}^i_{F_i}(p, x^i) := \frac{m^i(x^i) + \int_{\gamma_i(p, p, \omega_i)} F_i([v^i(y) - v^i(x)]^+)ydy}{1 + \int_{\gamma_i(p, p, \omega_i)} F_i([v^i(y) - v^i(x)]^+)dy}$$

is a competitive equilibrium of the pure exchange economy. Also every competitive equilibrium is a fixed point of these response maps.

**Proof:** We will prove the result by showing that it holds for the response maps given by (15).

We observe that if $((x^i)_{i \in N}, p^*)$ is a competitive equilibrium, then

$$[g_i(v^i(y)) - g_i(v^i(x^i))]^+ = 0$$

for all $y \in \gamma_i(p^*, p^*, \omega^i)$. Hence, from (15) it follows that

$$\hat{x}^i_{g_i}(p^*, (x^i)) = x^i$$

for every $i \in N$. This proves the first part of the claim.

Let $((\tilde{x}^i, \tilde{p})_{i \in N}, \tilde{p})$ be a fixed point of the response maps given by (15). Then

$$\tilde{x}^i \cdot \int_{\gamma_i(p, p, \omega_i)} [g_i(v^i(y)) - g_i(v^i(x))]^+ydy = \int_{\gamma_i(p, p, \omega_i)} [g_i(v^i(y)) - g_i(v^i(x))]^+ydy. \tag{17}$$

Now suppose that $((\tilde{x}^i)_{i \in N}, \tilde{p})$ is not a competitive equilibrium of the pure exchange economy. Then for some $i \in N$, there is an $\tilde{x}^i \in \gamma_i(\tilde{p}, \tilde{p}, \omega_i)$ such that

$$v^i(\tilde{x}^i) > v^i(\tilde{x}^i)$$

so that since $g_i$ is monotonic increasing, we have

$$g_i(v^i(\tilde{x}^i)) > g_i(v^i(\tilde{x}^i)).$$

From the quasi-concavity of $v^i$ and the monotonicity of $g_i$ it follows that the set

$$E_i = \{y | g_i(v^i(y)) > g_i(v^i(x))\}$$

is a nonempty convex set. Therefore, there is an $h \in \mathbb{R}^\ell$ such that

$$h.y > h.\tilde{x}^i \text{ for all } y \in E_i.$$

Since $E_i \cap \gamma_i(\tilde{p}, \tilde{p}, \omega_i) \neq \emptyset$, therefore

$$\int_{\gamma_i(p, p, \omega_i)} [g_i(v^i(y)) - g_i(v^i(x))]^+h.ydy > \int_{\gamma_i(p, p, \omega_i)} [g_i(v^i(y)) - g_i(v^i(x))]^+h.\tilde{x}^idy \tag{18}$$

which contradicts (24). Hence, $((\tilde{x}^i, \tilde{p})_{i \in N}, \tilde{p})$ is a competitive equilibrium. This completes the proof for the response maps given by (22). It should now be clear that a similar proof will show that the result holds for the response maps given by (23). This completes the proof. \qed

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Another question that naturally comes up in discussions involving the market equilibrium is the nature of the demand curve. A central postulate of economic theory is that the demand curve for a given commodity is downward-sloping. Becker in [2] argues that even some form of irrational behavior is consistent with a downward-sloping demand curve. While we do not quite make that claim here we do indicate that utility-improving behavior of the kind discussed here is also consistent with the usual downward-sloping property of the demand curve. The argument we make here is not a formal one but is given here to illustrate that many central results of economic theory would be consistent with consumers making utility improving choices rather than fully utility-maximizing ones.

Consider a consumer who buys two goods \( X \) and \( Y \). Let the initial budget line be given by \( AB \) in figure 2 and the let \( E \) be the commodity bundle chosen on that budget line. The commodity bundle \( E \) lies on the indifference curve \( IC_1 \) that gives a certain utility to the consumer. Now consider a fall in the price of good \( X \). As a result the budget line rotates out along the \( x \)-axis to \( AB' \). The utility-maximizing bundle on the new budget line \( AB' \) is \( E^* \). However, any commodity bundle between points \( C \) and \( D \) will give higher utility than the bundle \( E \). If the consumer chooses a utility-improving bundle according to the better-response map, namely a weighted average of all the bundles that give higher utility than the bundle \( E \), then the consumer will end up choosing a bundle between \( C \) and \( D \).
up choosing a commodity bundle like $E'$ that lies on the new budget line $AB'$. The resulting amount of good $X$ consumed will then increase from $x$ to $x_1$. This would then indicate that a fall in the price of good $X$ causes the quantity purchased of good $X$ showing that better-response choices are consistent with the downward-sloping property of the demand curve. It should be clear that there is no guarantee that the consumer will indeed always increase the purchase of good $X$ when its price falls, but as we know, that is also not guaranteed by utility-maximizing behavior. But the normal bias is possibly towards a downward-sloping demand curve as in the case of utility-maximizing behavior. Indeed, it is worth recalling at this point that the bundle $E'$ is a weighted average of all the bundles that have higher utility than the bundle $E$, with the weights increasing in the utility gains, thus skewing the choice of a commodity bundle towards $E^*$ and resulting in a downward-sloping demand curve.

6 Conclusion

The traditional view of the competitive equilibrium as the result of a process in which the consumers react to market prices by demanding the optimal consumption bundle begs the question as to whether the competitive equilibrium is consistent with behavior that is not fully utility-maximizing. The results here show that as long as the consumers respond to market prices by demanding a commodity bundle that is utility-improving but not necessarily utility-maximizing, and the market price adjusts to the excess demand in a similar fashion, then the rest points of this process is exactly the set of competitive equilibrium points. The results here can thus be viewed as showing that the competitive equilibrium is much more robust to behavior norms than the existing existence results would indicate. Indeed, one can now go a step further and make the point that when we observe price-taking behavior in which the consumers respond to market prices by changing the consumption vector, we do not really know whether we are observing utility-maximizing behavior, or simply some weaker form of rational behavior. Note that the better-response demand functions that we have here still imply that consumers are rational since they look for consumption bundles that give them higher utilities. The results here would thus indicate that the competitive equilibria are the fixed points of a large class of better-response demand functions, and therefore, quite robust to a variety of responses by consumers to the market price.

While we have focused on the properties of the equilibrium points, we have not discussed convergence to the equilibrium points. As the literature on the tatonnement process has shown, the convergence to equilibrium entails a very different set of issues. In fact it is reasonable to conjecture that the convergence process is sensitive to the response maps used by the consumers and the market participant. If the equilibrium point is stable
then the convergence could occur faster or slower depending on the response maps. If the equilibrium point is unstable then there may not be convergence to the equilibrium point but to something else. This particular phenomenon is amply illustrated in [4], although in the context of the game of Matching Pennies rather than a market. The paper studies the stability properties of equilibrium in games and the convergence of dynamic processes in which agents respond using the Nash map. It is possible that similar dynamic processes can be used to study the stability properties of equilibrium prices.

In this paper we have discussed the results for a pure exchange economy. However, given the rather general nature of the results and the role of the social equilibrium it would be surprising if the results do not hold for the more general cases of a private-ownership economy with production. In addition, one could also ask whether the results will hold for consumption sets that are not necessarily subsets of finite-dimensional Euclidean spaces. Since the response maps here depend on there being enough convexity, it seems plausible that the results presented here will hold for quite general cases.

In addition to showing that the competitive equilibrium is robust to perturbations of behavior norms, the response maps used here could prove useful in finding algorithms to compute the competitive equilibrium of an economy. As these response maps are continuous functions (and not correspondences like the best response maps), the competitive equilibrium points can be found by searching for the fixed points of these continuous maps. One may be able to use general versions of the Lemke-Howson algorithm as in [13] or in [21] for computing the fixed points of continuous functions. Computing the competitive equilibrium points has been part of the research in general equilibrium theory. Recently [19] has shown that it may not be possible to compute the exact competitive equilibrium using Turing machines. [12] provides a way of computing the error when using an algorithm that analyzes first-order conditions to compute an approximate competitive equilibrium. Using Sperner’s Lemma to find the cell (the area of the cell can be arbitrarily small) containing the fixed point would be an alternative approach. This method would most readily apply to the fixed points of continuous maps like those used here and in [3]. That this might indeed prove a useful approach in finding algorithms to compute fixed points has been observed by others, see for example [7].

Further in many experimental studies of markets, see for example [25], the convergence to equilibrium prices after a few rounds of trading is fairly quick and a question that could be posed is whether the traders always compute correctly the prices that would generate the most surplus, or simply start by quoting a range of prices that generate large positive surpluses, much like the utility-improving bundles of the kind described here. A trader in these settings trades even though the trade may not maximize the gains from trade. Similarly, the theory here seems to suggest that while out-of-equilibrium trades are quite possible, it shows that even if traders are not maximizing surplus when trading,
the trades in different rounds lead to a fixed point of the process, namely the equilibrium price.

The results here indicate that behavior that is broader than tight utility-maximization is consistent with the market process, and remarkably the competitive equilibrium. It goes to show that broad forms of rational behavior lead to results similar to those that are generated by careful utility-maximizing behavior. Thus it could be that general equilibrium has much to say about situations that go well beyond utility-maximization and that the market process can be consistent with behavior in which the agents do not always quote their utility-maximizing bundle.
Appendix

Lemma 4 Let $X^i$ for $i = 1, \ldots, n$ be nonempty, compact and convex subsets of a Euclidean space $\mathbb{R}^k$. Let $X := \prod_{i=1}^n X^i$ and $F^i : X \to X^i$ be a correspondence that is both upper semi-continuous and lower semi-continuous and is non empty and closed for every $x \in X$. Let $\lambda$ denote the Lebesgue measure on $\mathbb{R}^s$ where $s = \dim F^i(x)$ so that $s \leq k$. Then as $x_k \to x$

$$
\lim_{k \to \infty} \lambda(F^i(x_k) \setminus F^i(x)) = 0 \quad \text{and} \quad \lim_{k \to \infty} \lambda(F^i(x) \setminus F^i(x_k)) = 0.
$$

Proof: CASE 1: $\lambda(F^i(x)) = 0$.

Since $F^i : X \to X^i$ is upper semi-continuous, for any sequence $\{x_k\}$ and any open set $G$ such that $F^i(x) \subset G$, there is a $k_1$ such that for all $k \geq k_1$, $F^i(x_k) \subset G$. Since $\lambda(F^i(x)) = 0$, $G$ is compact and $F^i$ is closed, we can choose $G$ such that $\lambda(G) < \frac{1}{2}$. Hence, for all $k \geq k_1$, $\lambda(F^i(x_k) \setminus F^i(x)) < \frac{1}{2}$ and $\lambda(F^i(x) \setminus F^i(x_k)) < \frac{1}{2}$. But this shows that $\lim_{k \to \infty} \lambda(F^i(x_k) \setminus F^i(x)) = 0$ and $\lim_{k \to \infty} \lambda(F^i(x) \setminus F^i(x_k)) = 0$.

CASE 2: $\lambda(F^i(x)) > 0$.

Suppose that the sequence $\{\lambda(F^i(x_k) \setminus F^i(x))\}_k$ does not converge to 0. Then there is a $\delta > 0$ such that for every $k$ there is an $\ell > k$ for which

$$
\lambda(F^i(x_k) \setminus F^i(x)) > \delta. \quad (19)
$$

Now consider an open set $G$ such that $F^i(x) \subset G$ and $\lambda(G) < \lambda(F^i(x)) + \frac{\delta}{2}$. Then since the correspondence $F^i(.)$ is upper semi-continuous there is a $k_1$ such that for every $\ell > k_1$, $F^i(x_\ell) \subset G$. But that means for all $\ell > k_1$, $(F^i(x_\ell) \setminus F^i(x)) \cup F^i(x) \subset G$. Therefore,

$$
\lambda(F^i(x_\ell) \setminus F^i(x)) + \lambda(F^i(x)) < \lambda(G),
$$

so that

$$
\lambda(F^i(x_\ell) \setminus F^i(x)) < \lambda(G) - \lambda(F^i(x)) < \frac{\delta}{2}. \quad (20)
$$

We thus get a contradiction to (22). Thus this proves

$$
\lim_{k \to \infty} \lambda(F^i(x_k) \setminus F^i(x)) = 0.
$$

Now suppose that $\{\lambda(F^i(x) \setminus F^i(x_k))\}_k$ does not converge to 0. Then there is a $\delta > 0$ such that for every $k$ there is an $\ell > k$ for which $\lambda(F^i(x) \setminus F^i(x_k)) > \delta$. Therefore, for every $k$ there is an $\ell > k$ and an open set $G_\ell \subset (F^i(x) \setminus F^i(x_\ell))$ for which $\lambda(G_\ell) \geq \frac{\delta}{2}$. But this now implies that for every $k$ there is an $\ell > k$ and an open set $G_\ell$ such that $G_\ell \cap F^i(x) \neq \emptyset$ and $G_\ell \cap F^i(x_\ell) = \emptyset$. But this contradicts the fact that the correspondence
Proof: We will first show that the map \( g : X \to X^i \) defined as
\[
g(x) = \int_{F^i(x)} \phi^i(x, a)da
\]
is continuous. Let \( \{x_n\} \) be a sequence in \( X \) such that \( x_n \to x \). For any open set \( G \) such that \( F^i(x) \subseteq G \) we have,
\[
\| \int_{F^i(x_n)} \phi^i(x_n, a)da - \int_{F^i(x)} \phi^i(x, a)ada \| \\
\leq \| \int_{F^i(x_n)} \phi^i(x_n, a)da - \int_G \phi^i(x_n, a)ada \| + \| \int_G \phi^i(x_n, a)ada - \int_G \phi^i(x, a)ada \| \\
+ \| \int_G \phi^i(x, a)ada - \int_{F^i(x)} \phi^i(x, a)ada \|. \tag{21}
\]
Since \( \phi^i : X \times X^i \to \mathbb{R} \) is continuous and \( X \) and \( X^i \) are compact subsets of an Euclidean space, there exists a finite \( K > 0 \) such that \( |\phi^i(x, a)a| \leq K \) for all \( (x, a) \in X \times X^i \). Therefore,
\[
\| \int_{F^i(x_n)} \phi^i(x_n, a)ada - \int_G \phi^i(x_n, a)ada \| = \| \int_{(G \setminus F^i(x_n))} \phi^i(x_n, a)ada \| < \lambda(G \setminus F^i(x_n))K, \tag{22}
\]
where \( \lambda \) is the Lebesgue measure of the measurable subset \( G \setminus F^i(x_n) \) of the Euclidean Space.

Now given an \( \epsilon > 0 \), consider an open set \( G \) such that \( \lambda(G \setminus F^i(x)) < \frac{\epsilon}{K} \). Then noting that \( (G \setminus F^i(x_n)) = ([G \cap F^i(x)] \setminus F^i(x_n)) \cup ([G \setminus F^i(x)] \setminus F^i(x_n)) \), and by the result in lemma 3 of the appendix, there is an \( n_1 \) such that for all \( n \geq n_1 \), we have
\[
\lambda(G \setminus F^i(x_n)) = \lambda([G \cap F^i(x)] \setminus F^i(x_n)) + \lambda([G \setminus F^i(x)] \setminus F^i(x_n)) < \lambda(F^i(x) \setminus F^i(x)) + \lambda(G \setminus F^i(x)) < \frac{\epsilon}{K} + \frac{\epsilon}{K}. \tag{23}
\]
Therefore from (4) and (5) it follows that for all \( n \geq n_1 \), we have
\[
\| \int_{F^i(x_n)} \phi^i(x_n, a)ada - \int_G \phi^i(x_n, a)ada \| < 2\epsilon. \tag{24}
\]

Further, since \( \phi^i(.) \) is bounded, by Lebesgue’s dominated convergence theorem, there is an \( n_2 \) such for all \( n \geq n_2 \),
\[
\| \int_G \phi^i(x_n, a)ada - \int_G \phi^i(x, a)ada \| < \epsilon. \tag{25}
\]

\( F^i(.) \) is lower semi-continuous. This concludes the proof. \( \square \)
Finally, given an $\epsilon > 0$, and an open set $G$ such that $\lambda(G \setminus F^i(x)) < \frac{\epsilon}{K}$, we note that
\[
\| \int_G \phi^i(x,a)ada - \int_{F^i(x)} \phi^i(x,a)ada \| = \| \int_{G \setminus F^i(x)} \phi^i(x,a)ada \| < \frac{\epsilon}{K} K = \epsilon.
\] (26)

Let $n_3 = \max\{n_1, n_2\}$. Then from equation (3) and the results in (6), (7) and (8) we have for any $\epsilon > 0$ there is an $n_3$ such that for all $n \geq n_3$,
\[
\| \int_{F^i(x_n)} \phi^i(x_n,a)ada - \int_{F^i(x)} \phi^i(x,a)ada \| < 4\epsilon.
\] (27)

This proves the claim that $g : X \to X^i$ is continuous. This also shows that the function $g' : X \to X^i$ defined by
\[
g'(x) = \int_{F^i(x)} \phi^i(x,a)da
\]
is also continuous. Finally, we have already observed that the map $m^i : X \to X^i$ is continuous if the correspondence $F^i$ is compact-valued, convex-valued and continuous.

We can now conclude that the better-response function of player $i$, $b_i : X \to X^i$ given by
\[
b_i = \frac{m^i(x^i) + g(x)}{1 + g'(x)}
\]
is continuous. This completes the proof. □

The better-response function $b : X \to X$ of the Abstract economy is the combined better-response function of all the players and is defined by
\[
b(x) = (b_1(x), \cdots, b_n(x)).
\]

The next result shows that the social equilibrium points of the abstract economy are given by the fixed points of this better-response function of the Abstract Economy.

**Theorem 4** If for all $i = 1, \cdots, n$, $X^i$ is a nonempty, compact and convex subset of an Euclidean space, the correspondences $F^i : X \to X^i$ are all upper semi-continuous, lower semi-continuous and nonempty-valued, compact-valued and convex-valued, and $u^i(x,.)$ is quasi-concave on $F^i(x)$ for every $x \in X$, then the better-response function has a fixed point. Further, every fixed point of the better-response function is a social equilibrium of the Abstract Economy.
Therefore, since \( h \) there is a vector \( x^* \) such that
\[
b(x^*) = x^*.
\]
We show that such a fixed point of the better-response function is a Social Equilibrium of the Abstract Economy. We first claim that \( x^* \in F^i(x^*) \) for all \( i = 1, \cdots, n \). Note that since \( b_i(x) \in F^i(x) \) and \( x^*_i = b_i(x^*) \), we get \( x^*_i = b_i(x^*) \in F^i(x^*) \).

We now show that \( u^i(x^*, x^{i*}) \geq u^i(x^*, \xi^i) \) for all \( \xi^i \in F^i(x^*) \). For every \( x \in X \), define
\[
B_i(x) = \{ a \in F^i(x) : \phi^i(x, a) > 0 \}.
\]
Note that if \( B_i(x^*) = \emptyset \) then for all \( a \in F^i(x^*) \), \( \phi^i(x, a) = 0 \), that is \( u_i(x^*, x^*_i) \geq u_i(x^*, x_a) \).
Thus, if \( B_i(x^*) = \emptyset \) then \( x^* \) is a social equilibrium. Hence in order to complete the proof it is sufficient to show that for every player \( i \), \( B_i(x^*) = \emptyset \).

Suppose not. Then, \( B_i(x^*) \) is a nonempty set which by lemma 1 is a convex subset of \( F^i(x) \). This is clearly a Borel subset of an Euclidean space and it has positive Lebesgue measure as the continuity of \( \phi^i(x^*, .) \) implies that \( B_i(x^*) \) contains a nonempty open set.

By the definition of \( B_i(x^*) \), if \( x^{i*} \notin B_i(x^*) \), then \( ri\{x^{i*}\} \cap ri\{B_i(x^*)\} = \emptyset \), where \( ri\{A\} \) is the relative interior of a convex set \( A \). Since \( B_i(x^*) \) is convex by the Support Theorem (see for example Theorem 1.5.3 of [11] and theorem 11.3, page 97 of [20]) \(^{15}\), there is a vector \( h \neq 0 \) in the Euclidean space containing \( X^i \) such that
\[
h. x^{i*} < h. a, \text{ for all } a \in B_i(x^*).
\]
Therefore, since \( \phi^i(x^*, a) > 0 \) for all \( a \in B_i(x^*) \), and \( ri\{B_i(x^*)\} \neq \emptyset \), we have
\[
\int_{B_i(x^*)} \phi^i(x^*, a) h. x^{i*} da < \int_{B_i(x^*)} \phi^i(x^*, a) h. ada. \tag{28}
\]
Now since \( x^* \) is a fixed point of the map \( b : X \to X, b_i(x^*) = x^{i*} \), and
\[
x^{i*} = \frac{x^{i*} + \int_{F^i(x^*)} \phi^i(x^*, a) ada}{1 + \int_{F^i(x^*)} \phi^i(x^*, a) da}.
\]
Since \( \phi^i(x^*, a) = 0 \) for \( a \in F^i(x^*) \setminus B_i(x^*) \), therefore we can rewrite the above equation as
\[
x^{i*} = \frac{x^{i*} + \int_{B_i(x^*)} \phi^i(x^*, a) ada}{1 + \int_{B_i(x^*)} \phi^i(x^*, a) da}. \tag{29}
\]
\(^{14}\)This proof closely follows the proof of theorem 2 of [3], but makes use of the Support Theorem rather than a Separation Theorem as is the case in [3]. This makes the proof here a little more direct. \(^{15}\)This result is part of the general results that are derived from the Hahn-Banach theorem but as in [11] and [20] are also often referred to as a Support theorem.
From (11) it now follows that

\[ x^i \int_{\mathcal{B}_i(x^*)} \phi^i(x^*, a) da = \int_{\mathcal{B}_i(x^*)} \phi^i(x^*, a) da \]

so that,

\[ \int_{\mathcal{B}_i(x^*)} \phi^i(x^*, a) h.x^i da = \int_{\mathcal{B}_i(x^*)} \phi^i(x^*, a) h da. \quad (30) \]

But (12) contradicts (10). Hence, \( \mathcal{B}_i(x^*) = \emptyset \). This concludes the proof of the theorem. □
References


