Endogenous Bourse Structures

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Abstract. Being inspired by the recent wave of demutualizations in the securities exchange industry, and knowing the importance of such institutions to the market, we provide an equilibrium setting where traders sort in bourses to trade their securities. In our model, where bourses may range in size from one-person to the entire population, equilibrium exists generically. We provide a micro-founded characterization of the emergence of a global trading platform, and also show how large exchanges are ill-suited if traders’ complementarities are poor. We also give characterizations and examples that illustrate how traders’ attributes and/or bourse formation costs explain the market (in)completeness.


Keywords : bourse structures; traders’ complementarities; bourse formation cost; bourse size; unique bourse; multiple bourses; endogenous (in)complete markets.

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1 Introduction

Bourses (or stock exchanges) have origins distant in history. Any group of agents who agree to trade their securities in fact constitutes a bourse. Over time bourses evolved and large trading organizations emerged. By the 20th century, bourses were linked to their respective national countries, where outsiders were charged high trading fees in order to gain access to their liquidity. But the evolution of bourses has never been as dramatic as in the last decade. In 2007 the European Commission enacted the Market in Financial Instruments Directive (MiFid) to facilitate competition across the region. The competitors are known as MTFs (Multilateral Trading Facilities). Similarly, the U.S. have encouraged fragmentation with Regulation National Market System.\(^1\) The change in regulation and the new electronic trading technologies are driving the old-style stock exchanges out of existence, and they are being progressively replaced by new global trading organizations. Any bourse can now be created, at convenient low formation costs, to trade securities with self-picked traders.\(^2\) The following factors inspire the current research.

Fact Set 1 (Toward a unique global bourse?): In October 2006 the Chicago Mercantile Exchange (CME) and the Chicago Board of Trade (CBOT) created the world’s largest futures exchange. In May 2006 the New York Stock Exchange (NYSE), itself a product of the union between the New York Stock Exchange and American Stock Exchange, acquired Euronext, in turn the result of the merger between Paris, Amsterdam, and Brussels stock exchanges. In April 2007 Deutsche Börse acquired the International Securities Exchange (ISE).

Fact Set 2 (Competition): Competition from small exchanges, offshore centers, banks trading consortia, and dark liquidity pools led to an increase in fragmentation in 2008 when the number of trading platforms increased. By 2009 MTFs accounted for about 20% of the trading in the largest shares across European main markets. In the U.S., trading volume in listed stocks at NYSE fell from around 80% in 2003 to 20% in 2011 due to the entrance of new trading venues.\(^3\)

The recent wave of demutualization implies that traders can move now freely from their pre-assigned bourses (e.g., national bourses) to their most preferred ones, without any restriction other than paying the corresponding membership fee. Knowing the importance

of such institutions to the market, we provide an equilibrium model of endogenous bourse formation. We consider an economy with three periods. Traders form bourses in period 0 and trade securities in their respective bourses in period 1. In period 2 there are several states of nature where securities pay returns. Traders evaluate bourses in period 0 by their risk sharing possibilities associated with periods 1 and 2, and given these evaluations each trader chooses the bourse he wants to belong to.

A bourse is a local public good that allows traders to diversify risk by trading their securities with the other bourse members. This is an important conceptualization of an exchange, as it allows us to use the theory of club formation to price the bourse memberships and analyze competition among exchanges. Our main technical contribution with respect to the previous club literature in general is that in our model the utility provided by the club good (bourse) is endogenous. Whereas in the previous literature the club good (e.g., park or swimming pool) entered directly as a primitive in the individual’s utility function, in our approach the club good is a facility that permits social interactions (i.e., security trading), whose equilibrium outcome, referred here as “trading equilibrium”, is embedded into the trader’s utility function. Then, the utility that the trader assigns to bourse memberships depends on his evaluation of the risk sharing possibilities achieved in equilibrium in these bourses.

The literature of club theory offers many different models, and although we could use many of them to design our bourse formation process of period 0 in a more or less realistic environment, we find the model of Allouch and Wooders [2008] (hereinafter AW) appropriate for our purpose. Their setup allows us to incorporate explicitly in our model three important characteristics of the bourse industry: 1) large and finite number of traders, and 2) unbounded bourse sizes and increasing gains from trade in larger bourse sizes. Below we justify the necessity of using AW’s model by explaining in detail the importance of incorporating each of these features in a model of bourse formation. One should notice here that we do not simply apply prior results. There are fundamental differences between security markets and commodity markets. In club theory, typically the preferences of an individual are independent of prices; prices enter through constraint sets. In bourse formation with security markets, as studied in the present paper, preferences over bourses depend on the security prices, which ultimately depend on the complementarities in preferences and endowments among the traders that form the bourse. Bourses are formed endogenously by the traders’ choices, given the cost of bourse formation, and then

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4The results of AW are themselves based on Wooders (1983) showing that cooperative games with many players have nonempty approximate cores. These results are the basis for a number of influential works, including Wooders (1994), among others.
the security equilibrium prices are determined.

In our economy there is only one good and a trader can belong to one bourse or none, and thus there are no interactions among traders who belong to different bourses by trading in the spot or the securities market. In this context, a trader is only affected by the bourses he belongs to.\(^5\) Equilibrium consumption in periods 1 and 2 may differ from the initial trader’s good endowment if the trader diversifies risk by buying or selling securities with the other members of the bourses where he belongs to. A trader then evaluates a bourse by incorporating the equilibrium consumption of periods 1 and 2 into his utility function. As in many other models of financial equilibrium, in our model there can be more than one security trading equilibrium.\(^6\) We take the simplest approach and assume that traders agree on the type of trading equilibrium realization. We are able to show that there exists a measurable selector of this trading equilibrium with which any trader can evaluate his bourse membership. We refer to the resulting indirect utility function as the “equilibrium bourse utility function”, which is itself a function of the trader’s private good consumption in period 0 (yet to be determined) and the equilibrium consumption selection for periods 1 and 2. Moreover, we show that equilibrium consumption selection for periods 1 and 2 is continuous in period 0 consumption. Any deviation by a subset of traders incorporates the new equilibrium risk sharing possibilities that the new candidate bourses provide, and therefore, any deviation must have different membership prices. An equilibrium for this bourse formation process is called a “bourse structure equilibrium”. For a given bourse structure equilibrium, an off-equilibrium deviation consists of a subset of traders such that for each of these traders, the membership fees associated with the new deviating bourses exceed the trader’s good endowment net of his consumption.

Considering a large number of traders portrays the idea that any type of trader has many substitutes in the process of bourse formation. Traders take bourse membership prices as given, but we allow membership prices to depend on the type of trader, e.g., hedge fund, pension fund, and investment bank. In fact, when a trader applies to become a member of a bourse (in period 0), some type of identification process must be filled and the type of trader may be revealed. The alternative scenario, where the bourse membership is anonymous (i.e., traders identities are not observable), can be considered in our model of bourses in the form of a poll fee. Notwithstanding, we chose to emphasize the role of non-anonymous bourse memberships to highlight that trading platforms can be

\(^5\)A model with multiple memberships would make trader’s utility dependent on the whole bourse structure through the equilibrium security prices, something incompatible with AW’s framework.

\(^6\)However, it is easy to provide examples of economies (as we do in Section 4) where the trading equilibrium is uniquely determined.
priced as any other local public good. Once bourses are formed, security trading occurs within each bourse. We assume that trading is done electronically, so traders’ identity may no longer be revealed, and the market clearing is Walrasian for any bourse of any size - i.e., in equilibrium all traders of a bourse face a single clearing security price. Other possible trading mechanisms, such as bilateral trading where the security price depends on the pair of traders, can also be adapted in our model (see discussion in Section 5).

Another feature of our model is that it allows for unbounded bourse sizes and increasing gains from trade in larger bourse sizes. Certainly, one of the questions we want to explore is whether a large unique bourse can arise in equilibrium - Fact Set 1 and anecdotal evidence shows that this was a common belief until 2006. For this, we have to remove any assumption that bounds bourse sizes. In doing so, one has to be careful in dealing with our bourse economy since the local public good that the bourse provides - the facility of security trading - might generate increasing returns from larger bourses, which might be at odds with existence of equilibrium. For example, a trader may benefit from being in a larger bourse because it provides him with more information (the utility function can be written accordingly). The application of AW’s existence result, which permits unbounded bourse sizes, is not immediate since some of their assumptions rely on the individual’s utility function being evaluated at the club good, which in our model is endogenous. The technical hurdle to implement a model of bourse formation in this context is to show that there exists an open and dense set of economies where the trading equilibrium for any given bourse structure is continuous in traders’ utility and endowment attributes. The existence of this “generic” set implies that equilibrium fails to exist in our model only under special “negligible” combinations of parameters of the system. Roughly speaking, we can say that the non-existence of equilibrium is an “accident” in our model.

To the best of our knowledge, this paper is the first that seeks to analyze the specific market micro-structure of trading in bourses under the lens of club theory. In particular, we contribute with a different viewpoint to the market microstructure literature that analyzes the issues of concentration and fragmentation of trade across markets - see Pagano

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8Allouch and Wooders [2008] allow for unbounded club sizes. Konishi, Le Breton, and Weber (1998) could also be used for our purposes (it allows an arbitrary number of jurisdictions). Other papers in the club / local public goods literature - for instance, Cole and Prescott [1997] and Ellickson et al. [2001], and Wooders [1980, 1978] - do not allow clubs to be unbounded in size. There are other papers that do allow unbounded club sizes and also consider price-taking equilibrium – see Wooders [1983, 1997] and, more indirectly, Wooders [1994]. Allouch et al. [2009] also allow unbounded club sizes, but require that all gains can be realized by coalitions strictly bounded in size.
[1989] and related works - and the impact of trading costs on trading behavior.\(^9\) We depart from that literature in that we allow for every possible subset of traders to form a bourse, which contrasts with the two-bourse analysis of Pagano.\(^{10}\) Our club theory approach offers a new powerful tool to analyze equilibrium structures of exchanges, different from the classical models of vertically differentiated duopolies.\(^{11}\) We also depart from the market microstructure literature in that we do not limit our analysis to only one security (i.e., stock or bond). Instead, in our model there can be different security structures, with more than one security, but possibly incomplete. In equilibrium the security structure endogenously arises once traders sort in bourses.

In our analysis of bourse formation we focus on an important type of assets: securities (stocks and bonds). Key to this type of assets is the natural role for security “possession”. The notion of a “box constraint”, that requires the physical amount of security titles that the trader possess to be non-negative, is borrowed from Bottazzi, Luque and Pascoa (2012). In that paper the “box” was introduced in a general equilibrium framework to understand security borrowing, lending, and leverage. In the present paper, the “box constraint” plays a new role: it keeps track of the amount of security titles that a trader brings to a bourse. The box constraint restricts in a natural way the type and quantity of securities that can be traded in a given bourse - the securities available in a bourse are only those that traders with membership in that bourse bring to the platform. Our general equilibrium model of the security exchange industry provides a new tool to understand how competition affects the bourses and associated structures of securities. Our theory fills an important gap of the empirical literature that failed to find evidence that guide researchers and policy makers in the understanding of the demutualization of securities exchange industry. For instance, Treptow (2006) analyzes the micro- and macroeconomic drivers of the demutualization decision of the securities exchange industry during the last decade, but does not find enough empirical support that helps understand how competition forces affects the exchange industry.

The endogeneity of the market incompleteness has been an old question in the last decades. Recently, a main stream in this literature has focused on models of financial

\(^9\)See, for example, Lo, Mamaysky, and Wang [2004] for an equilibrium model with fixed transaction costs.

\(^{10}\)Restrictions on agents’ participation have been recurrent in the literature of adverse selection - Dubey, Geanakoplos, and Shubik (2005) and Guerrieri, Shimer, and Wright (2010)-, although with different purposes. There, a small number of agents with the highest type is assumed to search for each contract.

\(^{11}\)See Gabszewicz and Thisse (1979) for a leading paper on vertically differentiated oligopolies, and Colliard and Foucault (2012) and Pagnotta and Philippon (2012) for models where investors can choose between two platforms trading one security and exchanges compete \textit{a la} Bertrand.
innovation by intermediaries - see Duffie and Rahi (1995) for a literature survey, and also the more recent paper by Carvajal, Rostek, and Weretka (2012) and references therein. It is widely accepted that intermediaries played an important role in the security innovation process, and therefore these models are important to understand the incompleteness of the markets. Our paper provides a different viewpoint. We believe that one cannot dismiss the possibility that a mere group of traders, e.g., consortia of banks, hedge funds, or investors, can create their own bourse in order to trade their securities. In this respect, our model is able to pin down the formation of these coalitions of traders and associated security structures in equilibrium. Trading complementarities appear to be a sufficient explanation of why a group of traders want to sort in a bourse with an incomplete security structure. As it is well known in the literature, and our Example 5 clarifies, a platform with an incomplete security structure is not necessarily an inefficient outcome. However, previous literature assumed the set of traders as a primitive of the model. In our model of bourse formation an efficient bourse with incomplete markets can be an equilibrium result (“bourse structure equilibrium”), once we let traders sort in bourses. The technology of bourse formation, on the other hand, also stands as a decisive factor that can dictate, by itself, whether a bourse with an incomplete security structure forms. In fact, we find that, in a simple economy, a sufficient condition for the incompleteness of the security structures is that bourse formation costs are non-linear and strictly increasing in the number of securities. Thus, “constrained efficiency” may arise in equilibrium if the cost of forming a bourse with complete markets is too expensive compared with a bourse with incomplete markets.

Somehow related to the previous issue on market incompleteness is the conexion of our new model with the literature of restricted participation - see Balasko, Cass, and Siconolfi (1990) for one of the earliest contribution to this literature in the field of general equilibrium, and Rocheteau and Wright (2005) for a rich comparison of three market structures (search equilibrium, competitive equilibrium, and competitive search equilibrium) in an environment where frictions take the form of exogenous participation constraints. While in previous Walrasian models of restricted participation the trading

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12 For anecdotal evidence, see “Exchanges appear ready to go over to the dark side”, Financial Times, January 9, 2008.
13 The existence issue for (simple) real numeraire assets markets is established by Polemarchakis and Siconolfi in a few different papers around 1995-2000.
14 In the Walrasian model of Rocheteau and Wright (2005), the size of the subset of traders that can participate in the Walrasian market is restricted exogenously, and traders in this subset participate in the bargaining according to some exogenous random probability. This convenient way to introduce search frictions into an otherwise Walrasian model can be thought as a generalized version of Lucas and Prescott (1974).
coalitions (understood in terms of participation constraints) are exogenously imposed, in our paper there is a pre-trading stage where bourses are endogenously formed (with some costs).\(^\text{15}\) The endogeneity of the bourse structure permits us to give positive predictions regarding bourse sizes. In particular, we find conditions on preferences and the technology of bourse formation where a single global bourse forms in equilibrium. This result is important in our attempt to establish a proper theory of endogenous bourse formation because it rejects the conjecture that with sufficient heterogeneity among traders, all possible exchanges are active. Also, inspired by Fact Set 2, we provide a result where multiple bourses emerge in equilibrium if bourse formation costs are sufficiently low, which is according to the recent empirical study on market fragmentation by O’Hara and Ye (2011). Even more interesting is our result that traders’ complementarities (in endowments and preferences) can \textit{solely} determine the equilibrium bourse sizes.

The outline is as follows. The model is presented in Section 2. Section 3 states the equilibrium concept and establishes the existence result. In Section 4 we provide characterizations and examples regarding the formation of a global bourse, multiple bourses, complete markets, and incomplete markets. Section 5 discusses how to extend our model to account for bilateral trading, and also establishes a parallelism between our theory of bourse formation and a theory of endogenous trading networks. Section 6 concludes.

\section{Bourse economies}

\subsection{Bourse structures}

An exchange participant or \textit{trader} is a corporation that may trade on or through the exchange and is licensed under the ordinance of the corresponding exchange financial regulator to carry on security trading activity. The set of traders is \(I \equiv \{1, \ldots, n\}\), with \(n\) assumed to be large but finite. We write \(\Theta\) to denote the set of traders’ characteristics, endowed with a metric \(d\). An attribute function is a mapping \(\alpha : I \rightarrow \Theta\), so that \(\alpha(i) = \theta\), with \(\theta \in \Theta\), describes trader \(i\)’s endowments and preferences (described below). An economy is represented by a pair \((I, \alpha)\). To establish a proper pricing theory of bourse

\(^{15}\) As mentioned in Section 3 below, the existence proof of a trading equilibrium does not use the standard technique of taking a single Walrasian auctioneer that chooses all prices in the simplex. In that context, the lower semicontinuity of each trader’s admissible strategy correspondence would fail. Instead, we require an auctioneer for each security in each bourse that chooses its price (relative to the commodity price) in a compact set. In this context, the problem reduces to show that bourse clearing security prices cannot be “too large” in equilibrium.
memberships, we assume that traders’ characteristics are observable at the time when applying for a bourse membership.

Our economy lasts for three periods, 0, 1, and 2. In period 0 traders form bourses; in period 1 security trading occurs in each bourse;\(^{16}\) and in period 2 securities pay returns. The set of states of uncertainty in the last period is \(\Xi \equiv \{1, ..., \Xi\}\), with representative element \(\xi\). Traders are assumed, for all effects, to be price takers in all periods. There is only one commodity in each period, whose price we normalize to 1. Trader \(i\) is endowed with a finite and strictly positive vector of the perfectly divisible private commodity \(\omega^i = (\omega^i_0, \omega^i_1, (\omega^i(\xi), \xi = 1, ..., \Xi))\). We denote trader \(i\)’s non-negative consumption bundle by \(x^i = (x^i_0, x^i_1, (x^i(\xi), \xi = 1, ..., \Xi))\). We write \(x^I_0 \equiv (x^i_0 : i \in I)\) to denote traders’ consumption bundles in period 0, while \(x^I = (x^i : i \in I)\) denotes traders’ consumption bundles in the three periods.

Traders are endowed with some securities out of a finite set of securities, denoted by \(J \equiv \{1, ..., J\}\). We denote trader \(i\)’s endowment of security \(j\) by \(e^i_j \geq 0\). Given these security endowments, there is a natural mapping from the set of traders that constitutes a bourse, \(S\), to the bourse \(S\)’s set of available securities, denoted by \(J(S) = \{j \in J : \exists i \in S \quad \text{with} \quad e^i_j > 0\}\). We denote security \(j\)’s return at state \(\xi\) by \(a_j(\xi)\) (in terms of the good). Then, we can write bourse \(S\)’s associated payoff matrix as \(A(S) = [a_j(\xi)]_{\Xi \times J(S)}\).\(^{17}\) We assume that \(A(I) = [a_j(\xi)]_{\Xi \times J(I)}\), that is, the payoffs of the available securities in the economy are linearly independent.\(^{18}\) The pair \((S, A(S))\) describes a bourse, which can be seen as a club good that allows traders to diversify risk by offering the specific activity of trading their securities. As such, a bourse becomes a source of liquidity.

A bourse structure is denoted by \(F(I) = \{(S_1, A(S_1)), ..., (S_k, A(S_k)), ..., (S_K, A(S_K))\}\). We write \(F(I)\) to denote the set of all possible bourse structures. A trader can belong to none or one bourse, and therefore, a bourse structure is necessarily a partition, \(S_k \cap S_{k'} = \emptyset\).

A deviation by a set of traders \(N \subset I\) from a bourse structure \(F(I)\) determines a bourse structure that contains \(F(N) = \{(S_k', A(S_k'))\}_{k=1}^{K'}\) and also those bourses in \(F(I) = \{(S_k, A(S_k))\}_{k=1}^{K}\), but without those traders belonging to \(N\). Formally, we write

\(^{16}\)For simplicity we assume that trading occurs only once in period 1, although period 1 could have been modeled as a period that permits multiple trading rounds.

\(^{17}\)Alternatively, we could assign to each bourse \(S\) a subset of assets (in the broad sense) \(J(S) \subseteq J\), with associated (full rank) payoff matrix \(A(S) = [a_j(\xi)]_{\Xi \times J(S)}\), that traders in bourse \(S\) agree to issue for posterior trading. We leave the possibility of issuing assets for future research.

\(^{18}\)This assumption is standard in the literature, and just requires that there are no redundant securities in the economy. As in Geanakoplos and Polemarchakis [1986], we need this assumption to show that the set of security trading equilibria is a continuous differentiable function of both the commodity endowments.
\[ \tilde{F}_{F(N)}(I) = F(N) \cup \{(S_k \cap (I \setminus N)), A(S_k \cap (I \setminus N))\}_{k=1}^K. \] 

In the equilibrium Definition 2 below, the bourse structure \( \tilde{F}_{F(N)}(I) \) stands for the off-equilibrium deviation of the set of traders \( N \). Notice that the set of possible deviations that we are considering is “large” in the sense that we allow the traders in \( N \) to form any type of bourse structure among them.

Now, given an economy \( (I, \alpha) \) and a bourse structure \( F(I) \), let us denote by \( F[i; I] \) the bourse in \( F(I) \) where trader \( i \) has membership. Trader \( i \) can only trade securities with those traders in \( F[i; I] \). This implies that, to diversify risk, traders not only choose a bourse because of its securities available for trade, but also because of the other traders’ endowments and preferences.\(^{19}\) We denote trader \( i \)’s trading of security \( j \in J(S) \) in period 1 by \( y^j_i \). As usual, \( y^j_i > 0 \) denotes a purchase of security \( j \), while \( y^j_i < 0 \) denotes a sale of this security. A necessary condition for trader \( i \) to trade \( j \in J(S) \) is that \( S = F[i; I] \). Thus, we may also write \( y^j_i(F[i; I]) \) to denote the trade of security \( j \in J(F[i; I]) \) in bourse \( F[i; I] \) where \( i \) belongs to. We write \( y^I_i = (y^j_i \in \mathbb{R}^{J(F[i; I])} : i \in I) \) to denote the vector of security positions that all traders have in their respective bourses. Here, \( q(S) \in \mathbb{R}^{J(S)}_+ \) stands for the vector of security prices corresponding to bourse \( S \). In our Walrasian bourse economy there is a unique market clearing price for each security trading in a bourse. This interpretation captures to some extent the essentiality of dark pools of liquidity. In these trading platforms participants are protected against adverse share price movements from outsiders since the trades are privately negotiated (within the bourse) and outsiders are not allowed to trade their securities. However, the trading mechanism in a dark pool of liquidity is not necessarily Walrasian. Section 5 suggests that one further step for future research is to extend our model to other types of trading environments.

### 2.2 Formation costs, communication costs, and transaction fees

In order to adopt the technology of security trading and build the trading platform, a bourse \( S \) faces (fixed) formation costs \( z(S) \in \mathbb{R}_+ \) - in terms of the private good, e.g., installation charges.\(^{20}\) Since the creation of a bourse involves some costs, it is natural to require the number of bourse memberships of every trader to be bounded (also in previous club literature). To fulfill the participantship requirement, all exchange participants are

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\(^{19}\)To put this in another way, consider a bourse where all traders are identical, i.e., all have the same endowments and want to consume in the same state of nature. In this case no trade of securities is possible.

\(^{20}\)See the HKEx security trading infrastructure at http://www.hkex.com.hk/eng/market/sec_tradinfra/CMTradInfra.htm
required to hold a bourse membership (or trading right) of the respective exchanges.\textsuperscript{21} The acquisition of a bourse membership (here in period 0) usually involves a commitment for trading in the bourse for a long period of time - usually, the exchange participant stays in the bourse since its year of accession.\textsuperscript{22} The membership fee that trader \( i \) pays to participate in a bourse \( S \), denoted by \( \pi^i(S) \), is dependent on trader’s own type \( \alpha(i) \) (e.g., broker-dealer and retail investor participants). Then, a participation price system is a set \( \Pi = \{\pi^i(S) \in \mathbb{R} : i \in S \text{ and } S \subseteq I\} \). According with our interpretation of a bourse as a club that provides the securities trading facility (local public good), we allow membership fees to be non-anonymous, so that these fees reflect the traders’ valuation of each bourse. Anonymous membership fees can easily be adapted to our theory by making \( \pi^i(S) = \pi(S) \) for all \( i \in S \), where \( \pi(S) = z(S)/|S| \) is the poll tax in bourse \( S \).

For simplicity we do not model trading fees, whose importance has declined substantially since the implementation of the MiFid regulation,\textsuperscript{23} neither we consider transaction fees (e.g., stamp duty).\textsuperscript{24} However, we assume that traders face communication costs in period 0 if they wish to move to different bourses. These communication costs for a bourse of size \( |S| \) are given by \( c(\varepsilon_0) = \varepsilon_0|S| \), \( \varepsilon_0 > 0 \).

### 2.3 Constraints and preferences

Given a bourse structure \( F(I) \) and the bourse membership price for a trader of type \( \alpha(i) \) with membership \( F[i; I] \), trader \( i \)’s budget constraint in period 0 is

\[
x_0 - \omega_0^i + \pi^i(F[i; I]) \leq 0.
\]

The price to acquire bourse memberships can be seen as the opportunity cost, in terms of period 0 consumption, that allows the trader to get access to risk sharing in periods 1 and

\textsuperscript{21}As stated in the Hong Kong Exchange (HKEx) rules, any broker-dealer intending to operate a brokerage business for products available on HKEx, using the trading facilities of the Stock Exchange and/or Futures Exchange, must be admitted and registered as an Exchange Participant of that Exchange. This membership fee is HK$500,000 (US$64,100).

\textsuperscript{22}See, for example, the MICEX list of participants: http://www.micex.com/markets/stock/members/list.

\textsuperscript{23}See Colliard and Foucault [2012] for a survey on this literature and for an analysis of the trading fees on the efficiency of the markets.

\textsuperscript{24}It would be easy to prove equilibrium existence if we incorporate the transaction fee, although notation would be more complicated. We notice here that in a different economy where multiple memberships are allowed, the transaction fee would play a role in showing that trader \( i \)’s demand of the commodity is continuous in security prices, since in that context security payoffs can be collinear among different bourses. See Faias [2008] for a similar application of a transaction fee in a context of optimal security design.
2. Notice that communication costs do not enter in constraint (1) since these costs are paid only if the trader decides to move to another bourse. Below, we will argue that these communications costs are only relevant off the equilibrium path. The profits of bourse $S$ are given by $\sum_{i \in S} \pi^i(S) - z(S)$.

In period 1 a trader faces two constraints: the “box constraint” and the budget constraint. Given traders’ bourse memberships and security prices $q$, trader $i$’s budget constraint in period 1 is

$$x_1 - \omega^i_1 + \sum_{j \in J(F[i; I])} q_j y_j \leq 0. \tag{2}$$

The “box constraint” for security $j \in J(F[i; I])$ says that the amount of security $j$ with which the trader is endowed plus the trading on this security in the bourse he belongs to must be non-negative:

$$e^i_j + y^i_j \geq 0. \tag{3}$$

We always refer to $e^i_j + y^i_j$ as the trader $i$’s position on security $j \in J(F[i; I])$, whereas we refer to $y^i_j$ as the trader $i$’s trade of security $j$. The interaction between the box and budget constraints can be read as follows: a trader $i$ can sell the amount $y^i_j$ of security $j \in J(F[i; I])$ if he is endowed with at least $e^i_j \geq -y^i_j$; if this sale satisfies the box constraint, the sale is possible and the trader would give $y^i_j$ units of the security to the buyer (these units of the security disappear from his box constraint), and receive $-q_j y^i_j > 0$ (which enters as income in the trader’s budget constraint of period 1).

Finally, trader $i$’s budget constraint in period 2 and node $\xi \in \Xi$ is such that his consumption, net of his good endowment, is bounded by the returns on his security positions.\footnote{\cite{Santos and Scheinkman 2001}}

$$x(\xi) - \omega^i(\xi) \leq \sum_{j \in J} a_j(\xi) e^i_j + \sum_{j \in J(F[i; I])} a_j(\xi) y_j. \tag{4}$$

Trader $i$’s utility function $u^i : \mathbb{R}_+^{2+\Xi} \rightarrow \mathbb{R}$ evaluated on the consumption bundle $x \in \mathbb{R}_+^{2+\Xi}$ is denoted by $u^i(x_0, x_1, x(1), ..., x(\Xi))$. For simplicity, we consider a utility function that reflects the idea that bourse participants are inherently forward-looking: traders, subsequent to their purchase of bourse memberships, buy and sell securities in their

\footnote{\cite{Santos and Scheinkman 2001}.

\cite{Bottazzi, Luque and Pascoa 2012}. This extension of the model is left for future research.

\cite{Santos and Scheinkman 2001}.

\cite{Santos and Scheinkman 2001} for a leading model with default and two clearing-houses.
respective bourses depending only on their preferences to share risk between periods 1 and 2 and across states of nature. Mathematically speaking, we restrict our analysis to the set of utility functions where the maximum value of $u^i(x_0, \cdot)$ does not depend on $x_0$. If that were the case our proof of equilibrium existence would be far more involved, as it would require to show that there is a random selection of the trading equilibrium point that is continuous in $x_0$. The details for such extension are explained in Section 5.

### 2.4 Assumptions

**A1.i** For every $x_0 \in \mathbb{R}_+$, $u^i(x_0, \cdot)$ is twice continuous differentiable, increasing, and has the matrix of second derivatives, $D^2u^i_1(x_0, \cdot)$, negative semidefinite.

The following impatience assumption is needed to guarantee that the security market clears for each bourse.

**A1.ii** For any $x_0 \in \mathbb{R}_+$ and for any $i \in I$, given $\theta \in (0, 1)$, there exists $\rho : \mathbb{R}_+^{I+\Xi} \to \mathbb{R}$, such that $u^i(x_0, x_1 + \rho(x_1), \theta x(1), \ldots, \theta x(\Xi)) > u^i(x_0, x_1)$, where $x_1 = (x_1, x(1), \ldots, x(\Xi))$.

Assumption (A1.ii) just says that we can always find a large consumption in period 1 such that the trader is better off with this extra consumption in period 1 but less consumption in every state of period 2. As we explain in detail in Section 3 below, we need Assumption Set A1 to show that a trading equilibrium exists for a given bourse structure.

Another set of assumptions on $u^i$ is needed to prove existence of an equilibrium bourse structure.

**A2.i** For all $i \in I$, $\omega^i_0 > \tau$ with $\tau > 0$, and given $\varepsilon > 0$, there exists $\lambda > 0$ such that for any set $I$ and pair of economies $(I, \alpha)$ and $(I, \beta)$, if $d(\alpha(i), \beta(i)) \leq \lambda$ for any $i$, then $\omega^\alpha_0(i) \leq \omega^\beta_0(i) + \varepsilon$. Also, $u^i(\cdot, x_1)$ is continuous, increasing, and strictly quasiconcave.

Assumption (A2.i) requires endowments to be uniformly bounded away from 0, and for near economies traders’ endowments should not differ significantly. This assumption already appeared in AW. The assumptions on trader $i$’s utility in period 0, in the second part of (A2.i), are standard.

**A2.ii** Given $\varepsilon > 0$, there exists $\lambda > 0$ such that, for any bourse $S$ and any attribute functions $\alpha$ and $\beta$, if $d(\alpha(i), \beta(i)) \leq \lambda$ for every $i \in S$, then $z^\alpha_{S^\alpha} \leq z^\beta_{S^\beta} + \varepsilon$.

Assumption (A2.ii) imposes a continuity condition on $z$ with respect to attributes, needed for our existence result below. There, $S^\alpha$ and $S^\beta$ are the bourses comprising the same traders, but characterized by attributes $\alpha$ and $\beta$, respectively. The remaining
assumptions differ slightly from AW, since the club good associated with each bourse, interpreted here as the securities trading facility, is endogenous in our model. We refer the reader to the proof of our Lemma 3 in the Appendix, for an exhaustive relationship between our assumptions and those of AW.

(A2.iii) If $u^i(\omega^i_0 - \tau_i, \omega^i_1, \omega^i(1), ..., \omega^i(\Xi)) < u^i(x_0, x_1, x(1), ..., x(\Xi))$, then $x_0 > 0$.

Assumption (A2.iii) guarantees that endowments of period 0 are desirable. This assumption is needed in the proof of Lemma 3.

(A2.iv) Given $\varepsilon > 0$, there exist $\rho_\varepsilon$ and $\gamma > 0$, such that, for any set $I$ and pair of economies $(I, \alpha)$ and $(I, \beta)$, if $d(\alpha(i), \beta(i)) \leq \gamma$, then $u^{\alpha(i)}(x^i_0, x_1, x(1), ..., x(\Xi)) + \rho_\varepsilon < u^{\beta(i)}(x^i_0 + \varepsilon, x_1, x(1), ..., x(\Xi))$ for any $i$ and any $x^i_0$.

Assumption (A2.iv) says that the private good in period 0 is valuable. In particular, each trader’s “marginal” utility of consumption in period 0 is uniformly bounded away from zero and this bound is the same for near economies.

3 Equilibrium

Consistent with our argument that the acquisition of bourse membership (in period 0) usually involves a commitment for trading in that bourse for a long period of time, whereas the security trading activity occurs only when bourses have been formed, we distinguish between the evaluation of bourses (for each bourse structure traders assess the risk sharing attained in their respective bourses) and the process of bourse formation (traders form bourses taking these evaluations as given).

3.1 Bourse evaluation

Definition 1 (Security trading equilibrium for a given bourse structure): Given the bourse structure $F(I)$, a price taking security trading equilibrium consists of a system $(x^I_1, x^I(1), ..., x^I(\Xi), y^I, q)(F(I))$, such that,

(D1.i) given trader’s bourse membership $F[i; I]$, the trader optimally chooses his periods 1 and 2 consumption and securities trading, i.e., $(x^i_1, x^i(1), ..., x^i(\Xi), y^i)(F(I)) \in \arg \max u^i(x_0, x_1, x(1), ..., x(\Xi))$, subject to constraints (2), (3) and (4).

(D1.ii) commodity markets clear in periods 1 and 2, i.e.

• $\sum_{i \in I} (x^i_1 - \omega^i_1 + \sum_{j \in J(F[i; I])} g_{ij}(y^i_j)) = 0$. 


\[ \sum_{i \in I} \left( x^i(\xi) - \omega^i(\xi) - \sum_{j \in J} a_j(\xi)e^j_i \right) = 0, \text{ for all } \xi \in \Xi. \]

(D1.iii) the security market clears for each bourse, i.e. \( \sum_{i \in S} y^i_j = 0, \forall j \in J(S), \forall S \in F(I) \).

We denote the set of security trading equilibria, given a bourse structure \( F(I) \), by \( E(F(I)) \).

Let us now provide a simple example of a security trading equilibrium when the bourse structure is exogenously given. Example 1 illustrates how the bourse structure affects welfare and trading volume through varying complementarities among different sets of traders. The traders’ indirect utility functions in a given bourse, which capture their trading complementarities, will then be used in our examples below to portrait the formation of a bourse structure. Our terminology should give no space for confusion since we always give a name to each set of numbers, indicating its nature. For example, the set \( S^1 = (1, 2) \) indicates a bourse with traders 1 and 2, the set \( \Xi = \{1, 2\} \) indicates that the states of nature under consideration are 1 and 2, and \( A(S) = \{(1, 1), (1, 2)\} \) indicates that there are two securities brought to bourse \( S \) by its members, the first security paying one unit in each of the two states of nature, whereas the second security paying one unit in the first state of nature and two units in the second state. All examples below are valid for any \( N \)-replication of these economies (see explanations in Example 3).

**Example 1 (Bourse structure affects welfare):** Our objective in this first example is to compare three different economies, one with trader set \( S^1 = (1, 2) \), another with \( S^2 = (1, 2, 3) \), and another with \( S^3 = (1, 2, 4) \). In this simple example, we shall focus on the bourse evaluation process, which is necessary for the analysis of period 0 in the subsequent examples. Commodity and security trading occur as described in the model above. We consider two securities \( J = \{1, 2\} \) and two states of nature in period 2, \( \Xi = \{1, 2\} \). Security endowments are \( e^1_1 = 6, e^1_2 = 6 \), and \( e^2_1 = e^2_2 = e^3_1 = e^3_2 = e^4_1 = e^4_2 = 0 \). Security payoffs are \( a_1 = (1, 0) \) and \( a_2 = (0, 1) \). Thus, the security structure is complete for each bourse \( S = S^1, S^2, S^3 \), \( A(S) = \{(1, 0), (0, 1)\} \). Let trader \( i \)'s utility be as follows\(^{27}\)

\[ u^i(x_0, x_1, x(1), x(2)) = u^i_0(x_0)u^i(x_1, x(1), x(2)), \tag{5} \]

where \( u^i(x_1, x(1), x(2)) = \alpha_1^i \ln x_1 + \alpha^i(1) \ln x(1) + \alpha^i(2) \ln x(2) \) and \( u^i_0(x_0) = (1/2) \ln x_0 \). This utility functional form is also assumed in the next examples of the paper. The

\(^{27}\) Under the utility function (5), the maximum value of \( u^i(x_0, \cdot) \) is independent of \( x_0 \). Other types of utility function \( u^i \) are also admissible, as long as period 0 consumption and periods 1 and 2 consumptions are not additive, i.e., \( u^i_0(x_0) + u^i_1(x_1, x(1), x(2)) \); if so, the decision of choosing a bourse would not reflect the trading opportunities associated with a bourse.
relevant information for this first example is the following: Traders 1 and 2’s endowments and preference parameters are $(\omega_1, \omega^1(1), \omega^1(2)) = (2, 2, 0)$, $(\alpha_1, \alpha^1(1), \alpha^1(2)) = (1, 1, 0)$, $(\omega_2, \omega^2(1), \omega^2(2)) = (2, 0, 2)$ and $(\alpha_2, \alpha^2(1), \alpha^2(2)) = (1, 0, 1)$, respectively. We perform comparative statics with the three economies. Trader 3 is rich today and prefers to consume today, i.e., $(\omega_3, \omega^3(1), \omega^3(2)) = (6, 2, 2)$ and $(\alpha_3, \alpha^3(1), \alpha^3(2)) = (1, 1/2, 1/2)$. Trader 4 is rich today and prefers to consume tomorrow, i.e., $(\omega_4, \omega^4(1), \omega^4(2)) = (6, 2, 2)$ and $(\alpha_4, \alpha^4(1), \alpha^4(2)) = (1/2, 1, 1)$. Observe that trader 4 allows traders 1 and 2 to transfer wealth to those nodes where consumption is more valued to them. Trader 3 is not bound to make such transfers, as trader 3 has high endowments in the node where he most values consumption (period 1). Thus, trader 4 has better complementarities (in endowments and preferences) with traders 1 and 2 than trader 3 has. Moreover, trader 4 allows traders 1 and 2 to better diversify their risk in bourse $S^3$ than if they were alone in bourse $S^1$. Let us abbreviate notation and redefine trader $i$’s utility $u_i^1$ evaluated at the equilibrium point $\bar{x}(F[i;I]) = (\bar{x}, 1, \bar{x}(2))(\bar{x}(F[i;I]))$ as follows: $u_i^1(\bar{x}(F[i;I])) = U_i^1(S)$, with $i \in S$ (in this example each trader belongs to only one bourse). The following tables give the traders’ indirect utilities and security trading for different bourses.

<table>
<thead>
<tr>
<th>$S^1 = (1, 2)$</th>
<th>$S^2 = (1, 2, 3)$</th>
<th>$S^3 = (1, 2, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$U_1^1(S^1) = 2.7726$</td>
<td>$U_1^1(S^2) = 2.8904$</td>
<td>$U_1^1(S^3) = 3.0754$</td>
</tr>
<tr>
<td>$U_2^1(S^1) = 2.7726$</td>
<td>$U_2^1(S^2) = 2.8904$</td>
<td>$U_2^1(S^3) = 3.0754$</td>
</tr>
<tr>
<td>n.a.</td>
<td>$U_3^1(S^2) = 2.7726$</td>
<td>$U_3^1(S^3) = 3.3965$</td>
</tr>
</tbody>
</table>

Table 1

<table>
<thead>
<tr>
<th>$(y_1^1(S), y_2^1(S))$</th>
<th>$S^1 = (1, 2)$</th>
<th>$S^2 = (1, 2, 3)$</th>
<th>$S^3 = (1, 2, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(6, -6)$</td>
<td>$(4, -6)$</td>
<td>$(3.2941, -6)$</td>
<td></td>
</tr>
<tr>
<td>$(-6, 6)$</td>
<td>$(-6, 4)$</td>
<td>$(-6, 3.2941)$</td>
<td></td>
</tr>
<tr>
<td>n.a.</td>
<td>$(2, 2)$</td>
<td>n.a.</td>
<td></td>
</tr>
<tr>
<td>n.a.</td>
<td>n.a.</td>
<td>$(2.7059, 2.7059)$</td>
<td></td>
</tr>
</tbody>
</table>

Table 2

Table 2 shows the security trading for each trader in the different bourses - notice that security positions $e^1_j + y^1_j$ satisfy the box constraint for every $(i, j)$. In Table 1 we can see that traders $i = 1, 2$’ indirect utilities $U_i^1$ are substantially higher when they trade in bourse $S^3$ with trader $i = 4$ than when they trade in the bourse $S^2$ with trader $i = 3$. These observations indicate that trading complementarities are, indeed, an important determinant of traders’ welfare and security trading. ✽
3.2 Bourse formation

Given the bourse structure $F(I)$ and our specification of the utility function (1), we can define trader $i$’s utility via an equilibrium point $\tilde{x}(F[i; I]) \equiv (x_1, x(1), ..., x(\Xi))(F[i; I])$ as follows

$$V^i(x_0, F[I; i]) \equiv u^i(x_0, \tilde{x}(F[i; I]))$$

(6)

Lemma 2 in the Appendix guarantees that we can always choose a security trading equilibrium point $\tilde{x}(F[i; I])$ where we evaluate the utility $u^i(\cdot)$, and therefore, trader $i$’s indirect utility function is well defined, so it is trader $i$’s utility in period 0. Observe that the evaluation of trader’s bourse membership enters indirectly into his utility $u^i$ through the access to income and the risk sharing that he gains from trading the securities offered in the bourse he belongs to. We find appropriate to refer to function (6) as the trader $i$’s “equilibrium bourse utility function”.

It is well known that different beliefs among traders on the equilibrium realizations may lead to a problem of non-existence. To avoid this possibility we impose the standard “rational expectations hypothesis”; that is, traders agree on the realization of prices at each state (consensus) and simultaneously believe that there is a single possible price in each state (degenerate beliefs).\textsuperscript{28} Thus, traders’ beliefs about the realization of prices in each state are self-fulfilling. Moreover, multiplicity of trading equilibria can occur in our financial economy given a bourse structure $F(I)$. Instead of trying to impose strong assumptions that guarantee uniqueness of the trading equilibrium for a given bourse structure, we assume that there is consensus among traders on the equilibrium realization.\textsuperscript{29} In other words, the trading equilibrium realization is self-fulfilling. At this point, it is important to notice that we can select any security trading equilibrium, and in particular the self-fulfilling trading equilibrium, associated with a bourse structure (as shown in the Appendix), and evaluate the utility function $u^i$ in that equilibrium selection $\tilde{x}(F[i; I])$. Thus, $V^i(x_0, F[I; i])$ is uniquely defined.

Now observe that nothing prevents $u^i(x_0, \tilde{x}(F[i; I]))$ to exhibit ever-increasing gains from larger bourse sizes. One could even consider a more involved utility function that explicitly incorporates the bourse structure as a source of utility. For example, we can consider a trader $i$ with single membership in a bourse $S^r$ and the following utility function: $V^i(x_0, S^r) = \left(x_0 - \frac{4}{|S^r|}\right) \tilde{x}(S^r)$, where $S^r$ is an $r$-replica of a bourse $S$ formed by $n$ types of traders (i.e., bourse $S^r$ has $r$ traders of each type), and $|S^r|$ denotes the number

\textsuperscript{28}No information problems are considered here. For such problems in a context of general equilibrium see Radner (1978).

\textsuperscript{29}See Section 4 below for examples of economies with a unique trading equilibrium.
of traders in $S^*$. A plausible interpretation of this utility function is that the trader not only gets utility from the good consumption (itself depending on the risk sharing attained in the bourse), but also his utility increases the bigger the bourse because it provides him with more information.\footnote{Anecdotal evidence on the importance of data collected by the exchanges, see “NYSE Euronext invites rivals to use space at data centres”, Financial Times, November 26, 2012. See Ready (2012) for the impact of guarding information on platforms’ costs.} Multiple Listing System is a good example of an exchange for real estate where brokers share a database with information about their properties that are available for sale. Similarly, in exchanges for securities, such as the NYSE or NASDAQ, a security trader can see the price of a stock put offered on sale by other brokers through the exchange. We impose the following assumption to show that equilibrium exists in a context where bourses have no apriori bound on size:\footnote{In order to prove existence of equilibrium in a local public good economy using the technique of core decentralization, one must assure that the core converges as the number of replica economies increases. In the case of purely private goods, the basic intuition is that convergence takes place because the market power of an individual diminishes (Debreu-Scarfi (1963)). But if one naively inserts a pure public good, then as the size of the economy increases, the intrinsic value of the public good increases indefinitely, which in particular is at odds to the diminishing market power condition (even if the convergence takes place, one cannot say if it is due to diminishing individual market power or due to overwhelming value of the public good). The main novelty of AW is to show that a Debreu-Scarf like core convergence result holds in local public good economies. Assumption A3 assures that the “Desirability of wealth” assumption of AW holds for our economy.} 

\textbf{(A3)} There is a bundle of goods $x^*_0 \in \mathbb{R}_+$ such that for any economy $(I, \alpha)$, any consumer $i \in I$, and any $x^i_0 \in \mathbb{R}_+$, we have

$$u^i(x^*_0 + x^i_0, \omega^i(1), \ldots, \omega^i(\Xi)) \geq u^i(x^i_0, \sum_i (\omega^i(1), \ldots, \omega^i(\Xi))). \tag{7}$$

Assumption A3 permits ever-increasing gains from larger “bourses” while, at the same time, allows for small bourses. The assumption says that, even in the worst case scenario where trader $i$ cannot diversify risk in any bourse and, as a consequence, consumes his endowment $(\omega^i(1), \ldots, \omega^i(\Xi))$, the trader prefers to consume a very large amount of the private good in period 0, $x^*_0 + x^i_0$, rather than consuming the aggregate endowments in periods 1 and 2. Notice that the consumption $x^*_0 + x^i_0$ can be very large and even unfeasible for the trader - we require only the existence of such a large consumption $x^*_0$.

To guarantee equilibrium existence we need that those traders who are similar in attribute space are near-substitutes in the economy. In club economies, the local public good enters exogenously in the utility function, and, therefore, this property can be written as an assumption (see, for example, AW’s condition f). For our bourse economy, this
property cannot be taken as an assumption, because trader $i$’s utility from the local public is the equilibrium consumption vector of periods 1 and 2, $\bar{x}(F[i; \mathbf{I}])$, which is endogenous to the model. This in turn makes the utility $u^i(x_0, \bar{x}(F[i; \mathbf{I}]))$ dependent on the securities trading equilibrium. Proposition 1 below asserts that the securities trading equilibrium $\bar{x}(F[i; \mathbf{I}])$ is continuous in traders’ attributes (endowments and utilities). We explicitly refer to this result as a proposition, as it stands as a contribution to the theory of clubs. We write $\mathcal{I}^\alpha$ to refer to an economy $(\mathbf{I}, \alpha)$, and call an open and dense set with null complement a generic set. Notice that for simplicity, we have considered a finite dimensional manifold of utility functions.

Proposition 1: There exists a generic set of economies for which, given $\lambda > 0$, there is a $\gamma > 0$ such that, for any pair of economies $(\mathbf{I}, \alpha)$ and $(\mathbf{I}, \beta)$, if $d(\alpha(i), \beta(i)) \leq \gamma$ for any $i$, then $|V^{\alpha(i)}(x_0, F[i; \mathbf{I}^\alpha]) - V^{\beta(i)}(x_0, F[i; \mathbf{I}^\beta])| < \lambda$.

The proof of Proposition 1, in the Appendix, makes use of the Transversality theorem. There are two subtleties in this proof. The first is that we have to show that the securities trading equilibrium is a continuous differentiable function of commodity and security prices. The second subtlety to prove Proposition 1 has to do with equilibrium regularity. Precisely, we have to show that there exists a generic set of endowments and utilities such that the set of security trading equilibria is a continuously differentiable function of the endowment and utility assignment. For that purpose, Geanakoplos and Polemarchakis [1986] offer good guidance. However, our framework is different and therefore we must adapt their proof to an economy where: 1) the security market clears for each bourse; 2) the trading period accounts for the commodity and securities; and selling a security requires having enough endowment of that security (the box constraint must be satisfied).

Definition 2 (c($\varepsilon_0$)-equilibrium of the bourse formation): A price taking $c(\varepsilon_0)$-bourse structure equilibrium for period 0 is an ordered triple $((x_0^i, F(\mathbf{I})), \Pi)$ that consists of a consumption vector $x_0^i$, a bourse structure $F(\mathbf{I})$, and a participation price system $\Pi$ such that,

(D2.i) $\sum_{i \in \mathbf{I}}(x_0^i - \omega_0^i) + \sum_{S_k \in F(\mathbf{I})} z(S_k) \leq 0$.

(D2.ii) For each $S \subset \mathbf{I}$, profits are non positive, i.e., $\sum_{i \in S} \pi^i(S) - z(S) \leq 0$.

(D2.iii) For each $i \in \mathbf{I}$, any $N \subset \mathbf{I}$ with $i \in N$, and any bourse structure deviation $\tilde{F}_{F(N)}(\mathbf{I})$, if $V^i(y_0^i, \tilde{F}_{F(N)}[i; \mathbf{I}]) > V^i(x_0^i, F[i; \mathbf{I}])$, then $y_0^i - \omega_0^i + \pi^i(\tilde{F}_{F(N)}[i; \mathbf{I}]) > -\varepsilon_0$.

32Extending our framework to consider instead an infinite dimensional manifold of utility functions would complicate substantially the proofs, and thus is left for some future research.
The equilibrium condition (D2.iii) deserves three remarks. First, it says that, if a set of traders $N$ deviates and forms another bourse structure with the remaining traders of $I$ staying in the same bourses as in $F(I)$ but without the traders belonging to the set $N$, then the budget constraints of those traders in $N$ are violated in excess of the communication cost. This communication cost consists of small frictions in the economy, and affect the opportunities to change bourse memberships.

Second, any deviation by a subset of traders $N$ incorporates the new risk sharing possibilities that these traders have access to in the new candidate bourses, and therefore, must have different membership prices. An off-equilibrium (self-fulfilling) deviation, $\tilde{F}_{F(N)}(I)$, consists of a subset of traders such that their respective membership fees associated with the new deviating bourses exceed their respective good endowment net of consumption. By the same argument as above, the function $V^i(y_0^i, \tilde{F}_{F(N)}[i; I])$ in condition (D2.iii) is also uniquely defined.

Finally, notice that it may occur that, depending on the composition of the set of traders, some traders cannot be accommodated in their preferred bourses. But since our economy is large, these traders constitute only a small proportion of the total population.\[33\]

### 3.3 Equilibrium for the bourse economy

**Definition 3** (*c*(\(\varepsilon_0\))-equilibrium of the bourse economy): We say that a vector \((x_0^I, F(I), \Pi, (x_1^I, x^I(1), ..., x^I(\Xi), y^I, q)(F(I)))\) constitutes a price taking *c*(\(\varepsilon_0\))-equilibrium for our bourse economy if

\[(D3.i) \ (x_1^I, x^I(1), ..., x^I(\Xi), y^I, q)(F(I)) \text{ is a security trading equilibrium for } F(I).\]

\[(D3.ii) \text{ Given } (x_1^I, x^I(1), ..., x^I(\Xi), y^I, q)(F(I)), \ (x_0^I, F(I), \Pi) \text{ is a *c*(\(\varepsilon_0\))-equilibrium for the bourse formation.}\]

### 3.4 Existence

**Theorem 1:** Let us assume A1, A2, and A3. If there are sufficiently many traders with attributes represented in the economy, there exists a generic set of bourse economies for which there is a price taking *c*(\(\varepsilon_0\))-equilibrium with possibly ever-increasing gains from larger bourses.

\[33\text{We prefer to avoid further notation and refer to the original paper of AW for the refinement of accommodating the equilibrium notion by taking into account these reminders.}\]
The proof of Theorem 1 is left for the Appendix. There, we explain in detail every step. Here, we just state the steps of the proof and the main subtleties in each step. First, we show that, under Assumption set A1, a trading equilibrium exists for a given bourse structure. The proof of existence of a trading equilibrium has some novel aspects. In the previous literature of general equilibrium with incomplete markets and a unique trading platform, there is one auctioneer that chooses both the commodity price and the security prices in the simplex. In our model with multiple bourses, it may happen that this auctioneer chooses the price of one security equal to 1, and therefore, the remaining commodity and security prices would be zero. But then, there would be bourses whose traders face commodity and security prices equal to zero, and thus their budget constraints would hold with equality, and lower semicontinuity of the budget correspondence would fail. In our proof of existence of a trading equilibrium (given a bourse structure), the commodity price is normalized to 1 and each security relative price belongs to a different compact sets, which allows us to guarantee the property of lower semicontinuity of the budget correspondence. But in this framework we need Assumption (A1.ii) to guarantee that security prices cannot be very large, so that the security market clears in each bourse.

Second, we show that there exists a measurable selector of the set of security trading equilibria, at which traders evaluate their bourse memberships.

Third, we show that, under Assumption set A2 and A3, a $c(\varepsilon_0)$-equilibrium of the bourse formation exists, given the bourse evaluation at the selected trading equilibrium. For the latter, we apply AW [2008, Theorem 2], which says that: “Given communication costs, for all sufficiently large economies the core is nonempty and the set of $c(\varepsilon_0)$-price-taking equilibrium outcomes (similar notion than our Definition 2) is equivalent to the core”. In our setting, which also has a large and finite number of “individuals”, we need to assure that all assumptions that lead to this theorem are satisfied.\textsuperscript{34}  The trickiest part is to guarantee continuity on traders’ attributes for a generic set of economies. Proposition 1 addresses this issue (see the details of this proof above). Then, for each economy in a generic set of endowments and utilities, we have to find a compact subset (this set exists because the generic set is open), and use this set to extend replica economies.\textsuperscript{35}

\textsuperscript{34}Namely, (a)-(h) and “desirability of wealth” in AW - see our proof of Lemma 3 in the Appendix for details.

\textsuperscript{35}AW’s result relies on the use of a compact subset of a generic set of traders’ attributes where the continuity property on attributes holds.
4 Positive predictions

For the following normative analysis, let us work with a more tractable environment, where there are only two states of nature in period 2 and the utility function is as follows:

\[ u^i(x_0, x_1, x(1), x(2)) = u^i_0(x_0)U^i_1(F[i; I]), \]

where \( U^i_1(F[i; I]) \equiv u^i_1(\tilde{x}(F[i; I])) \) stands for the trader \( i \)'s indirect utility attained in periods 1 and 2 with membership \( F[i; I] \). We will provide characterizations of different bourse structures in terms of this indirect utility function. Also, we will offer examples and discussions that illustrate how the indirect utility \( U^i_1(F[i; I]) \) depends on the bourse members' attributes (endowments and preferences for consumption).

4.1 Large unique bourse Vs. multiple small bourses

What makes a single global exchange a natural monopoly? This questions has been often debated in the financial press.\(^36\)

**Proposition 2:** Let us assume that, for all \( i \in I \), the inverse of \( u^i_0 \) is homogeneous of degree \( \eta > 0 \). Then, a large unique bourse \( S^* \), with \( |S^*| = |I| \), forms in equilibrium if and only if, for any smaller bourse \( S \in F(I) \), the following condition holds for all \( i \in S \cap S^* \):

\[
\frac{U^i_1(S)^\eta}{\sum_{i \in S} U^i_1(S)} \left( \sum_{i \in S} \omega^i_0 - z(S) \right) < \frac{U^i_1(S^*)^\eta}{\sum_{i \in S^*} U^i_1(S^*)} \left( \sum_{i \in S^*} \omega^i_0 - z(S^*) \right)
\]

(C1)

Condition (C1) is necessary and sufficient for the formation of a large unique bourse. This condition identifies the technology of bourse formation and traders’ characteristics (endowments and bourses’ valuation) as the key drivers that drive the bourse formation process. A single global bourse is likely to emerge in equilibrium if the formation cost of bourse \( S^* \) does not far exceed the cost of forming the other bourse \( S \). Thus, the technology of bourse formation stands as a crucial determinant on the equilibrium number of bourses. Even more interesting is the fact that when period 0 aggregate consumption is the same in the two bourses, i.e., \( \sum_{i \in S} \omega^i_0 - z(S) = \sum_{i \in S^*} \omega^i_0 - z(S^*) \), then it is the relative trader’s valuation of the bourses (itself a function of traders’ complementarities within a bourse) what shapes the size and number of bourses. In particular,

**Corollary 1:** If \( \sum_{i \in S} \omega^i_0 - z(S) = \sum_{i \in S^*} \omega^i_0 - z(S^*) \), the single global bourse forms in equilibrium if and only if, for all \( i \in S \cap S^* \), trader \( i \)'s relative value of belonging to

\(^36\)See The Economist, March 25, 2006 (http://www.economist.com/node/6978712): “Liquidity and technology will inevitably make trading a natural monopoly”.
the big global bourse \( S^* \) is higher than the relative value of belonging to any other smaller bourse \( S \), i.e.,
\[
\frac{\sum_{i \in S} U_i^1(S)}{\sum_{i \in S} U_i^1(S)} < \frac{\sum_{i \in S^*} U_i^1(S^*)}{\sum_{i \in S^*} U_i^1(S^*)}.
\] (C2)

Examples 2 and 3 below illustrate this point. In both examples, all possible bourses have a formation cost function proportional in bourse size and with the same factor (e.g., \( z(S) = 3|S| \)), so formation costs are not discriminatory.\(^{37}\) Example 2 shows that a large bourse forms in equilibrium when bourse formation costs are proportional in size, as long as there exist good complementarities among traders. Example 3 provides the opposite case, where a small bourse provides more welfare given the poor complementarities in a larger bourse, with formation cost again proportional in size.

**Example 2 (Large bourses are optimal if trading complementarities are good):** Being inspired by Fact Set 1, we illustrate here how good trading complementarities are enough for a large bourse to emerge. Let us add to the set-up presented in Example 1 an initial period 0 where bourses form. We assume no uncertainty between periods 0 and 1. Our framework is again characterized by non-anonymity and market completeness. We restrict our attention to the set of traders \( \mathbf{I} = \{1, 2, 4\} \). The possible bourses are \( S^1 = \{1, 2\} \), \( S^3 = \{1, 2, 4\} \), \( S^4 = \{1, 4\} \), and \( S^5 = \{2, 4\} \). Good endowments in period 0 are \( \omega_0^1 = 7 \), \( \omega_0^2 = 7 \), and \( \omega_0^4 = 5.9 \). In period 0 traders must pay for the membership fee to have access to the bourse’s trading facility. Trader \( i \)’s membership fee in bourse \( S \) is denoted by \( \pi^i(S) \).

We compute the non-anonymous membership fees by considering a welfarist agent that maximizes the weighted sum of indirect utilities subject to individual budget constraints in period 0, and such that membership fees cover the bourse formation cost. It can be shown that the optimal membership fee for a trader \( i \) in a two-traders bourse \( S = (i, k) \) is given by
\[
\pi^i(S) = \frac{U_{i}^k(S)\omega_0^i - U_i^1(S)\omega_0^k}{U_i^1(S) + U_i^k(S)} + \frac{U_i^1(S)}{U_i^1(S) + U_i^k(S)}z(S)
\] (8)
whereas if it is a three-traders bourse \( S = (i, j, k) \) we would have
\[
\pi^i(S) = \frac{\omega_0^j [U_{i}^j(S) + U_i^k(S)] - U_i^j(S)(\omega_0^j + \omega_0^k)}{U_i^1(S) + U_i^j(S) + U_i^k(S)} + \frac{U_i^1(S)}{U_i^1(S) + U_i^j(S) + U_i^k(S)}z(S)
\] (9)

These formulas give an efficient characterization of the non-anonymous bourse membership pricing.\(^{38}\) Notice that in both cases the membership fee equations (8) and (9)

\(^{37}\) Denoting by \( |S| \) the cardinality of \( S \), we say that two bourses \( S^1 \) and \( S^2 \) with \( |S^1| < |S^2| \) have bourse formation costs proportional in size if \( z(S^1)/|S^1| = z(S^2)/|S^2| \).

\(^{38}\) An anonymous pricing context would make better (worse) off those traders who value more (less) the bourse than if the context were non-anonymous (not all surplus can be subtracted when pricing is anonymous).
consist of the sum of a pure transfer (first term on the right hand side) and a proportional tax (second term). The pure transfer reflects the trader’s valuation of the trading opportunities in bourse $S$. The tax is such that all traders share the bourse $S$’s formation cost $z(S)$. We consider bourse formation costs of the form $z(S) = 3|S|$. These costs are proportional to the bourse size because we seek to emphasize the role of complementarities in traders’ attributes (we wish to see the complementarities as the driving force that determine the bourse composition). The membership fee values are

\begin{align*}
\pi^1(S^1) &= 3, \\
\pi^2(S^2) &= 4.0022, \\
\pi^3(S^3) &= 1.9978, \\
\pi^4(S^4) &= 3.4889 \\
\pi^5(S^5) &= 2.0222.
\end{align*}

The indirect utilities are

\begin{align*}
V^1(\omega^1) &= 1.9218, \\
V^2(\omega^2) &= 1.3418, \\
V^3(\omega^3) &= 1.3488, \\
V^4(\omega^4) &= 2.0253, \\
V^5(\omega^5) &= 2.1662.
\end{align*}

Table 3

<table>
<thead>
<tr>
<th></th>
<th>$S^1 = (1, 2)$</th>
<th>$S^4 = (1, 4)$</th>
<th>$S^5 = (2, 4)$</th>
<th>$S^3 = (1, 2, 4)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trader 1</td>
<td>$V^1(S^1)$ = 1.9218</td>
<td>$V^1(S^4)$ = 1.3418</td>
<td>$V^1(\omega^1) = 1.3488$</td>
<td>$V^1(S^3) = 1.9312$</td>
</tr>
<tr>
<td>Trader 2</td>
<td>$V^2(S^1)$ = 1.9218</td>
<td>$V^2(\omega^2) = 1.3488$</td>
<td>$V^2(S^5) = 1.3418$</td>
<td>$V^2(S^3) = 1.9312$</td>
</tr>
<tr>
<td>Trader 4</td>
<td>$u^4(\omega^4) = 2.0253$</td>
<td>$V^4(S^4) = 2.1662$</td>
<td>$V^4(S^5) = 2.1662$</td>
<td>$V^4(S^3) = 2.3016$</td>
</tr>
</tbody>
</table>

In this example with complete markets and proportional bourse formation costs, we find that the three traders prefer to sort in the largest possible bourse, $S^3 = (1, 2, 4)$. This occurs even if traders 1 and 2 pay a higher membership fee in the three-trader bourse. Thus, good complementarities in preferences and endowments are sufficient here to obtain a large bourse in equilibrium. ♠

**Example 3 (Size versus tailored efficiency):** This example illustrates the opposite case to Example 2, that poor complementarities alone can lead a small bourse to form in equilibrium - in accordance with Fact Set 2, which suggests that traditional large exchanges are ill-suited to certain types of institutions. Again, we assume that markets are complete. For this example it is worth considering larger replica bourses with an increasing number $N$ of traders of each type. For example, in the three-traders bourse case, the bourse $S^2_N = (1, 2, 3)^N$ denotes a bourse composed by $N$ traders of each type. Again, we assume that bourse formation costs are proportional to bourse size in order

\[39\] The indirect utility of a trader who does not belong to a bourse is given by his utility $u^i$ evaluated in his good endowments of period 0 and the equilibrium consumption point that the trader get in a given bourse in periods 1 and 2 (recall that there is only one good and $u^i$ is strictly increasing in the consumption of the good).

\[40\] The properties of self-picked traders motivate the expression "tailored-efficiency." Observe that for any given bourse $S_N$ with $m$ types of traders and $N$ traders of each type, the trader $i$’s indirect utility given in Example 1 is such that $U^i(S) = U^i(S_N)$, for any $N \in \mathbb{N}$. This remains true for all the examples in the paper. To see this, notice that commodity and security market clearing equations in $S_N$ are the same as in $S$. Also, the membership pricing expressions (8) and (9) hold
to isolate the role of complementarities on the equilibrium outcome. In particular, we assume $z(S_N) = 3N|S_N|$, for any bourse $S_N$. Trader 3’s endowment in period 0 is $\omega_3^0 = 5.9$. Our objective here is to compare the small bourse $S^1 = (1,2)$ with the big bourse $S^2_N = (1,2,3)^N$, with $N$ large, where trader $i = 3$ has poor complementarities with traders $i = 1,2$. The indirect utilities in these bourses are

\[
\begin{array}{|c|c|c|}
\hline
\text{Trader} & S^1 = (1,2) & S^2_N = (1,2,3)^N \\
\hline
\text{Trader 1} & V^1(S^1) = 1.9218 & V^1(S^2_N) = 1.8843 \\
\text{Trader 2} & V^2(S^1) = 1.9218 & V^2(S^2_N) = 1.8843 \\
\text{Trader 3} & u^3(\omega^3) = 2.0253 & V^3(S^2_N) = 1.7498 \\
\hline
\end{array}
\]

Table 4

In Table 4 we see that each trader $i$’s indirect utility in bourse $S^1 = (1,2)$ is higher than in the larger bourse $S^2_N = (1,2,3)^N$, for all $i$ and any $N$. This result contrasts with the values given in Table 1, where traders 1 and 2 prefer to be in a larger bourse with trader 3 (there, only the bourse evaluation process was under consideration). However, in the bourse formation process, the benefit for traders 1 and 2 of enlarging the bourse with a third trader with poor complementarities (like trader 3) is not enough to compensate the cost of paying a higher membership fee - in bourse $S^1$ the membership fees are $\pi^1(S^1) = \pi^2(S^1) = 3$, while in bourse $S^2_N$ the membership fees are $\pi^1(S^2_N) = \pi^2(S^2_N) = 3.3166$ and $\pi^3(S^2_N) = 2.3667$ for trader 3. We conclude that poor complementarities between trader $i = 3$ and traders $i = 1,2$ make traders $i = 1,2$ prefer the small bourse $S^1$ to the larger bourse $S^2_N$, for any $N$-replica. This result identifies poor traders’ complementarities (in preferences and endowments) as an important force against the tendency toward a unique bourse.

\section*{Remark 2:}

With the specific logarithmic utility function considered in Example 1, condition (C1) in Proposition 2 can be rewritten in a quite similar way:

\[
\left( \frac{U^i_1(S)}{\sum_{i \in S} U^i_1(S)} \sum_{i \in S} (\omega^0_i - z(S)) \right)^{U^i_1(S)} \leq \left( \frac{U^i_1(S^*)}{\sum_{i \in S^*} U^i_1(S^*)} \sum_{i \in S^*} (\omega^0_i - z(S^*)) \right)^{U^i_1(S^*)}
\]

Examples 2 and 3 verify condition (C3).
Remark 3: What is the effect of the implementation of a Tobin tax on certain (but not all) bourses? It can easily be shown that for two possible bourses - subject to legislation by regulators - with the same traders and security structure, traders will participate in the bourse with lower execution rates for their trades. This result adds to the current debate on international financial transactions (Stiglitz [1989]).

To see this, notice that higher execution rates in a bourse, due to a Tobin tax (see Tobin [1984]), can be accommodated in the form of higher formation cost, which penalizes traders in that bourse through higher memberships. Since traders are free to move to their most preferred bourse, we can infer that the bourse with high execution rates (or equivalently, high formation costs) may not be formed. In other words, if traders are free to choose their preferred bourses, then a Tobin tax on the financial transactions in some but not all bourses may not be effective.

4.2 Endogenous market incompleteness

Proposition 3: Let us assume that, for all $i \in I$, the inverse of $u^i_0$ is homogeneous of degree $\eta > 0$, and let security endowments be such that bourse $S^*$ has complete markets, while bourse $S$ has incomplete markets. Then, bourse $S^*$ emerges in equilibrium if and only if condition (C1) holds for any $S \neq S^*$.

An immediate application of Propositions 2 and 3 is that:

Corollary 2: A unique large bourse with complete markets emerges in equilibrium if and only if condition (C1) holds for the pair of bourses $I$ and $S$, where $I$ is characterized by complete markets, and $S \subset I$ by incomplete markets. Otherwise, a small bourse with incomplete markets emerges in equilibrium.

Condition (C1) in Proposition 3 depends on both the technology of bourse formation, $\gamma(S)$, and the trading complementarities (through $U^i(S)$). Now we will isolate each factor and examine the role of 1) the technology of bourse formation and 2) the trading complementarities, in the formation of a bourse with an incomplete security structure.

CASE 1: Let us examine first how the technology of bourse formation plays a crucial role in determining the incompleteness of the markets. In order to illustrate this point with an intuitive bourse formation cost function, let us start by illustrating a simple economy where two group of traders have good complementarities in the security endowments.

In this economy there are two sets of traders $I'$ and $I''$ ($I = I' \cup I''$), the first one $I'$ with preferences over consumption only in the first $E' = \{1, \ldots, E'\}$ states of nature, and

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42For anecdotal evidence of this debate, see “Tobin tax talk not without merit”, Financial Times, December 21, 2009.
the the second one $I''$ with preferences over consumption only in the subsequent states of
nature, $\Xi'' = \{\Xi' + 1, \ldots, \Xi\}$. The partition is such that $\Xi = \Xi' \cup \Xi''$. Let us assume
that the two sets of traders do not want to get together because a trading motive in the
good market. For this we just need assign each trader zero endowments of the good in
those nodes where he does not value consumption, i.e., $\omega^i(\Xi' + 1) = \ldots = \omega^i(\Xi) = 0$, if
$i \in I'$, and $\omega^i(1) = \ldots = \omega^i(\Xi') = 0$, if $i \in I''$. The two sets of traders, $I'$ and $I''$, have
good complementarities in security endowments if the first set of traders have securities
that only pay in one of the states of $\Xi''$, while the second set of traders have security
endowments that pay in one of the states of $\Xi'$. In particular, let $r_j$ be the column
vector with all coordinates equal to 0 except coordinate $j$ which is 1, and consider the
following security structures: if $S \subseteq I'$ then $A(S) = A' = [r_1 \ldots r_{\Xi'}]$; if $S \subseteq I''$ then
$A(S) = A'' = [r_{\Xi' + 1} \ldots r_{\Xi}]$; otherwise, there is complete markets, that is, $A(I) = [r_1 \ldots r_{\Xi}]$.
In this context, a sufficient condition for the formation of multiple bourses with incomplete
markets is:

**Proposition 4:** If $z(S, A(S)) = |S|f(|A(S)|)$ for all possible bourses, where function $f$ is non-linear and strictly increasing in the number of securities, then the equilibrium
bourse structure is characterized by multiple bourses with incomplete markets.

The issue of equilibrium optimality is important and can be related to previous results
in the literature of general equilibrium with incomplete markets. To make the connexion
clear, recall the well known result in that literature: equilibrium efficiency is “constrained”
to the market incompleteness. Moreover, given that any $\varepsilon_0$-equilibrium is in the core for
an economy with sufficiently many traders of each attribute (see AW), we can also assert
that an equilibrium of the bourse formation process is also efficient. Therefore, we can
conclude that a $c(\varepsilon_0)$-equilibrium of the bourse economy is efficient if condition (C1) in
Proposition 4 holds. Otherwise, it is constrained efficient, given the frictions of bourse
formation in the economy. Proposition 4 clarifies that constrained efficiency may arise
in equilibrium if the cost of forming a bourse with complete markets is too expensive
compared with a bourse with incomplete markets.

**CASE 2:** What is perhaps more interesting is the second possibility, where trading
complementarities are isolated and become the only force that drives the incompleteness
of the market. The next corollary, that follows Proposition 2, asserts that this occurs when
period 0 aggregate consumption is the same in both the bourse with complete markets
and the bourse with incomplete markets.

**Corollary 3:** Let again security endowments be such that bourse $S^*$ has complete
markets, while bourse $S$ has incomplete markets. Then, if $\sum_{i \in S} \omega^i_0 - z(S) = \sum_{i \in S^*} \omega^i_0 - z(S^*)$,
the bourse with complete markets forms in equilibrium if and only if, for all \( i \in S \cap S^* \),

\[
\frac{U_i(S^n)}{\sum_{i \in S} U_i(S^n)} < \frac{U_i(S^*)^n}{\sum_{i \in S^*} U_i(S^*)^n}.
\]

(C4)

This result in Corollary 3 is interesting since it points out to the indirect utilities \( U_i \) as the explanatory force that explains the incompleteness/completeness of the markets. The next example illustrates this point.

**Example 4 (Endogenous market incompleteness):** We consider an economy with two states of nature in \( t = 2 \) and two existing securities, \( j = 1, 2 \), with respective payoffs \( a_1 = (1, 0) \) and \( a_2 = (0, 1) \). Traders’ identities are \( i = 5, 6, 7 \). The utility function \( u_i^j(x_1, x(1), x(2)) = x_1^{\alpha_i^j}x(1)^{\alpha_i^1}x(2)^{\alpha_i^2} \) is such that preference parameters are \( (\alpha_1^5, \alpha_5^5(1), \alpha_5^5(2)) = (0, 1, 0) \), \( (\alpha_1^6, \alpha_6^5(1), \alpha_6^5(2)) = (1, 0, 0) \) and \( (\alpha_7^1, \alpha_7^7(1), \alpha_7^7(2)) = (0, 0, 1) \).Trader 5’s good endowments are \( (\omega_5^5, \omega_5^5(1), \omega_5^5(2)) = (1, 0, 0) \). Traders \( i = 6, 7 \) have no security endowments neither at \( t = 1 \) nor at \( t = 2 \). However, trader \( i = 6 \) owns 1 unit of security \( j = 1 \) \( (e_i^6 = 1) \), which trader \( i = 5 \) is willing to buy and trader \( i = 6 \) is willing to sell. Trader \( i = 7 \) owns 1 unit of security \( j = 2 \) \( (e_i^7 = 1) \). However, neither trader \( i = 5 \) and \( i = 6 \) are interested to buy this security, nor trader \( i = 7 \) wants to sell it, since it pays in the state where trader \( i = 7 \) most values consumption. Given these security endowments, we have that bourses \( S^6 = (5, 6) \) and \( S^7 = (5, 7) \) have each of them incomplete markets (IM), while bourses \( S^8 = (6, 7) \) and \( S^9 = (5, 6, 7) \) are characterized by complete markets (CM). Traders \( i = 5, 6 \) perfectly share risk between each other with only security \( j = 1 \), while trading complementarities with trader 7 are so poor that no security trading is made at all, even when the securities available in bourse \( S^9 = (5, 6, 7) \) complete the market. Equilibrium consumptions, and thus indirect utilities \( U_i(S) \), can be shown to be:

<table>
<thead>
<tr>
<th>( S^6 = (5, 6), \text{IM} )</th>
<th>( S^7 = (5, 7), \text{IM} )</th>
<th>( S^8 = (6, 7), \text{CM} )</th>
<th>( S^9 = (5, 6, 7), \text{CM} )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( U^5(S^6) = 1 )</td>
<td>( U^5(S^7) = 0 )</td>
<td>( U^5({5}) = 0 )</td>
<td>( U^5(S^9) = 1 )</td>
</tr>
<tr>
<td>( U^6(S^6) = 1 )</td>
<td>( U^6({6}) = 0 )</td>
<td>( U^6(S^8) = 0 )</td>
<td>( U^6(S^9) = 1 )</td>
</tr>
<tr>
<td>( U^7({7}) = 1 )</td>
<td>( U^7(S^7) = 1 )</td>
<td>( U^7(S^8) = 1 )</td>
<td>( U^7(S^9) = 1 )</td>
</tr>
</tbody>
</table>

Table 5

In Table 5 we can see that bourses \( S^6 \) and \( S^9 \) are the candidates to emerge in equilibrium. We will point out now that, in the process of bourse formation in \( t = 0 \), bourse \( S^9 \) is in fact blocked by trader \( i = 7 \), because paying the bourse membership price makes him worse off than if he were alone - this occurs as long as \( z(S) \) is greater than 0. As a
result, the bourse that emerges in equilibrium is indeed the bourse with incomplete markets \( S^6 \). Let \( u^*_i(x_0) = x_0^{1/2} \), \( z(S) = |S| \) for every possible bourse \( S \), and \( \omega_0^5 = \omega_0^7 = 2 \) and \( \omega_0^7 = 1 \) (notice that this parameter values satisfy the condition of Corollary 3). We can then apply the membership price formulas (8) and (9),\(^{43}\) and compute the equilibrium “bourse” indirect utilities:

<table>
<thead>
<tr>
<th>( S^6 = (5, 6) )</th>
<th>( S^7 = (5, 7) )</th>
<th>( S^8 = (6, 7) )</th>
<th>( S^9 = (5, 6, 7) )</th>
</tr>
</thead>
<tbody>
<tr>
<td>( V^5(S^6) = 1 )</td>
<td>( V^5(S^7) = 0 )</td>
<td>( V^5({5}) = 0 )</td>
<td>( V^5(S^9) = \sqrt{2}/3 )</td>
</tr>
<tr>
<td>( V^6(S^6) = 1 )</td>
<td>( V^6({6}) = 0 )</td>
<td>( V^6(S^8) = 0 )</td>
<td>( V^6(S^9) = \sqrt{2}/3 )</td>
</tr>
<tr>
<td>( V^7({7}) = \sqrt{2} )</td>
<td>( V^7(S^7) = \sqrt{2} )</td>
<td>( V^7(S^8) = \sqrt{2} )</td>
<td>( V^7(S^9) = \sqrt{2}/3 )</td>
</tr>
</tbody>
</table>

Table 6

As we indicated before, the bourse with complete markets \( S^9 \) does not form in equilibrium because trader \( i = 7 \) gets a lower payoff than if he remains alone.\(^{44}\) This is in fact the result that Corollary 3 applied to Table 5 would predict. To sum up, this example shows that trading complementarities alone are able to drive the incompleteness of the market structures. This outcome is captured by our model of bourses formation.

\(^{43}\)Non-anonymous membership price formulas (8) and (9) imply \( \pi^5(S^6) = \pi^6(S^6) = 1 \), \( \pi^5(S^7) = \pi^6(S^8) = 2 \), \( \pi^7(S^7) = \pi^7(S^8) = 0 \), \( \pi^5(S^9) = \pi^6(S^9) = 4/3 \), and \( \pi^7(S^9) = 1/3 \).

\(^{44}\)Notice that the result in this example is independent of whether the bourse membership fees are anonymous or non-anonymous (poll taxes). The only difference between anonymous and non-anonymous bourse membership pricing is that when memberships are anonymous equilibrium bourse indirect utilities remain the same as in Table 6, except \( V^5(S^9) = V^6(S^9) = V^7(S^9) = 1 \), and, therefore, bourse \( S^6 \) still forms.

5 Extensions

**Competing trading mechanisms:** In our model a *bourse* is understood as a central market where security prices were determined by a Walrasian auctioneer, and thus were common to all traders in the bourse without the need of intermediaries (e.g., Trading-point Stock Exchange). But our model can be extended to account for other market mechanisms, such as direct search, competitive search, Nash bargaining, auctions, etc - see, for example, Rocheteau and Wright (2005) or Lagos and Wright (2007). All we need to do to use AW’s result is to guarantee that Proposition 1 holds. The enrichment of the model would give a broader sense to the notion of a bourse and is left for future research. Our use of club theory, where the trader’s utility function incorporates the equilibrium achieved by trading in the bourse, can be used in future research to model competition...
among different trading mechanisms - see Hernando-Veiciana (2005) for a model of competition among auctioneers in a large markets, Biais (1993) for a model of competition between markets under different trading rules, and and Peters (2012) for a recent survey of competing mechanisms. The trading mechanisms that “survive” would be those chosen by the trading platforms that belong to the equilibrium bourse structure.

**Bilateral trading and endogenous trading networks:** Our model can be modified to capture bilateral trading. In such framework a security can have a different price depending on the pair of traders (say $q_{ij}^k$). As a first approach, one can consider a structure where all traders within a bourse are linked with each other, and communication between a pair of traders is costless. Then, a bourse can be seen as a *connected network* where every pair of traders can directly trade securities. A trading relationship exists between two traders $i$ and $j$ if they agree to transact a security (i.e., $|y_{ij}^k| \neq 0$ for some $j$). We then can distinguish between a *complete trading network* if all traders can trade with each other, and an *incomplete trading network* if not all pairs can trade. Our approach of embedding the outcome of social interactions (i.e., security trading) in the trader’s utility function gives a new perspective on how to establish a model that endogenizes the network structure. It is important to observe that previous models of endogenous network formation, including Page and Wooders (2007, 2012), were not able to explain the endogenous formation of a *complete / incomplete* network structure because these models did not explicitly capture the equilibrium social interactions among the members of a club. In a bourses model with bilateral trading a link between two traders would arise endogenously in equilibrium when the pair of traders engages in a security transaction.

**Default and clearing houses:** We also think that our model paves the way to study other interesting questions, like the formation of clearing houses, which act as an intermediary insuring traders against the possibility of default. For that economy, traders should not only choose the trading platform, but also the clearing house.

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^45 We can then accommodate bilateral trading into our trading setup by modifying the box and budget constraints, respectively, as follows: $e_j^i + \sum_{k \in F_{[i]:I}} y_{ij}^k \geq 0$, $x_1 - \omega^i + \sum_{k \in F_{[i]:I}} \sum_{j \in J(F_{[i]:I})} \sum_{s \in S} q_{ij}^k y_{ij}^{sk} \leq 0$, and $x(\xi) - \omega(\xi) \leq \sum_{j \in J} a_j(\xi) e_j^i + \sum_{k \in F_{[i]:I}} \sum_{j \in J(F_{[i]:I})} \sum_{s \in S} a_{ij}(\xi) y_{ij}^{sk}$.

^46 The latter can occur if we consider a transaction fee for some pairs of traders, say $g_{ij}^k (y_{ij}^k)$, that is high enough that prevents any security transaction between them. The period 1 budget constraint would then be written as follows: $x_1 - \omega^i + \sum_{k \in F_{[i]:I}} \sum_{j \in J(F_{[i]:I})} \sum_{s \in S} (q_{ij}^k y_{ij}^k + g_{ij}^k (y_{ij}^k)) \leq 0$.

^47 Page and Wooders (2007, 2012) follow the seminal work by Jackson and Wolinsky (1996), which focus exclusively on strategic considerations in club network formation. Our process of bourse formation is cooperative rather than strategic, and this distances our paper from the emerging literature of endogenous networks that study the interaction between link formation and action choice in a social game with two-way links, as in Jackson and Watts (2002) and Hojman and Szeidl (2006).
6 Conclusions

This paper is pioneering in examining financial market structures and their welfare properties under the new perspective of club theory. The attractiveness of a bourse is evaluated in the light of the complementarities, in preferences and endowments, among the members of a bourse and the bourse formation costs. Then, given these bourse evaluations, traders cluster in bourses. In our opinion, our club theory approach to finance is powerful for obtaining new insights on the functioning of financial markets, in the same way that the theory of networks (Allen and Babus (2009), Bloch and Jackson (2007), Gofman (2012), and Jackson (2005, 2008)) and search theory (the list is vast; see, for instance, Duffie, Garleanu, and Pedersen (2005), Guerrieri, Shimer, Wright (2010), Lagos and Wright (2007), and Weill (2007)) contribute so much to the understanding of different issues in finance. Our novel approach using the theory of group formation provides a useful framework to model endogenous participation in trading platforms and analyze important issues, such as, the optimal bourse sizes and the completeness/incompleteness of the bourses’ security structures. In this paper we provide several characterizations and examples that shed light on these issues. Traders’ complementarities and/or the technology of bourse formation stand as the leading forces that shape the structure of bourses. Our descriptive analysis, on the other hand, shows that our new equilibrium setting, where bourse and bourses may range in size from one-person to the entire population, satisfies generic existence. We believe that the machinery developed in this paper is powerful in addressing important issues in the future, such as the role of default in the formation of clearing houses, the optimal structure of trading mechanisms that arises through competition among trading venues, and the emergence of endogenous trading networks. The present paper therefore stands as a first step in a broad research agenda.

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7 Appendix

Proof of Theorem 1: This theorem follows by Lemmas 1, 2, and 3 below. For convenience of exposition we present the proof of Proposition 1 immediately after the proof of Lemma 1.

Lemma 1: Let us assume (A1.i) and (A1.ii). Then, for a fixed bourse structure, there exists a security trading equilibrium.

Proof of Lemma 1: Let us fix a bourse structure $F(I)$, and consider a generalized game in which players maximize payoffs in truncated compact sets. The players are the traders and one auctioneer for the security markets. Let $\mathbf{V} = 2 \max_{j \in J} \sum_{i \in I} e_j^i$, $\mathbf{W} = \max \{ \sum_{i \in I} w_i, \sum_{i \in I} w_i(1), \ldots, \sum_{i \in I} w_i(\Xi) \}$ and denote the aggregate endowment of period 1 by $W_1 = \sum_{i \in I} \omega_i^1$. Given a constant $X_1$, consider the closed cube: $K(X_1) = [0, X_1] \times [0, 2\mathbf{W}] \times \Pi_{S_k \in F[j,i]} [-\mathbf{V}, \mathbf{V}]^{[J(S_k)]}$. We choose $K(X_1)$ to be large in order to guarantee that the equilibrium of the generalized game is an interior point.

48The auctioneer for the market of the good chooses the price in the simplex, and since there is only one good, this price is naturally 1.
Each trader chooses a vector \((x^i_1, x^i(1), ..., x^i(\Xi), y^i)(F(I))\) on \(K^i(X_1)\) subject to constraints (2), (3) and (4). Now fix \(Q \in \mathbb{R}_+\) and let, \(Q = [0, Q]^{\sum_{s \in F(I)}|J(S)|}\). The auctioneer chooses securities prices in \(Q\) in order to maximize

\[
\sum_{i \in I}(x^i_1 - \omega^i_1) + \sum_{s_k \in F(I)} \sum_{j \in J(s_k)} q_j \sum_{i \in s_k} y^i_j.
\]  

(10)

We denote the generalized game by \(G(Q, X_1)\). The generalized game has an equilibrium since it satisfies all the assumptions of Debreu’s [1952] theorem. In fact, the auctioneer’s objective function (10) is linear in their respective price variables and the choice set is compact. Traders’ utilities are continuous and strictly concave by (A1.i), and their choice variables \((x_1, x(1), ..., x(\Xi), y)\) belong to non-empty, convex and compact sets. Moreover, for each security price vector chosen by the auctioneer the traders’ choice set has an interior point. For instance, the worst possible case is when \(q_j > 0\) and \(e_j > 0\). But then one can always choose \(x^i_1 = 0, x^i(\xi) = 0, \forall \xi\), and \(y^i_j > 0\) small enough such that \(e_j^i + y^i_j > 0\) and \(\sum_{j \in J(F[I])}(q_j y^i_j + g_j(y^i_j)) < \omega^i_1\), as good endowments are assumed to be strictly positive. This assures the lower hemicontinuity of the admissible set strategy correspondence of each trader when security market clearing occurs in each bourse.

We now prove that for vectors \((Q, X_1) \in \mathbb{R}^2\) with large enough components, any equilibrium of the generalized game is a security trading equilibrium for our economy. First, we show that security prices are uniformly bounded under the impatience assumption (A1.ii). For this, fix \(q \in \mathbb{R}^{\sum_{s \in F(I)}|J(S)|}\) and suppose that, for any trader \(i \in I\) there is an optimal solution \((x^i_1, x^i(1), ..., x^i(\Xi), y^i)(F(I))\) for his individual problem such that \(x^i_1 \leq W_1\) and \(x^i(\xi) \leq 2\bar{W}, \forall \xi \in \Xi\). We now prove that there exists \(\bar{Q} > 0\) such that \(\max_{j \in J(F[I])} q_j < \bar{Q}\). Fix \(j \in J(F[I])\). Consider a trader \(i \in I\) with \(S \in F[i; I]\) and security endowment \(e_j > 0\) who sells \(\mu_j < 0, j \in J(S)\), such that \(e_j + \mu_j \geq 0\), \(-a_j(\xi)\mu_j \leq \kappa \equiv \min_{i \in I, \xi \in \Xi} \omega^i(\xi)/2\), for any \(\xi \in \Xi\). This trading position on security \(j\) is admissible. Moreover, it allows trader \(i\) to consume the bundle \(\omega^i_1 - q_j \mu_j\) and, therefore, we can write \(u^i_1(\omega^i_1 - q_j \mu_j, 0.5 \omega^i(1), ..., 0.5 \omega^i(\Xi)) \leq u^i_1(x^i_1, x^i(1), ..., x^i(\Xi))\). Moreover, \(u^i_1(x^i_1, x^i(1), ..., x^i(\Xi)) < u^i_1(W_1, 2\bar{W}, \xi = 1, ..., \Xi)\). On the other hand, by the impatience assumption (A1.ii), we can show that there exists a constant \(\theta > 0\) such that \(u^i_1(W_1, 2\bar{W}, \xi = 1, ..., \Xi) < u^i_1(\omega^i_1 + \theta, 0.5 \omega^i(1), ..., 0.5 \omega^i(\Xi))\). Indeed, consider \(\theta \in (0, 1)\) such that \(2\bar{W} \theta < \kappa\). Impatience assumption implies that there is \(g(W_1, 2\bar{W}, \xi = 1, ..., \Xi)\) such that \(u^i_1(W_1, 2\bar{W}, \xi = 1, ..., \Xi) < u^i_1(W_1 + g(W_1, 2\bar{W}, \xi = 1, ..., \Xi), (2\bar{W}, \xi = 1, ..., \Xi))\). The choice of \(\theta\) implies \(u^i_1(W_1 + g(W_1, 2\bar{W}, \xi = 1, ..., \Xi), (2\bar{W}, \xi = 1, ..., \Xi)) < u^i_1(W_1 + g(W_1, 2\bar{W}, \xi = 1, ..., \Xi), 0.5 \omega^i(1), ..., 0.5 \omega^i(\Xi))\). And finally we can write \(u^i_1(\omega^i_1 -
with the optimality of
\( q_j \tilde{y}_j, 0.5 \omega^i(1), \ldots, 0.5 \omega^i(\Xi) < u^i_1(\omega^i_1 + \tilde{y}, 0.5 \omega^i(1), \ldots, 0.5 \omega^i(\Xi)), \) where \( \tilde{y} = g(W_1, (2 W, \xi = 1, \ldots, \Xi)) + W_1 - \omega^i_1. \) Thus, \( q_j < \tilde{Q}_j = \tilde{y}/(-\tilde{y}_j) \) (recall that \( \tilde{y}_j < 0 \)). Since there is a finite set of securities, we conclude that there exists \( \tilde{Q} > 0 \) such that \( \max_{j \in J(F[I])} q_j < \tilde{Q}. \)

Next, we show that if \( (Q, X_1) \gg (\tilde{Q}, \tilde{X}_1), \) where \( \tilde{X}_1 = 2 \tilde{W} + \tilde{J} \tilde{Q} \tilde{Y}, \) then the equilibrium of the generalized game \( G(Q, X_1), \) denoted by \( (x^i_1, x^i(1), x^i(2), \ldots, x^i(\Xi), y^i)(F[I]) \), is a trading equilibrium of the economy.

**Markets clear:** Let us show that \( \sum_{i \in S_k} y^i_j \leq 0, \) for all \( S_k \in F[I]. \) If for some security it holds that \( \sum_{i \in S_k} y^i_j > 0, \) the auctioneer would choose the maximum possible price for this security, \( q_j = Q > \tilde{Q}, \) which is a contradiction with our previous result that security prices are uniformly bounded by \( \tilde{Q}. \) Now, we show that there is no excess demand in the market of the good. Suppose not, \( \sum_{i \in I} (x^i_1 - \omega^i_1) > 0. \) But then the auctioneer would choose \( q = 0, \) and this would contradict the aggregation of the first period traders’ budget constraints \( \sum_{i \in I} (x^i_1 - \omega^i_1) + \sum_{i \in I} \sum_{j \in J(F[I])} q_j y^i_j \leq 0. \)

It remains to show that there is no excess supply in the good and in the security markets. If there were excess supply in the market of the good, we must have at least one trader with his budget constraint with strict inequality, but this contradicts strict monotonicity of trader’s preferences. Thus, we conclude that \( \sum_{i \in I} (x^i_1 - \omega^i_1) = 0. \) Finally, if \( \sum_{i \in S_k} y^i_j < 0, \) the auctioneer would choose \( q_j = 0, \) again a contradiction with the strict monotonicity of preferences. Actually, if \( q_j = 0 \) every trader would choose \( y^i_j = 0. \) Otherwise, the trader has to pay a trading fee and gets no income from the short sale. The proof that there is no commodity excess supply or demand in each node \( \xi \) of period 2 follows by similar arguments as above (the only difference is that in period 2 we need to add the returns on security endowments to the trader’s good endowment).

We conclude this proof by showing that the bundle allocations are optimal: Market clearing in each market implies that \( x^i_1 < X_1 \) and \( x^i(\xi) < 2 \tilde{W}, \) for any, \( i \in I \) and \( \xi = \ldots, \Xi. \) Moreover \( |y^i_j| < \tilde{Y}. \) Thus, for any \( i, (x^i_1, x^i(1), \ldots, x^i(\Xi), y^i) \) belongs to the interior of \( K^i(X_1). \) Suppose that there exists another allocation \( (\hat{x}^i_1, \hat{x}^i(1), \ldots, \hat{x}^i(\Xi), \tilde{y}^i) \) such that \( u^i_1(\hat{x}^i_1, \hat{x}^i(1), \ldots, \hat{x}^i(\Xi)) > u^i_1(x^i_1, x^i(1), \ldots, x^i(\Xi)). \) For \( \lambda \) small enough, \( \lambda(\hat{x}^i_1, \hat{x}^i(1), \ldots, \hat{x}^i(\Xi), \tilde{y}^i) + (1 - \lambda)(x^i_1, x^i(1), \ldots, x^i(\Xi), y^i) \) belongs to \( K^i(X_1), \) is budget feasible, and \( u^i_1(\lambda(\hat{x}^i_1, \hat{x}^i(1), \ldots, \hat{x}^i(\Xi)) + (1 - \lambda)(x^i_1, x^i(1), \ldots, x^i(\Xi), y^i) > u^i_1(x^i_1, x^i(1), \ldots, x^i(\Xi)), \) which is a contraction with the optimality of \( (x^i_1, x^i(1), \ldots, x^i(\Xi), y^i). \)

**Proof of Proposition 1:** This proof is a consequence of the following four steps.

**Step 1:** \( \bar{x}^i(F[i; I]) \) is a \( C^1 \) function in prices \( q. \)
The first order necessary and sufficient conditions for an interior optimum of a trader $i$’s problem with some of his box-constraints being non-binding are:

\[
\begin{align*}
D_1 u^i - \tilde{\beta}_1 &= 0 \\
D_\xi u^i - \tilde{\beta}(\xi) &= 0, \xi = 1, ..., \Xi \\
-x(\xi) + \omega^i(\xi) + A^i(\epsilon^i + y) &= 0, \xi = 1, ..., \Xi \\
\tilde{\beta}^T A^i - \tilde{\beta}_1 q + \tilde{\mu}^i &= 0 \\
-x_1 + \omega^i_1 - qy &= 0 \\
\epsilon^i + y &= 0
\end{align*}
\]

where $T$ refers to the transpose of a matrix. The shadow price vectors for the budget constraints in period 1 and node $\xi$ of period 2 are $\tilde{\beta}_1$ and $\tilde{\beta}(\xi)$, respectively, whereas the shadow price vector for the box constraints is denoted by $\tilde{\mu}$. The columns of the return matrix $A^i = |\cdots A(S) \cdots|$ are those $A(S)$ with $S \in F[i, I]$. Then, the element $A^i_\xi$ denotes the line $\xi$ of the return matrix $A^i$. The Jacobian matrix with the second order derivatives with respect to $(x_1, x(\xi), \tilde{\beta}(\xi), y, \tilde{\beta}_1, \tilde{\mu})$, where $x(\xi)$ and $\tilde{\beta}(\xi)$ are generic elements of the corresponding $\Xi$-vector, is:

\[
J = \begin{bmatrix}
D_1^2 u^i & 0 & 0 & 0 & -1 & 0 \\
0 & D_\xi^2 u^i & -1 & 0 & 0 & 0 \\
0 & -1 & 0 & A^i & 0 & 0 \\
0 & 0 & A^i T & 0 & -q & -1 \\
-1 & 0 & 0 & -q^T & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0
\end{bmatrix}
\]

It is easy to see that the matrix $J$ is non-singular (i.e., invertible). For this, we need to show that if $Jz = 0$, where $z = (\tilde{x}_1, \tilde{x}(\xi), \tilde{\beta}(\xi), \tilde{y}, \tilde{\beta}_1, \tilde{\mu})$, then $z = 0$. So let $z$ be such that $Jz = 0$. Then, $\tilde{y} = 0$. We also have that $z^T Jz = 0$, and by using $\tilde{y} = 0$, it reduces to

\[
\tilde{x}_1^T (D_1^2 u^i) \tilde{x}_1 + \tilde{x}(\xi)^T (D_\xi^2 u^i) \tilde{x}(\xi) = 0.
\]

This last equality can be written as

\[
\begin{bmatrix}
\tilde{x}_1^T & \tilde{x}(\xi)^T
\end{bmatrix}
\begin{bmatrix}
D_1^2 u^i & 0 \\
0 & D_\xi^2 u^i
\end{bmatrix}
\begin{bmatrix}
\tilde{x}_1 \\
\tilde{x}(\xi)
\end{bmatrix} = 0
\]

which implies $\tilde{x}_1 = 0$, $\tilde{x}(\xi) = 0$ by negative definiteness of $D^2 u^i$. Then, back to $Jz = 0$, we obtain $\tilde{\beta}_1 = 0$, $\tilde{\beta}(\xi) = 0$, and $\tilde{\mu} = 0$. Therefore, by the implicit function theorem, we conclude that a trader’s excess demand with binding box constraints is a $C^1$ function on security prices.

If instead a trader $i$ has his box constraints non-binding, then we just have to remove the last first order condition ($\epsilon^i + y = 0$), and also the last column and last row of the previous Jacobian matrix. This modified matrix is also non-singular. To see this, let $z$ be
such that $Jz = 0$. Then, $z^TJz = 0$, and using some of the equations of the system $Jz = 0$, $z^TJz = 0$ reduces to $\hat{x}_1^T(D_1^2u_1^1)\hat{x}_1 + \hat{x}(\xi)^T(D_2^2u_1^1)\hat{x}(\xi) = 0$. Notice that this last equality can be written as before, and we again can conclude that $\hat{x}_1 = 0$, $\hat{x}(\xi) = 0$, and $\hat{y} = 0$ by negative definiteness of $D^2u_1^1$. Then, back to $Jz = 0$, we obtain $\beta(\xi) = 0$. Finally, again with $Jz = 0$, $\beta_1 = 0$. Therefore, by the implicit function theorem, we conclude that trader’s excess demand with non-binding box constraints is a $C^1$ function on security prices.

**Step 2:** For any choice of utilities $U \equiv (u^i)_{i \in I}$ in a given utility space $U$, there exists a generic set $W(U)$ of endowments of periods 1 and 2, such that for every economy $(I, \alpha)$, $\alpha(i) (i \in I)$ characterized by $u^i$ and $(\omega^j)_{j \in I} \in W(U)$, the set of security trading equilibria is a continuous differentiable function of the endowments of periods 1 and 2.

Denote the price domain in period 1 by $M_1 = \mathbb{R}_{+}^{\sum_{s \in F(I)} J(S)}$. Also, denote by $f : U \times W \times M_1 \rightarrow \mathbb{R}_{+}^{\sum_{s \in F(I)} J(S)}$ the aggregate excess demand function of securities, given utilities, endowments and security prices. Let us fix the utilities to $U$. We want to show that $f$ restricted to $U$, denoted by $f_{|U}$, is transverse to 0 in order to apply the implicit function theorem. That is, if for all $(\omega, q) \in W \times M_1$ with $f_{|U}(\omega, q) = 0$, the Jacobian matrix $D_{(\omega, q)}f_{|U}$ has full rank. This amounts to show that there exists a set of independent vectors of directional derivatives that has dimension $\sum_{s \in F(I)} J(S)$. Consider a bourse $S$ and choose a trader $k$ that belongs to a bourse $S$. We can decrease $\omega^k(\xi)$ by $a_{j(S)}(\xi)$, for all $\xi$, and increase $\omega_1^k$ by $q_{j(S)}$, and budget constraint still holds. The only effect on trader $k$’s demand is an increase in security $j(S)$ by 1 unit, and the net effect on the aggregate excess demand of security $j(S)$ is now $(0, \ldots, 1, \ldots, 0)$. This proves Step 2.

**Step 3:** There exists a generic set $U' \times W'$, such that for every economy $(I, \alpha)$, $\alpha(i) (i \in I)$ with $(u^i)_{i \in I} \in U'$ and $(\omega^j)_{j \in I} \in W'$, the set of security trading equilibria is a continuous differentiable function of both the endowment and the utility assignment.\(^{49}\)

Fix a trading equilibrium $(x_1^I, x_1^I(1), \ldots, x_1^I(\Xi), y_1^I,q)$ for bourse structure $F(I)$. Now choose a bourse $S \in F(I)$ and a trader $k$ in $S$. Trader $k$’s budget constraints in this bourse are $x_1 - \omega_1^k + \sum_{j \in J(S)} q_j y_j = 0$ and $x(\xi) - \omega^k(\xi) = \sum_{j \in J} a_j(\xi) e_j^k + \sum_{j \in J(S)} a_j(\xi) y_j$. Since the payoffs of the securities available in bourse $S$ are independent, there exists a submatrix with complete rank: we consider without loss of generality that the first $J(S)$ rows are independent and we fix $B$ as the non-singular $J(S) \times J(S)$ submatrix with the first $J(S)$ rows. Then, we can obtain the portfolio vector as a function of the commodity bundle, $y^k = B^{-1}[x^k(\xi) - \omega^k(\xi) - \sum_{j \in J} a_j(\xi) e_j^k]_{\xi = 1, \ldots, J(S)}$, and write the following single

\(^{49}\)A set is a continuously differentiable function if all its elements are continuously differentiable functions.
budget constraint
\[ x_1^k - \omega_1^k + q(S) \cdot B^{-1}[x^k(\xi) - \omega^k(\xi) - \sum_{j \in J} a_j(\xi)e_j^{kJ(S)}] = 0, \tag{11} \]

where \( q(S) \) is the vector of security prices in \( S \). Consider the following utility function for trader \( k : u^k(x_0, x_1, x(1), ..., x(J(S)), x^k(J(S) + 1), ..., x^k(\Xi)) \), which is a function of \((x_1, x(1), ..., x(J(S)))\) and where \((x_0^k, x^k(J(S) + 1), ..., x^k(\Xi))\) are assumed to be fixed. The previous single budget constraint (11) can be written as follows,
\[ x_1 - \omega_1 + p \cdot [x(\xi) - \omega^k(\xi) - \sum_{j \in J} a_j(\xi)e_j^{k}]^{J(S)}_\xi = 0, \tag{12} \]

where \( p = q(S)B^{-1} \). Now, let \( \hat{x} = (\hat{x}(1), ..., \hat{x}(J(S))) \) be the trader \( k \)'s commodity demand function using the single budget constraint (12). Following Geanakoplos and Polemarchakis [1986], we can now perturb trader \( k \)'s utility in such a way that \( \hat{x} \) is does not change even if \( D_p\hat{x} \) changes. To see this, we have \( D_q(y^k) y^k = D_y g^k D_p \hat{x}^k D_q(S)p = B^{-1} D_p \hat{x}^k [B^{-1}]^T \) since \( y^k = B^{-1}[x^k(\xi) - \omega^k(\xi) - \sum_{j \in J} a_j(\xi)e_j^{k}]^{J(S)}_\xi = 0, \) where \( B^{-1} \) and \([B^{-1}]^T\) are invertible (since \( B \) is invertible).\(^50\) Thus, by applying this procedure to all bourses it is possible to prove that \( D_q f \) (a function of utilities, endowments and prices) is invertible (as in Geanakoplos and Polemarchakis [1986]), and then strong regularity follows.

Step 4: There exists a generic set of economies for which, given \( \lambda > 0 \), there is a \( \gamma > 0 \) such that for any set \( I \) and pair of economies \((I, \alpha)\) and \((I, \beta)\), if \( d(\alpha(i), \beta(i)) \leq \gamma \) for any \( i \), then \( |V^{\alpha(i)}(x_0, F[i; I_i]) - V^{\beta(i)}(x_0, F[i; I_i])| < \lambda \).

The proof of this last step follows by the continuity of \( u^i \) (by A1.i) and the continuity of security trading equilibria (Step 3). \( \blacksquare \)

Now, at this part of the proof of Theorem 1, we must observe that, given a bourse structure, the security trading equilibrium may not be unique. This would imply that there is an indirect utility \( u^i(x_0, \bar{x}(F[i; I])) \) for each equilibrium solution \( \bar{x}(F[i; I]) \), and thus more than one function \( V^i(x_0, F[i; I]) \). Existence of an equilibrium for the bourse economy would require choosing a measurable selector of the equilibrium correspondence \( E(\cdot) \). Then, we have to consider that for each bourse structure \( F(I) \), the utility \( u^i \) is evaluated at the equilibrium selection \( \bar{x}^i(F[i; I]) \). The next proposition asserts that this measurable selection exists.

**Lemma 2:** There exists a measurable selection \( \bar{x}^i(F(I)) = (\bar{x}^i(F[i; I]) : i \in I) \) for the equilibrium correspondence \( E(F(I)) \).

\(^50\)We denote by \([B^{-1}]^T\) the transpose of \( B^{-1} \).
Proof of Lemma 2: The proof follows by the Kuratowski-Ryll-Nardzewski measurable selection theorem (a weak measurable correspondence with non-empty closed values into a separable metrizable space admits a measurable selection).\textsuperscript{51} In fact, we have that $F(I)$ is a finite set, and therefore the equilibrium correspondence $E(\cdot)$ defined in $F(I)$ is trivially a weak measurable correspondence (see Aliprantis and Border [2006, p. 600]). The correspondence takes values in the positive coordinate subset of a finite dimensional space and therefore it follows immediately that it is a separable metrizable space.

The correspondence $E(\cdot)$ takes closed values, i.e., if $(x^{I,s}, x^{I,s}(1), \ldots, x^{I,s}(\Xi), y^{I,s}, p^s, q^s)$ is a sequence in $E(F(I))$ that converges to $(x^{I}, x^{I}(1), \ldots, x^{I}(\Xi), y^{I}, p, q)$, then $(x^{I}, x^{I}(1), \ldots, x^{I}(\Xi), y^{I}, p, q)$ also belongs to $E(F(I))$. Given an equilibrium sequence, if we consider the budget constraints of each trader and pass to the limit, we obtain that in the limit the budget constraint of each trader is satisfied. The same reasoning allows us to prove that the market clearing also holds in the limit.

Finally, it remains to show that in the limit each trader is maximizing his utility. Suppose not, so for a trader $i$ there exists another bundle $(\tilde{x}^i, \tilde{y}^i)$ which is budget feasible and such that $u^i(x_0, \tilde{x}^i) > u^i(x_0, \tilde{x}^i)$. Now, let $(\tilde{x}^i, \tilde{y}^i) = (\lambda \tilde{x}^{i,s} + (1-\lambda)\tilde{x}^i, \lambda y^{i,s} + (1-\lambda)\tilde{y}^i)$ with $\lambda \in [0, 1]$. Observe that $(\tilde{x}^i, \tilde{y}^i)$ is budget feasible for $s$ large enough and for $\lambda$ close to one. Moreover, by continuity we have that $u^i(\tilde{x}^i) > u^i(\tilde{x}^{i,s})$, for $s$ large enough. Then, the strict quasiconcavity implies that $u^i(x_0, \tilde{x}^i) = u^i(x_0, \lambda \tilde{x}^{i,s} + (1-\lambda)\tilde{x}^i) > u^i(\tilde{x}^{i,s})$. This is a contradiction because $((\tilde{x}^{i,s})_{i \in I}, y^{I,s}, p^s, q^s)$ was a trading equilibrium for the given bourse structure $F(I)$.

An immediate consequence of Lemma 2 is that $V^i(x_0, F[i; I])$ is well defined.

Lemma 3: Let us assume that A2 and A3 hold. Then there exists a generic set of bourse economies for which there is a $c(\varepsilon_0)$-equilibrium with possibly ever-increasing gains from larger bourses.

Proof of Lemma 3: To prove Lemma 3 we need to assure that all assumptions required in AW [2008, Theorem 2] are satisfied.

First, our assumption (A2.i) on $u^i(\cdot, x_1)$ implies AW’s assumptions (a) monotonicity, (b) continuity, and (c) convexity on $V^i(\cdot, F[i; I])$.

Second, AW’s condition (d) “Desirability of endowment” can be rewritten with our

\textsuperscript{51}We remark that the equilibrium correspondence is defined in the finite set of bourse structures, and therefore, a continuous measurable selector is not needed. Continuous selectors are in general used to construct continuous objective functions. Thus, they only fit if the correspondence is defined in a continuum set.
notation as follows: if \( V^i(\omega^i_0 - \tau, \{i\}) < V^i(x^i_0, F[i; I]) \), then \( x^i_0 > 0 \). Since, \( V^i(\omega^i_0 - \tau, \{i\}) = u^i(\omega^i_0 - \tau, \omega^i_1, \omega^i(1), ..., \omega^i(\Xi)) \), and \( V^i(x^i_0, F[i; I]) = u^i(x^i_0, \tilde{x}^i_1(1), ..., \tilde{x}^i(\Xi)) \), assumption (A2.iii) implies AW's condition (d).

Third, AW's condition (e) "Private goods are valuable" says that, given any attribute \( \theta \) and any \( \varepsilon > 0 \), there is \( \rho^\theta_\varepsilon > 0 \) such that, for all \( i \in I \) with \( \alpha(i) = \theta \) and all \( x^i_0 \in \mathbb{R}_+ \), \( V^i(x^i_0, F[i; I]) + \rho^\theta_\varepsilon < V^i(x^i_0 + \varepsilon, F[i; I]) \) holds. This assumption follows immediately from our assumption (A2.iv).

Fourth, AW's condition (g), "Continuity with respect to attributes 2", says that, given \( \varepsilon > 0 \), there exists \( \lambda > 0 \) such that for any set \( I \) and pair of economies \( (I, \alpha) \) and \( (I, \beta) \), if \( d(\alpha(i), \beta(i)) \leq \lambda \) for any \( i \), then \( \omega^\alpha_0(i) \leq \omega^\beta_0(i) + \varepsilon \). This is precisely what was assumed in (A2.i).

Fifth, AW's assumption (h), "Continuity with respect to attributes 3" is precisely our assumption (A2.ii).

Sixth, we have to show that AW's assumption (f) holds. This assumption says that given \( \varepsilon > 0 \), there exist \( \gamma > 0 \), such that, for any set \( I \) and pair of economies \( (I, \alpha) \) and \( (I, \beta) \), if \( d(\alpha(i), \beta(i)) \leq \gamma \), then \( V^\alpha(i)(x^i_0, F[i; I^\alpha]) < V^\beta(i)(x^i_0 + \varepsilon, F[i; I^\beta]) \), for any \( i \) and any \( x^i_0 \in \mathbb{R} \).

Proposition 1 asserts that, for each bourse structure \( F(I) \), there is a generic set \( U' \times W'(F(I)) \) where the securities trading equilibrium is continuous in traders' attributes. Now, the finite intersection \( U'' \times W'' \equiv \bigcap_{F(I) \in F(I)} (U' \times W'(F(I))) \) is a generic set, where the securities trading equilibrium is continuous in traders' attributes for every bourse structure. Given an economy \( (I, \alpha) \) belonging to the generic set \( U'' \times W'' \), we can find a compact subset of economies containing \( (I, \alpha) \) (since \( U'' \times W'' \) is open) where Proposition 1 holds. We now prove that assumption (f) of AW holds in this compact set of economies for \( x^i_0 \leq \sum \omega^i_0 + \varepsilon \), with \( \varepsilon > 0 \).\(^{52}\)

On the one hand, our assumption (A2.iv) implies that given \( \varepsilon > 0 \), there exist \( \rho_\varepsilon \) and \( \gamma_1 > 0 \), such that, for any set \( I \) and pair of economies \( (I, \alpha) \) and \( (I, \beta) \), if \( d(\alpha(i), \beta(i)) \leq \gamma_1 \), then \( u^\beta(i)(x^i_0 + \varepsilon, x_1, x(1), ..., x(\Xi)) - u^\alpha(i)(x^i_0, x_1, x(1), ..., x(\Xi)) > \rho_\varepsilon \). On the other hand, Proposition 1 implies that there exists \( \gamma_2 > 0 \) such that, if \( d(\alpha(i), \beta(i)) \leq \gamma_2 \), then \( |u^\alpha_1(x^i_0, \tilde{x}_1(F[i; I])) - u^\beta_1(x^i_0, \tilde{x}_1(F[i; I]))| < \rho_\varepsilon \). Then, \( V^\beta(i)(x^i_0 + \varepsilon, F[i; I^\beta]) - V^\alpha(i)(x^i_0, F[i; I^\alpha]) = u^\beta(i)(x^i_0 + \varepsilon, \tilde{x}_1(F[i; I^\beta])) - u^\alpha(i)(x^i_0, \tilde{x}_1(F[i; I^\alpha])) = u^\beta(i)(x^i_0 + \varepsilon, \tilde{x}_1(F[i; I^\beta])) - u^\beta(i)(x^i_0, \tilde{x}_1(F[i; I^\beta])) + u^\beta(i)(x^i_0, \tilde{x}_1(F[i; I^\beta])) - u^\alpha(i)(x^i_0, \tilde{x}_1(F[i; I^\alpha])). \) Thus, for \( \gamma = \min\{\gamma_1, \gamma_2\} \), bounded above by the aggregate endowments plus some \( \varepsilon > 0.\)

\(^{52}\) Observe that AW [2008, p. 271-272] only require assumption f) to be satisfied for a consumption \( x^i_0 \)
if \( d(\alpha(i), \beta(i)) \leq \gamma, V^\beta(i)(x'_0 + \varepsilon, F[i; g]) - V^\alpha(i)(x'_0, F[i; g]) > 0 \).

Finally, it remains to show that our economy satisfies AW’s assumption “Desirability of wealth”, which says that there is \( x'_0 \in \mathbb{R}_+ \) and an integer \( \eta \) such that for any economy \( (I, \alpha) \) and any \( i \in I \), there is a coalition \( S \in I \) with \( |S| \leq \eta \) and a club structure \( F(S) \) satisfying \( V^i(x'_0 + x^*_0, F[i; S]) \geq V^i(x'_0, F[i; I]) \), for any \( F(I) \) and any \( x^*_0 \in \mathbb{R}_+ \). Notice that by \( V^i(x'_0 + x^*_0, \{i\}) = u^i(x'_0 + x^*_0, \omega^i_1, \omega^i(1), ..., \omega^i(\Xi)) \). By (A3), \( u^i(x'_0 + x^*_0, \omega^i_1, \omega^i(1), ..., \omega^i(\Xi)) \geq u^i(x'_0, \sum_i(\omega^i_1, \omega^i(1), ..., \omega^i(\Xi)) \) and \( u^i(x'_0, \sum_i(\omega^i_1, \omega^i(1), ..., \omega^i(\Xi)) \geq V^i(x'_0, F[i; I]) \). Then, transitivity implies that \( V^i(x'_0 + x^*_0, \{i\}) > V^i(x'_0, F[i; I]) \), which satisfies the “Desirability of wealth” assumption for a bourse \( S = \{i\} \) (and therefore, for \( \eta = 1 \) in AW’s terminology).

**Proof of Propositions 2 and 3:** The following proof is valid for both Propositions 2 and 3. In Proposition 2 \( S^* \) denotes the large unique bourse, whereas in Proposition 3 it denotes a bourse with complete markets. Also, in Proposition 2 \( S \) denotes a bourse smaller than \( I \), whereas in Proposition 3 it denotes a bourse with incomplete markets. Bourse \( S^* \) forms in equilibrium if for any \( S \neq S^* \), \( u_0^i(\omega_0 - \pi^i(S))U^i_1(S) < u_0^i(\omega_0 - \pi^i(S^*))U^i_1(S^*) \), for all \( i \in S \cap S^* \). This inequality is equivalent to \( (\omega_0^i - \pi^i(S))U^i_1(S)^\eta < (\omega_0^i - \pi^i(S^*))U^i_1(S^*)^\eta \), since the inverse of \( u_0^i \) is strictly increasing and homogeneous of degree \( \eta > 0 \). The non-anonymous membership fees of any bourse \( \tilde{S} \) can be obtained from the optimization problem of a welfarist agent that maximizes \( \sum_{i \in \tilde{S}} u_0^i(x'_0)U^i_1(\tilde{S}) \) subject to \( x'_0 = \omega_0^i - \pi^i(\tilde{S}) \) for all \( i \in \tilde{S} \) and \( \sum_{i \in \tilde{S}} \pi^i(\tilde{S}) = z(\tilde{S}) \). The membership prices that solve this optimization problem are efficient and satisfy traders’ budget constraints with equality, and equilibrium conditions (D2.i) and (D2.ii). Moreover, from AW core decentralization result, we know that an efficient allocation is an equilibrium. The membership price solution for a trader \( i \) in a given bourse \( \tilde{S} \) can be written as follows:

\[
\pi^i(\tilde{S}) = \frac{\omega_0^i \sum_{k \neq i, k \in \tilde{S}} U^k_1(\tilde{S}) - U^i_1(\tilde{S}) \sum_{k \neq i, k \in \tilde{S}} \omega_0^k}{\sum_{i \in \tilde{S}} U^i_1(\tilde{S})} \sum_{i \in \tilde{S}} U^i_1(\tilde{S}) z(\tilde{S})
\]  

(13)

We just need to substitute expression (13) into the previous inequality, and after some algebra, condition (C1) follows.

**Proof of Proposition 4:** Suppose that in equilibrium there is a bourse \( S \subseteq I \) with complete markets. Consider another bourse structure where \( S \) splits into two bourses with incomplete markets, \( S' = S \cap I' \) and \( S'' = S \cap I'' \). The membership fees in these two scenarios must satisfy \( \sum_{i \in S} \pi^i(S) = z(S, A(S)), \sum_{i \in S'} \pi^i(S') = z(S', A(S')) \) and \( \sum_{i \in S''} \pi^i(S'') = z(S'', A(S'')) \). But \( z(S, A(S)) > z(S', A(S') + z(S'', A(S'')) \), and therefore, \( \sum_{i \in S} \pi^i(S) + \sum_{i \in S'} \pi^i(S') > \sum_{i \in S'} \pi^i(S') + \sum_{i \in S''} \pi^i(S'') \). Thus, \( \sum_{i \in S} \pi^i(S) > \sum_{i \in S'} \pi^i(S') \) or \( \sum_{i \in S''} \pi^i(S) > \sum_{i \in S''} \pi^i(S'') \). In any case, at least there must be one
trader that is better off in the bourse with incomplete markets than in the bourse with complete markets. Notice that given the characteristics of traders and the security structures, traders in $I'$ do not trade securities with traders in $I''$ because they want to share risk in complementary states of nature. Thus, when different types of traders belong to the same bourse, they cannot improve upon the existing partition of traders since the bourse $S$ splits in such a way that one bourse is contained in $I'$ and the other bourse is contained in $I''$. We conclude that the unique unique bourse does not form and that at equilibrium all bourses have incomplete markets.