Characterizing Markov-Switching Rational Expectations Models

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Abstract

Markov-switching rational expectations (MSRE) models can bring out fresh insights beyond what linear rational expectations (RE) models have done for macroeconomics as Davig and Leeper (2007) and Farmer, Waggoner and Zha (2009), among others, have noted and predicted. A lack of tractable methodological foundations, however, may have hindered researchers from uncovering the salient features of MSRE models. This paper improves the status quo to a level at which MSRE - inherently non-linear - models can be analyzed as easily and comprehensively as linear RE models. Specifically, we provide a solution method, determinacy conditions, and an economic solution refinement and completely characterize the set of RE equilibria for general MSRE models under determinacy and indeterminacy in the mean-square stability sense. These tasks are accomplished by simply solving a model forward and imposing the no-bubble condition for fundamental solutions. We apply our methodology to a New-Keynesian model subject to regime-switching in monetary policy and find some unforeseen but intuitive determinacy results. Markov-switching in the private sector is also shown to deliver potentially rich dynamics.

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1 Introduction

Only recently has Markov-switching - one of the most productive modeling techniques in time-series econometrics - been applied to structural rational expectations (RE) macroeconomic models. For instance, Davig and Leeper (2007) consider a regime-switching monetary policy in an otherwise canonical New-Keynesian model and provide a new perspective for understanding historical U.S. monetary policy. Their argument, expressed as the long-run Taylor principle (LRTP), is that the possibility of regime-switching between passive and active policy stances can expand the parameter space over which a unique bounded equilibrium exists. MSRE models can also be used to deal with time variations in important structural parameters governing optimal behaviors of private agents. Gordon and St-Amour (2000) and Melino and Yang (2003) study the regime-switching preferences of households, and Liu, Waggoner and Zha (2009) consider regime-switching in the optimal price-setting behavior of firms and derive a regime-switching New-Keynesian Phillips curve à la Calvo (1983). The optimal monetary policy of the central bank facing regime-shifting behaviors of private agents is also of great importance, as emphasized by Davig (2007). Other applied works in this area include Davig and Doh (2009), Bianchi (forthcoming), Liu, Waggoner and Zha (2011) and Baele et al. (2011).

While some progress has been made regarding solution properties, the following issues, which are essential for understanding MSRE models, have not yet been answered very well in the literature. First, under what economic circumstances would a MSRE model have a unique stable RE solution? This question, arguably the most important one pertaining to determinacy, has barely been resolved. Analogously to linear RE models, any solution to a MSRE model can be decomposed into a fundamental solution and a non-fundamental bubble component. So determinacy amounts to the case in which there is a unique stable fundamental solution and there is no stable bubble component associated with that solution. Unlike in linear models, however, different concepts of stability lead to different determinacy outcomes, as shown by Farmer et al. (2009), who propose mean-square stability. In contrast, boundedness is advocated by Davig and Leeper (2007). In any case, no tractable determinacy conditions have been developed even for very simple models without predetermined variables. This lack of knowledge may have been a major impediment to the progress of MSRE models.

Second, as for a solution method, Farmer et al. (2011) made an important contribution by proposing a numerical algorithm to find fundamental solutions to more general models.
with predetermined variables.\footnote{Farmer et al. (2011) use the term MSV solution instead of a fundamental solution. These terminologies are interchangeably used in the literature to denote a solution that depends on the minimum state variables. In contrast, Bennett T. McCallum uses the MSV solution to denote a fundamental solution that passes the solution selection criterion he proposed in McCallum (1983). To avoid confusion, we use the term ”fundamental” throughout this paper.} However, exactly how many fundamental solutions a MSRE model has is not yet generally known. The explicit functional form of the non-fundamental bubble components is also proposed by Farmer et al. (2009), but only for those models without predetermined variables. It turns out that the class of bubble solutions is too huge to solve for and examine their stability. These technical difficulties underlying MSRE models are due to the inherent non-linearities arising from regime-dependency, so standard solution techniques or determinacy conditions for linear models such as Blanchard and Kahn (1980) cannot be directly applied to MSRE models.

Third, economic refinements for RE solutions have hardly been discussed for MSRE models.\footnote{Farmer et al. (2011) propose a likelihood-based solution selection criterion, but it might conflict with existing selection refinements, if applied to linear models. We discuss this issue later in this paper.} Such refinements are much more important for MSRE models than they are for linear models because the set of stable bubble solutions that would emerge under indeterminacy is immensely large. While several economic refinements have been proposed for linear models, for instance, the MSV criterion of McCallum (1983), the E-stability of Evans and Honkapohja (2001), and the no-bubble condition of Cho and Moreno (2011), none of these have been applied to MSRE models.\footnote{Questions have also been raised about the economic validity of even a determinate solution, as argued by Bullard and Mitra (2002), and Cho and McCallum (2009). In a similar vein, McCallum (2012) argues that determinacy should mean a unique economically relevant equilibrium, not a single stable solution. This issue has not been discussed for MSRE models either.}

The main goal of this paper is to resolve these three difficulties and characterize the full set of rational expectations equilibria under determinacy and indeterminacy. We accomplish this goal by solving MSRE models forward and requiring RE solutions to satisfy two solution refinements: stability for defining determinacy and the no-bubble condition for fundamental solutions.\footnote{Throughout this paper, we use the term RE solution to denote a stochastic process consistent with a given RE model. By a rational expectations equilibrium (REE), we mean a RE solution that possesses a set of prescribed economic properties, which, in this paper, are stability and the no-bubble condition.} Below we briefly sketch the idea behind this result.

First, we derive some key properties of mean-square stability and use them to find a simple transition probability weighted matrix governing the non-existence of stable bubble components. Hence, it suffices to examine the maximum absolute eigenvalue of this matrix in order to establish one of the two conditions defining determinacy without
solving for all the bubbles and examining their stability. Mean-square stability also ensures that the variance of a solution will be bounded and hence enables econometric inference as Farmer et al. (2009) show. Our choice of stability concept over alternatives such as boundedness or stability in mean will be further discussed.

Second, we extend the forward method of Cho and Moreno (2011) developed for linear RE models to MSRE models and show that the forward method provides all the information to complete our methodology. When solved forward, a RE model can be written as the sum of the expectational term involving future endogenous variables and a function of state variables. As the forward recursion goes to infinity, the forward solution is defined as the unique limiting function of the state variables if it exists, and the no-bubble condition (NBC) requires the expectational term to disappear: the expectation of the endogenous variables far in the future should not affect the current endogenous variables. It seems obvious that a fundamental solution, often referred to as a bubble-free solution, should satisfy the NBC as its name indicates, and thus the NBC has typically been assumed to hold in the literature. However, Cho and Moreno (2011) show that this condition holds only for the forward solution. Therefore, the NBC rejects all of the other fundamental solutions as equilibria if any.

The uniqueness of the stable fundamental solution, the remaining condition for determinacy, is established by combining the two independently developed ideas described above. Using a property of mean-square stability again, we show that the stability of the forward solution and the non-existence of stable bubble components ensure that there is no other stable fundamental solution. Finally, this determinacy result and the fact that the forward solution is the unique fundamental solution satisfying the NBC fulfill our classification of the REEs to MSRE models: the forward solution is the unique stable REE under determinacy and the stable RE solutions associated with the forward solution are the REEs under indeterminacy. Our results are almost isomorphic to those for linear RE models shown by Cho and McCallum (2012), but the derivation is completely different and novel.

We apply our methodology to a standard New-Keynesian model subject to regime-switching in monetary policy stance. Our analysis shows that, indeed, a temporarily passive monetary policy can be admissible as a part of determinacy, as argued by Davig and Leeper (2007) and Farmer et al. (2009). However, we also find that if one regime is too active relative to the other even when both regimes are active, the additional volatility induced by regime-switching can actually lead the economy to indeterminacy.
We also show that a more general model with optimizing private agents whose preferences or technologies are regime-switching can exhibit quantitatively and qualitatively rich dynamics. Finally, using several examples in the literature, we show that not all of the solutions are economically relevant on the grounds of our solution refinement.

This paper is organized as follows. Section 2 presents two illustrative MSRE models. In section 3, we lay out a class of general MSRE models and RE solutions. Section 4 introduces mean-square stability and derives identifying conditions for determinacy, indeterminacy and the case of no stable solutions. Section 5 develops our forward method and the no-bubble condition. In section 6, we combine stability and the NBC to finally characterize the set of REEs under determinacy and indeterminacy. In section 7, we apply our methodology to several examples. Section 8 concludes.

2 Illustrative Examples

Two potentially representative examples of Markov-switching RE models are presented, which will be analyzed according to our methodology.

Example A Consider a Markov-switching monetary policy in a canonical New-Keynesian model studied by Davig and Leeper (2007), and Farmer et al. (2009):

\[
\begin{align*}
\pi_t &= \beta E_t \pi_{t+1} + \kappa y_t + z_{S,t}, \\
y_t &= E_t y_{t+1} - \frac{1}{\sigma}(i_t - E_t \pi_{t+1}) + z_{D,t}, \\
i_t &= (1 - \rho)(\phi_\pi(s_t) \pi_t + \phi_y y_t) + \rho i_{t-1} + z_{MP,t},
\end{align*}
\]

where \( \pi_t, y_t \) and \( i_t \) are inflation, the output gap and the short-term interest rate, respectively. \( s_t \) is a Markov chain that switches over two states. \( z_{S,t}, z_{D,t} \) and \( z_{MP,t} \) represent, respectively, the exogenous aggregate supply, demand and monetary policy shocks, which are assumed to be covariance-stationary. \( E_t[\cdot] \) is the mathematical expectation operator conditional on the information set at time \( t \), including the current regime \( s_t \). \( \rho \) captures the interest-rate-smoothing behavior by the central bank. \( \phi_\pi(s_t) \) represents the monetary policy stance against inflation, which is active (passive) if \( \phi_\pi(s_t) > 1 \) (\( \leq 1 \)).

Even for this simple ad hoc regime-switching model without predetermined variables

\footnote{Matlab codes for our methodology and the numerical examples analyzed in this paper can be found at http://web.yonsei.ac.kr/sc719.}
\( (\rho = 0) \), determinacy is not well understood. Corollary 1 of Farmer et al. (2009) presents their general determinacy result in the mean-square stability sense for this kind of purely forward-looking model, but it is hardly implementable in practice. Their proposition 1, presented for illustrative purposes, does not establish a complete determinacy result because only a subset of bubble solutions is examined for stability. The LRTP of Davig and Leeper (2007) has a very simple and intuitive form, but it is the determinacy condition for what they call a linear version of their original model, not for model (1).

Example B Preferences and technologies in the private sector may well be state-dependent. Gordon and St-Amour (2000) and Melino and Yang (2003) study the regime shifts of risk aversion, intertemporal substitution and a subjective discount rate in the asset pricing context. More recent works include a regime-dependent price adjustment cost considered by Davig (2007) and a regime-switching degree of inflation indexation by Liu et al. (2009).

The central aspect of these works is that rational agents take into account the time variation of future regime variables when making optimal decisions now. To illustrate the point, let us derive a simple regime-switching counterpart of (1b) from a standard intertemporal optimality condition of a representative household whose utility function is given by 
\[
U(C_t; s_t) = H_t(C_t^{1-\sigma(s_t)} - 1)/(1 - \sigma(s_t)) \]
where \( C_t \) is consumption and \( H_t \) is an exogenous preference shifter. This is a very special case of Gordon and St-Amour (2000) or the regime-dependent Epstein and Zin (1989) type model of Melino and Yang (2003), where \( \sigma \) measures the inverse of elasticity of intertemporal substitution or relative risk aversion. While preserving the non-linearity of the regime-switching parameter, a standard intertemporal optimality condition in this case can be log-linearized around the steady states as:

\[
y_t = E_t \left[ \frac{\sigma(s_{t+1})}{\sigma(s_t)} y_{t+1} \right] - \frac{1}{\sigma(s_t)} (i_t - E_t \pi_{t+1}) + \frac{1}{\sigma(s_t)} z_{D,t}, \tag{2}
\]

where \( y_t, i_t \) and \( \pi_{t+1} \) are the output gap (in logs), the nominal interest rate and inflation net of their steady states, and \( z_{D,t} = -E_t[\ln(H_{t+1}/H_t)] \). We have assumed that

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6 Davig and Leeper (2009) refer to equation (1) as a quasi-linear model to which the LRTP does not apply. Also, it is not clear whether a linear version can also be defined for the more general MSRE model with predetermined variables that we consider in this paper.

7 The model of Liu et al. (2009) would include a future regime variable if the degree of inflation indexation is assumed to depend on the current regime instead of the past regime.

8 \( U(C_t; s_t) \) is given by \( H_t \ln C_t \) if \( \sigma(s_t) = 1 \).
consumption equals output $Y_t$ in equilibrium. This equation, together with (1a) and a regime-independent policy rule (1c), constitutes a model of Markov-switching elasticity of intertemporal substitution.

The forward-looking term in equation (2) depends on the future regime-dependent parameter, which cannot be separated from expectations. It is this feature that we emphasize: MSRE models with micro-foundations may well involve the expectational effects of future regime variables in the optimal decision rules of economic agents. Equation (2) is in stark contrast to the intertemporal IS equation of the sort considered by Farmer et al. (2011), where $\sigma$ in the linear IS equation (1b) is assumed to depend on $s_t$, $y_t = E_t[y_{t+1} - \frac{1}{\sigma(s_t)}(i_t - E_t\pi_{t+1})] + z_{D,t}$.

3 Markov-Switching Rational Expectations Models

This section presents the class of general MSRE models and the full set of RE solutions, followed by the ideas and strategies for developing our methodology.

3.1 The Class of MSRE Models

The class of MSRE models we consider in this paper is given by:

$$x_t = E_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t,$$

$$z_t = Rz_{t-1} + \epsilon_t,$$

where $x_t$ is an $n \times 1$ vector of endogenous variables, measured from its steady state, which is assumed to be known to all agents in the model. $z_t$ is an $m \times 1$ vector of exogenous variables and $\epsilon_t$ is an $m \times 1$ asymptotically covariance-stationary vector such that $\epsilon_t \sim (0_{m \times 1}, D)$, where $D$ is the $m \times m$ variance-covariance matrix\footnote{Andolfatto and Gomme (2003) analyze a model with regime-dependent variances such that $D = D(s_{t-1})$. This specification can be easily handled in our framework and it does not alter our main results. For this reason, we do not explicitly consider regime-dependent variances for a compact exposition.} $s_t$ is an $S$-states ergodic Markov chain with the transition probability matrix $P$ where its $(i, j)$-th element is $p_{ij} = Pr(s_{t+1} = j|s_t = i)$ for all $i, j \in \{1, 2, \ldots, S\}$, $\mathcal{I}_t = \{x_{t-l}, z_{t-l}, s_{t-l}, l = 0, 1, 2, \ldots\}$ is the information set available at time $t$ and $E_t[\cdot] \equiv E[\cdot|\mathcal{I}_t]$ is the mathematical expectation operator conditional on $\mathcal{I}_t$. $A$, $B$ and $C$ at each state $s_t$ and $s_{t+1}$ are $n \times n$, $n \times n$ and $n \times 1$ respectively.
n × m matrices of the structural parameters, respectively. \( R \) is an \( m \times m \) stable matrix.10 The model is quite general in that \( A \) may depend on both current and future regimes and it can be singular for each state. The model also allows the presence of predetermined variables. Note that the examples in Section 2 can be written in the form of (3).

### 3.2 Rational Expectations Solutions

While there can be alternative ways of characterizing the complete set of RE solutions, we write a solution as a sum of the two parts: a fundamental solution and a non-fundamental component in the following way.

**Proposition 1** Any rational expectations solution to model (3) can be written as a sum of a fundamental solution that depends on the state vectors \( x_{t-1} \) and \( z_t \), and a non-fundamental component, \( w_t \), as:

\[
x_t = \left[ \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t \right] + w_t, \quad (4)
\]

\[
w_t = E_t[F(s_t, s_{t+1})w_{t+1}], \quad (5)
\]

where \( \Omega(s_t), \Gamma(s_t) \) and \( F(s_t, s_{t+1}) \) must satisfy the following conditions for all \( s_t \) and \( s_{t+1} = 1, 2, ..., S \):

\[
\Omega(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}B(s_t), \quad (6a)
\]

\[
\Gamma(s_t) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}C(s_t) + E_t[F(s_t, s_{t+1})\Gamma(s_{t+1})R], \quad (6b)
\]

\[
F(s_t, s_{t+1}) = \{I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})]\}^{-1}A(s_t, s_{t+1}), \quad (6c)
\]

under the regularity condition that \( I_n - E_t[A(s_t, s_{t+1})\Omega(s_{t+1})] \) is non-singular for all \( s_t \).

**Proof.** Plugging (4) into the model and rearranging it yield the formulae in (6). Since \( w_t \) represents either \( 0_{n \times 1} \) or any other stochastic process that is left over a fundamental solution, the set of solutions of the form (4) and (5) is exhaustive. Q.E.D.

The fundamental component of the general solution (4) with \( w_t = 0_{n \times 1}, \)

\[
x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t, \quad (7)
\]

10The model with regime switching \( R \) can also be modeled directly or represented in the form of (3) by redefining \( x_t \) so as to include \( z_t \).
will be referred to as a fundamental solution because it depends on the minimal set of the state variables $x_{t-1}$ and $z_t$, analogous to linear models, and $s_t$ through the coefficient matrices in a non-linear fashion. A non-zero stochastic process $w_t$ is referred to as a non-fundamental component - often a bubble or sunspot. The RE solution (4) with a stochastic $w_t$ will henceforth be referred to as a non-fundamental or bubble solution. It should be stressed that $w_t$ is restricted by the matrix $F(s_t, s_{t+1})$ in (5), which is in turn uniquely associated with a given $\Omega(s_t)$. Note also that $\Gamma(s_t)$ is uniquely determined by $\Omega(s_t)$. The explicit form of $\Gamma(s_t)$ will be provided by equation (27) below. For this reason, solving for $\Omega(s_t)$ is equivalent to solving for a fundamental solution and we will often denote a fundamental solution by $\Omega(s_t)$ for simplicity in what follows.

Unfortunately, it is hard to identify the full set of RE solutions to the MSRE model (3). On the one hand, solving for all $\Omega(s_t)$ in equation (6a) is difficult because the problem is non-homogeneous in the sense that (6a) involves cross-products of $\Omega(i)$ and $\Omega(j)$ for all i, j = 1, ..., S. One may rewrite (6a) as a regime-independent matrix quadratic form such that $\tilde{A}\tilde{\Omega}^2 + \tilde{B}\tilde{\Omega} + \tilde{C} = 0_{nS \times nS}$ where $\tilde{A}$, $\tilde{B}$ and $\tilde{C}$ are functions of the original matrices, $A$, $B$ and $C$ at all states. However, the generalized Schur decomposition theorem does not apply because $\tilde{B}$ is always singular and rank conditions are violated. To our knowledge, the existence and the number of solutions to such rank-deficient matrix quadratic equations remain unresolved in the mathematics literature. The numerical algorithm for finding multiple fundamental solutions to a MSRE model proposed by Farmer et al. (2011) indeed helps identify indeterminacy, but it is not sharp enough to pin down determinacy, and their method is not applicable to models with $A(s_t, s_{t+1})$.

On the other hand, Farmer et al. (2009) derive the explicit functional form of the non-fundamental components. Nonetheless, identifying all of the bubble components and examining their stability is extremely difficult. To demonstrate this point, we extend their Theorem 1 to the case where $F$ may depend not just on the current state $s_t$, but also on the future state $s_{t+1}$, and more important, $F$ may be singular for all states $(s_t, s_{t+1})$.

**Proposition 2** Any non-fundamental component $w_t$ in (5) can be written as:

$$w_{t+1} = \Lambda(s_t, s_{t+1})w_t + V(s_{t+1})V(s_{t+1})'\eta_{t+1},$$

where $V(s_t)$ is an $n \times k(s_t)$ matrix with orthonormal columns, $0 \leq k(s_t) \leq n$ and $k(s_t) > 0$

---

**Note:** $F(s_t, s_{t+1})$ collapses to $A(s_t, s_{t+1})$ if there is no predetermined variable in the model, i.e., $\Omega(s_t) = 0_{n \times n}$ for all $s_t$. When $A(s_t, s_{t+1}) = A(s_t)$ and this is invertible for all $s_t$, $\Gamma(s_t)$ in equation (1) of Farmer et al. (2009) corresponds to our $A(s_t)^{-1}$.
for some \( s_t \). \( \eta_t \) is an arbitrary \( n \times 1 \) innovation such that \( E_t[V(s_{t+1})V(s_{t+1})']\eta_{t+1} = 0_{n \times 1} \),
\[
\Lambda(s_t, s_{t+1}) = V(s_{t+1})\Phi(s_t, s_{t+1})V(s_t)' \quad \text{for some } k(s_{t+1}) \times k(s_t) \text{ matrix } \Phi(s_t, s_{t+1}) \text{ such that }
\]
\[
\sum_{j=1}^s p_{ij} F_{ij} V_j \Phi_{ij} = V_i, \quad \text{for } 1 \leq i \leq S,
\]
(9)
where \( V_i = V(s_t = i) \), \( \Phi_{ij} = \Phi(s_t = i, s_{t+1} = j) \) and \( F_{ij} = F(s_t = i, s_{t+1} = j) \).

**Proof.** See Appendix A. □

Consider the simplest possible multivariate system (5) such that \( S = n = 2 \) and \( F_{ij} \) is non-singular for all \( i, j = 1, 2 \). For the choice of \( k_i = k(s_t = i) = 2 \), the restriction (9) for \( i = 1 \) is \( p_{11} F_{11} \Lambda_{11} + p_{12} F_{12} \Lambda_{12} = I_2 \), where \( \Lambda_{ij} = \Lambda(s_t = i, s_{t+1} = j) \). This implies that there are 4 free parameters in \( \Lambda_{11} \) and \( \Lambda_{12} \). Thus, there are 8 free parameters to construct \( \Lambda(s_t, s_{t+1}) \). Moreover, we should also perform the same tasks for all of the cases with \( 0 \leq k_1, k_2 \leq 2 \), except for \( k_1 = k_2 = 0 \).

### 3.3 Ideas and Strategies

Cho and McCallum (2012) solve for all the REEs to linear RE models under determinacy and indeterminacy. The results for MSRE models in this paper will be shown to be surprisingly analogous to theirs and hence as tractable as those for linear models. Because of the aforementioned difficulties pertaining to Markov-switching, however, we need to come up with alternative strategies. Here we outline how we circumvent these problems and accomplish our goals. To proceed, we start with a formal definition of determinacy according to the characterization of the RE solutions in Proposition 1.

**Definition 1** The MSRE model (3) is said to be determinate if there exists a unique stable fundamental solution of the form (7), and there is no stable non-fundamental stochastic component \( w_t \) subject to (5) associated with that fundamental solution.

Note that this definition is consistent with linear or MSRE models or with any stability concept. For ease of exposition, we define the maximum absolute eigenvalue of a square matrix in the following way.

**Definition 2** The spectral radius of an \( n \times n \) matrix \( M \) is defined as
\[
|\sigma(M)| = \max_{1 \leq i \leq n}(|\lambda_i|),
\]
where \( \lambda_1, ..., \lambda_n \) are the eigenvalues of \( M \).
Note that linear RE models and their solutions are special cases of the MSRE counterparts described in equations (5) through (9) with all the coefficient matrices being regime-independent ($S = 1$ and $P = 1$). The main results of Cho and McCallum (2012) for linear models are based on the following properties.

- **P1**: there is no stable bubble $\omega_t$ associated with $F$ in (5) if and only if $r_\sigma(F) \leq 1$.

- **P2**: the eigenvalues of any particular $\Omega$ and the inverses of the eigenvalues of the associated $F$ constitute the generalized eigenvalues implied by a model.

According to P2, first observed by McCallum (2007), a fundamental solution $\Omega^{MOD}$ associated with the $n$ smallest generalized eigenvalues implies that the inverses of the eigenvalues of $F^{MOD}$ are the largest $n$ generalized eigenvalues. This observation leads to Proposition 2 of Cho and McCallum (2012), which classifies the RE models by stability into the following three cases, shown in Table 1:

<table>
<thead>
<tr>
<th>Class</th>
<th>Identifying Conditions</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determinacy</td>
<td>$r_\sigma(\Omega^{MOD}) &lt; 1$ and $r_\sigma(F^{MOD}) \leq 1$</td>
</tr>
<tr>
<td>Indeterminacy</td>
<td>$r_\sigma(\Omega^{MOD}) &lt; 1$ and $r_\sigma(F^{MOD}) &gt; 1$</td>
</tr>
<tr>
<td>No Stable Solution</td>
<td>$r_\sigma(\Omega^{MOD}) \geq 1$</td>
</tr>
</tbody>
</table>

Next, they apply the no-bubble condition (NBC) to obtain the set of economically relevant RE equilibria. This is automatically accomplished by applying the forward method proposed by Cho and Moreno (2011) because the forward solution is the only fundamental solution that satisfies the NBC. Putting all these results together, Cho and McCallum (2012) show that the stable forward solution is the unique equilibrium under determinacy and the REEs under indeterminacy are the stable solutions associated with the forward solution.

For MSRE models, Section 4 derives the condition for the non-existence of stable bubbles analogous to P1 under mean-square stability and classifies the MSRE models similar to the classes in Table 1. This is a very powerful result because it does not require us to solve for the explicit form of bubbles. But determinacy and indeterminacy cannot be established by stability alone because there is no direct relationship between
the eigenvalues of $\Omega(s_t)$ and $F(s_t, s_{t+1})$ analogous to $P2$. The forward method for MSRE models developed in Section 5 provides a solution method for the forward solution and the solution refinement, the no-bubble condition, independent of stability. Fortunately, in Section 6 we show that mean-square stability and the NBC provide all the information required to classify the REEs to MSRE models under determinacy and indeterminacy.

4 Mean-Square Stability and Determinacy

The definition of determinacy should be accompanied by a suitable concept of stability for MSRE models. Davig and Leeper (2007) and Benhabib (2009) adopt boundedness, whereas Farmer et al. (2009) propose mean-square stability, requiring a stochastic process to have finite first and second moments. Although both concepts of stability may well be admissible, we adopt mean-square stability for several reasons. First, we want to characterize determinacy for a larger set of stochastic processes that are covariance-stationary, not necessarily bounded. Second, most empirical studies and simulation exercises in macroeconomics assume covariance-stationary, unbounded processes such as normally distributed shocks. Third, mean-square stability leads to very tractable determinacy and indeterminacy conditions for MSRE models. In this section, we introduce the concept of mean-square stability, derive its key properties and classify MSRE models in three categories: determinacy, indeterminacy and no stable solution.

4.1 Mean-Square Stability

Consider the following $n \times 1$ stochastic process $y_{t+1}$:

$$y_{t+1} = G(s_t, s_{t+1})y_t + H(s_{t+1})\eta_{t+1}, \quad (10)$$

where $G(s_t, s_{t+1})$ and $H(s_{t+1})$ are $n \times n$, $n \times m$ matrices, respectively. $\eta_t$ is an arbitrary $m \times 1$ covariance-stationary (wide-sense stationary) process, independent of an ergodic Markov chain $s_t$ explained in Section 3.1. Both the fundamental solutions and bubble components can be cast in the form of $y_{t+1}$. Mean-square stability (MSS) amounts to the existence of the mean and variance of $y_t$. Formally,

**Definition 3** The process $y_{t+1}$ is mean-square stable if there exist an $n \times 1$ vector $\bar{y}$ and an $n \times n$ matrix $Q$ such that $\lim_{t \to \infty}(E(y_t) - \bar{y}) = 0_{n \times 1}$ and $\lim_{t \to \infty}(E(y_t y_t') - Q) = 0_{n \times n}$. 

11
A model similar to Equation (10) is analyzed by Petreczky and Vidal (2007), which has a more general form than the model of Costa et al. (2005) in that both \( G \) and \( H \) depend on \( s_{t+1} \) and \( \eta \) is measured at time \( t+1 \). But we show that both approaches yield identical results for MSS in Appendix B, so we can apply the results of Costa et al. (2005) directly to the model (10). Their Theorem 3.33 states that we have only to analyze the homogeneous part of (10), \( y_{t+1} = G(s_t, s_{t+1}) y_t \), in order to study its mean-square stability. Let \( G_{ij} = G(s_t = i, s_{t+1} = j) \). Now we define the following matrices:

\[
\Psi_G = [p_{ij}G_{ij}] = \begin{bmatrix} p_{11}G_{11} & \ldots & p_{1S}G_{1S} \\ \vdots & \ddots & \vdots \\ p_{1S}G_{1S} & \ldots & p_{SS}G_{SS} \end{bmatrix}, \quad \bar{\Psi}_G = [\bar{p}_{ij}G_{ij}] = \begin{bmatrix} p_{11}G_{11} & \ldots & p_{S1}G_{S1} \\ \vdots & \ddots & \vdots \\ p_{S1}G_{1S} & \ldots & p_{SS}G_{SS} \end{bmatrix},
\]

(11)

\[
\Psi_{G\otimes G} = [p_{ij}G_{ij} \otimes G_{ij}] = \begin{bmatrix} p_{11}G_{11} \otimes G_{11} & \ldots & p_{1S}G_{1S} \otimes G_{1S} \\ \vdots & \ddots & \vdots \\ p_{1S}G_{1S} \otimes G_{1S} & \ldots & p_{SS}G_{SS} \otimes G_{SS} \end{bmatrix},
\]

(12)

\[
\bar{\Psi}_{G\otimes G} = [\bar{p}_{ij}G_{ji} \otimes G_{ji}] = \begin{bmatrix} p_{11}G_{11} \otimes G_{11} & \ldots & p_{S1}G_{S1} \otimes G_{S1} \\ \vdots & \ddots & \vdots \\ p_{S1}G_{1S} \otimes G_{1S} & \ldots & p_{SS}G_{SS} \otimes G_{SS} \end{bmatrix},
\]

(13)

where \( \otimes \) denotes the Kronecker product. Define \( m^y_{i,t} = E[y_t(s_t)1_{\{s_t=i\}}] \) and \( m^y_i = [(m^y_{1,t})', \ldots, (m^y_{S,t})']' \), where \( 1_{\{s_t=i\}} \) is an indicator function that yields 1 when \( s_t = i \) and 0 otherwise. The second moment of \( y_t \) can also be defined as \( Q^y_{i,t} = E[y_t(s_t)y_t(s_t)1_{\{s_t=i\}}] \) and \( Q^y_i = [Q^y_{1,t}, \ldots, Q^y_{S,t}] \). By stacking \( m^y_{i,t} \) and \( vec(Q^y_{i,t}) \) over all \( i = 1, \ldots, S \), one can show that the first and second moments of the homogeneous part of (10) are given by:

\[
m^y_{i,t+1} = \bar{\Psi}_G m^y_i, \]

(14)

\[
v^y_{i,t+1} = \bar{\Psi}_{G\otimes G} v^y_i, \]

(15)

where \( v^y_i = vec(Q^y_i) \). (See chapter 3 of Costa et al. (2005) for details.) Finally, the first and second moments of \( y_t \) are defined as \( E(y_t) = \sum_{i=1}^S m^y_{i,t} \) and \( E(y_t y_t') = \sum_{i=1}^S Q^y_{i,t} \). Hence, it is easy to see that the existence of \( \bar{y} \) and \( Q \) can be expressed as \( r_\sigma(\Psi_G) < 1 \) and \( r_\sigma(\bar{\Psi}_{G\otimes G}) < 1 \) because equations (14) and (15) are regime-independent first-order vector autoregressions of order 1. For fixed regime models where \( G \) is regime-independent, \( r_\sigma(G) < 1 \) if and only if \( r_\sigma(G \otimes G) = [r_\sigma(G)]^2 < 1 \). Therefore, stability of \( y_t \) in mean is
equivalent to stability in variance. However, this is not true for MSRE models. Costa et al. (2005) show in their Proposition 3.6 that if $r_{\sigma}(\bar{\Psi}_{G\otimes G}) < 1$, then $r_{\sigma}(\Psi_{G}) < 1$, but the converse is not true. This implies that the volatility of $y_{t+1}$ in equation (10) can be amplified by regime-switching of $G(s_t, s_{t+1})$ such that while $y_{t+1}$ is bounded in mean, it may be unbounded in variance. Therefore, we have the following important theorem.

**Theorem 1** The process (10) is mean-square stable if and only if $r_{\sigma}(\bar{\Psi}_{G\otimes G}) < 1$.

**Proof.** See Proposition 3.9 of Costa et al. (2005). □

This theorem, a compact version of Theorem 3.9 of Costa et al. (2005), greatly simplifies examining the mean-square stability of RE solutions. Theorem 1 also hints as to why it would be difficult to define determinacy in mean stability, which will be discussed in Section 6.

### 4.2 RE Solutions and Mean-Square Stability

Recall that determinacy requires a unique mean-square stable fundamental solution and the non-existence of stable bubble components associated with that fundamental solution. We mostly elaborate on the latter condition below. Unless stated otherwise, the word stability will be used interchangeably with MSS in what follows.

#### 4.2.1 Fundamental Solutions

Equation (7) has the same form as (10). Therefore, from Theorem 1, a fundamental solution $x_t = \Omega(s_t)x_{t-1} + \Gamma(s_t)z_t$ is mean-square stable if

$$r_{\sigma}(\bar{\Psi}_{\Omega\otimes\Omega}) < 1,$$

where $\bar{\Psi}_{\Omega\otimes\Omega} = [p_{ji}\Omega_j \otimes \Omega_j]$. Note that $\Omega$ at time $t$ depends only on $s_t$. Hence, $\Omega_j = \Omega(s_t = j)$ for all $s_{t-1} = i = 1, ..., S$.

#### 4.2.2 Non-Fundamental Components

There are only two possibilities: there is either no stable bubble or a continuum of stable bubbles. We seek the condition under which there is no stable process $w_t$ subject to (5).
Thanks to Proposition 2, equation (5) - the restriction for \( w_t \) - can be written in a rather informal but more intuitive way involving \( F(s_t, s_{t+1}) \) and \( \Lambda(s_t, s_{t+1}) \) only:

\[
    w_t = E_t[F(s_t, s_{t+1})\Lambda(s_t, s_{t+1})]w_t. \tag{17}
\]

Now we present two key lemmas for establishing the non-existence of stable bubbles.

**Lemma 2** Consider two arbitrary processes of the same size, \( w_{t+1} = \Lambda(s_t, s_{t+1})w_t \) and \( u_{t+1} = F'(s_t, s_{t+1})u_t \) of the form (10). Let \( \bar{\Psi}_{\Lambda \otimes F'} = [p_{ji} \Lambda_{ji} \otimes F'_{ji}] \). The following holds.

1. If \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) < 1 \) and \( r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) < 1 \), then \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) < 1 \).
2. If \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1 \) and \( r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) \leq 1 \), then \( r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) \geq 1 \).

**Proof.** See Appendix C.

Assertion 1 of this lemma simply shows that the sum of any two mean-square stable processes is also stable and it must be true that \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) < 1 \). Assertion 2 is the converse relation of Assertion 1, but it should be emphasized that the second condition includes the case \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) = 1 \). This implies that when \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1 \), so that the sum \( w_{t+1} + u_{t+1} \) is not stable, \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \leq 1 \) becomes a sufficient condition for the non-existence of stable bubbles. Note that Lemma 2 does not impose any restrictions on the two matrices. The following lemma shows a pivotal role of \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \) in understanding the restriction of (17) and provides a clue for the non-existence of MSS bubbles.

**Lemma 3** For any process (8) subject to (9), \( \bar{\Psi}_{\Lambda \otimes F'} \) contains a root of 1, hence, \( r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1 \).

**Proof.** See Appendix D.

Note that \( \bar{\Psi}_{F' \otimes F'} = (\Psi_{F \otimes F})' \) from equations (12) and (13), hence, \( r_\sigma(\Psi_{F \otimes F}) = r_\sigma(\bar{\Psi}_{F' \otimes F'}) \).

Therefore, Assertion 2 of Lemma 2 and Lemma 3 lead to the following Proposition:

**Proposition 3** Consider equation (3). Suppose that the following condition holds:

\[
    r_\sigma(\Psi_{F \otimes F}) \leq 1. \tag{18}
\]

Then there is no stochastic mean-square stable process \( w_t \) satisfying (3).
Proof. Assertion 2 of Lemma 2 and Lemma 3 imply that as long as \( r_{\sigma}(\Psi_{F \otimes F}) \leq 1 \), \( r_{\sigma}(\Psi_{\Lambda \otimes \Lambda}) \geq 1 \) for any \( \Lambda(s_t, s_{t+1}) \) in (17). Q.E.D.

The most powerful consequence of Proposition 3 is that under condition (18), one need not solve for all \( w_t \) in order to confirm the non-existence of MSS bubbles satisfying (17).

Note, however, that (18) is a sufficient condition, and thus we need to examine whether there are additional cases of no stable bubbles when \( r_{\sigma}(\Psi_{F \otimes F}) > 1 \). To proceed, we derive a sharper result from Lemma 2 and 3 as follows.

**Lemma 4** For any process (8) subject to (9), \( r_{\sigma}(\bar{\Psi}_{\Lambda \otimes \Lambda}) \geq 1 = r_{\sigma}(\Psi_{F \otimes F}) \) for all \( \Lambda(s_t, s_{t+1}) \) in (17).

**Proof.** See Appendix E. ■

Lemma 4 states that for a given \( F(s_t, s_{t+1}) \), \( 1/[r_{\sigma}(\Psi_{F \otimes F})] \) is the lower bound of \( r_{\sigma}(\Psi_{\Lambda \otimes \Lambda}) \) for any \( \Lambda(s_t, s_{t+1}) \) in (17). Now define the minimum value of \( r_{\sigma}(\Psi_{\Lambda \otimes \Lambda}) \):

\[
\tau_2 = \min_{\Lambda(s_t, s_{t+1})} r_{\sigma}(\Psi_{\Lambda \otimes \Lambda}), \text{ subject to } w_t = E_t[F(s_t, s_{t+1})\Lambda(s_t, s_{t+1})]w_t. \tag{19}
\]

If \( \tau_2 = 1/[r_{\sigma}(\Psi_{F \otimes F})] \), then condition (18) becomes necessary and sufficient for the non-existence of stable bubbles. For linear models, this is always true because (17) collapses to \( w_t = F\Lambda w_t \) and we can always construct a \( \Lambda \) such that it contains the inverse of \( r_{\sigma}(F) \), implying \( \tau_2 = 1/[r_{\sigma}(F \otimes F)] \). For MSRE models, however, there is no such analytical result because the eigenvalues of \( F(s_t, s_{t+1}) \) and \( \Lambda(s_t, s_{t+1}) \) for each state are not directly related. So the question is whether there exists a case such that \( r_{\sigma}(\Psi_{F \otimes F}) > 1 \) but no stable bubble exists, i.e., \( \tau_2 \geq 1 > 1/[r_{\sigma}(\Psi_{F \otimes F})] \). Fortunately, Lemma 4 and Proposition 2 make it feasible to find \( \tau_2 \) by searching for \( \Lambda(s_t, s_{t+1}) \) subject to (9). Appendix F provides a detailed search procedure. Therefore, when \( r_{\sigma}(\Psi_{F \otimes F}) > 1 \), if \( \tau_2 \geq 1 \), no stable bubbles exist and if \( \tau_2 < 1 \), then there exist uncountably many stable bubbles.

How likely would the case arise that \( \tau_2 \geq 1 \) when \( r_{\sigma}(\Psi_{F \otimes F}) > 1 \)? As hinted at by linear models, the odds might be small. However, we construct an atheoretical example and find that such cases can arise, but only when a regime is absorbing, i.e., \( p_{ii} = 1 \) for some \( i \). We perform extensive numerical exercises including the economic models in this paper and others, and the relation \( \tau_2 = 1/[r_{\sigma}(\Psi_{F \otimes F})] \) is always confirmed. To
summarize, Proposition 3 and our search method complete the classification of the set of non-fundamental bubble components for all values of $r_\sigma(\Psi_{F\otimes F})$.

4.3 Classification of MSRE Models by MSS

Using the results derived above, we classify MSRE models by MSS into the categories of determinacy, indeterminacy and no stable solution. It should be stressed that the results explained below will not be applicable in general to MSRE models due to the lack of information about the uniqueness of a stable fundamental solution and the solution method. This critical step will be complemented by the forward method in Section 5.

The first determinacy result for MSRE models is presented as follows.

**Proposition 4** The MSRE model (3) is determinate in the mean-square stability sense if there exists a unique fundamental solution of the form (7) such that

$$r_\sigma(\bar{\Psi}_{\Omega\otimes\Omega}) < 1 \text{ and } r_\sigma(\Psi_{F\otimes F}) \leq 1.$$  \hfill (20)

**Proof.** The proof directly follows from the definition of determinacy and Proposition 3. Q.E.D.

Next, indeterminacy arises if one or more stable fundamental solutions exist and for at least one such a fundamental solution, $r_\sigma(\Psi_{F\otimes F}) > 1$ and $\tau_2 < 1$. If $r_\sigma(\bar{\Psi}_{\Omega\otimes\Omega}) \geq 1$ for all $\Omega$, then the model has no MSS solution. The remaining cases are as follows. The model is determinate if there is a unique stable fundamental solution and $\tau_2 \geq 1$ when $r_\sigma(\Psi_{F\otimes F}) > 1$. Indeterminacy could also arise if multiple stable fundamental solutions exist, but none of them is accompanied by stable bubbles. Our classification of MSRE models by stability is now complete and summarized in Table 2.

Cases 1, 2 and 3 for MSRE models are natural generalizations of the three cases in Table 1 for linear RE models: the relevant matrices are replaced with those weighted by transition probabilities. The information missing in MSRE models is the matrix $\Omega(s_I)$ corresponding to $\Omega^{MOD}$ in linear models. We have shown that Cases 1’ and 2’ could hardly arise for models without an absorbing state. In particular, the odds that Case 2’ arises would be extremely low, and this case is included just for completion of the some states, which is hardly justifiable. A model with an absorbing state would be less appealing as well, although we do not exclude such models a priori. For instance, such a specification implies that a MSRE model of two regimes would be a fixed regime model in a finite horizon.
Table 2: Classification of MSRE models by Stability

<table>
<thead>
<tr>
<th>Class</th>
<th>Identifying Conditions</th>
<th># of Ω</th>
</tr>
</thead>
<tbody>
<tr>
<td>Determinacy</td>
<td>Case 1: $r_\sigma(\Psi_{\Omega_{\otimes}\Omega}) &lt; 1$ and $r_\sigma(\Psi_{F_{\otimes}F}) \leq 1$</td>
<td>1</td>
</tr>
<tr>
<td></td>
<td>Case 1′: $r_\sigma(\Psi_{\Omega_{\otimes}\Omega}) &lt; 1$ and $r_\sigma(\Psi_{F_{\otimes}F}) &gt; 1$ and $\tau_2 \geq 1$</td>
<td>1</td>
</tr>
<tr>
<td>Indeterminacy</td>
<td>Case 2: $r_\sigma(\Psi_{\Omega_{\otimes}\Omega}) &lt; 1$ and $r_\sigma(\Psi_{F_{\otimes}F}) &gt; 1$ and $\tau_2 &lt; 1$</td>
<td>$\geq 1$</td>
</tr>
<tr>
<td></td>
<td>Case 2′: $r_\sigma(\Psi_{\Omega_{\otimes}\Omega}) &lt; 1$ and $r_\sigma(\Psi_{F_{\otimes}F}) &gt; 1$ and $\tau_2 \geq 1$</td>
<td>$&gt; 1$</td>
</tr>
<tr>
<td>No MSS solution</td>
<td>Case 3: $r_\sigma(\Psi_{\Omega_{\otimes}\Omega}) \geq 1$</td>
<td>All</td>
</tr>
</tbody>
</table>

The last column indicates the assumption about the number of fundamental solutions ($\Omega$) for each case.

classification of the models. Cases 1′, 2 and 2′ can be easily identified by the search method proposed in the previous subsection.

Proposition 4 and our classification, however, can be readily applied to an important subset of the MSRE models without predetermined variables. For such a class of models, the fundamental solution is of course unique and $r_\sigma(\Psi_{\Omega_{\otimes}\Omega}) = 0$ as $\Omega(s_t) = 0_{n\times n}$, and $F(s_t, s_{t+1}) = A(s_t, s_{t+1})$. Case 2′ drops out as well. Therefore, if $r_\sigma(\Psi_{F_{\otimes}F}) \leq 1$ in (20), for instance, the model is determinate and the determinacy solution is $x_t = \Gamma(s_t)z_t$.

5 The Forward Method

The forward method we develop here for MSRE models is essentially identical to the one that Cho and Moreno (2011) developed for linear RE models. We first present the solution methodology, followed by the solution refinement scheme.

5.1 Solution Methodology

The first step is to derive a forward representation implied by a given model such that the vector of current endogenous variables is related to the expectations of future endogenous variables and the current state variables recursively using the law of iterative expectations and, additionally, a Markov chain property. The forward method simply amounts to textbook-style “solving the model forward” as stated in the following:

**Proposition 5** Consider model $[\mathcal{M}]$. For a given set of states $x_t$, $x_{t-1}$, $z_t$ and an initial regime $s_t$ at time $t$, there exists a unique sequence of real-valued matrices ($\Omega_k(s_t)$,
\( \Gamma_k(s_t), F_k(s_t, s_{t+1}) \), \( k = 1, 2, 3, \ldots \) such that:

\[
x_t = E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1} + \Gamma_k(s_t)z_t, \quad (21)
\]

where \( \Omega_1(s_t) = B(s_t), \Gamma_1(s_t) = C(s_t), \) \( F_1(s_t, s_{t+1}) = A(s_t, s_{t+1}) \) and for \( k = 2, 3, \ldots \),

\[
\begin{align*}
\Omega_k(s_t) &= \{ I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}B(s_t), \quad (22a) \\
\Gamma_k(s_t) &= \{ I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}C(s_t) \\
&+ E_t[F_k(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})R], \\
F_k(s_t, s_{t+1}) &= \{ I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}A(s_t, s_{t+1}), \quad (22c)
\end{align*}
\]

if the following regularity condition is satisfied for all \( k > 1 \) and \( s_t = 1, 2, \ldots, S \):

\[
|I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]| \neq 0. \quad (23)
\]

**Proof.** See Appendix G \(^{13}\)**

The conditional expectations in equation (22) can be easily computed. For instance, \( E[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})|s_t = i] = \sum_{j=1}^{S} p_{ij} \Omega_{k-1}(j) \) for all \( s_t \) and for all \( k \geq 2 \) with \( \Omega_1(s_t) = B(s_t) \). Note that if the sequences of the matrices defined in (22) converge as \( k \to \infty \), their limiting functions fulfill (6). This suggests that the limiting functions of these sequences will define a fundamental solution and restrict the bubble components. Formally, the convergence property of these sequences is defined as follows.

**Definition 4** The MSRE model \(^3\) is said to satisfy the forward convergence condition (FCC) if the coefficients of the state variables, \( (\Omega_k(s_t), \Gamma_k(s_t)) \), in the forward representation of the model, (21), converge for every regime \( s_t \) as \( k \) tends to infinity. Under the FCC, the model implies:

\[
x_t = \lim_{k \to \infty} E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] + \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t, \quad (24)
\]

where \( \Omega^*(s_t) = \lim_{k \to \infty} \Omega_k(s_t) \) and \( \Gamma^*(s_t) = \lim_{k \to \infty} \Gamma_k(s_t) \) for every \( s_t \) \(^5\).

\(^{14}\)We show in the appendix that there exists a unique sequence \( M_k(s_t, s_{t+1}, \ldots, s_{t+k}) \) as well, but we do not present it since our methodology does not require us to compute the expectational term in (21).

\(^{15}\)Convergence of \( F_k(s_t, s_{t+1}) \) need not be included in the FCC since \( F^*(s_t, s_{t+1}) = \lim_{k \to \infty} F_k(s_t, s_{t+1}) \) exists if and only if \( \Omega^*(s_t) \) exists.
According to the FCC, the entire class of MSRE models is dichotomized into the two groups: the models that satisfy the FCC and those that violate the FCC. For ease of exposition, we denote these two groups as FC models and non-FC models, respectively. Note that the FCC holds if and only if $\Omega^*(s_t)$ and $\Gamma^*(s_t)$ exist, which define the forward solution to FC models as follows.

**Definition 5** For the forward convergent (FC) MSRE model (3), the forward solution is defined as the function of the state variables in the absence of the expectational effect in the forward representation in the limit, (24):

$$x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t. \quad (25)$$

By construction, the forward solution is a fundamental solution to model (3) because $(\Omega^*(s_t), \Gamma^*(s_t))$ satisfy (6a) and (6b) from Proposition 1. Moreover, the forward solution is unique for every single FC model because the sequence of $(\Omega_k(s_t), \Gamma_k(s_t))$ is uniquely constructed by the initial coefficient matrices given by the model. This solution is by itself a very natural fundamental equilibrium to any RE model, because it is the relation uniquely implied by the underlying model that is absent from the expectational effect of the endogenous variables far in the future on the current variables.

### 5.2 No-Bubble Condition as a Solution Refinement

As Proposition 5 highlights, the forward method is a solution algorithm to compute the forward solution. But it also provides the no-bubble condition, a key equilibrium characteristic that any fundamental solution should minimally possess. In this subsection, we show that the forward solution is the unique fundamental solution that passes the following no-bubble condition. This fact is particularly a great advantage for MSRE models because we do not need to know how many fundamental solutions a model has, which is actually unknown as we showed in Section 3, nor do we need to solve for them.

**Definition 6** A rational expectations solution to the MSRE model (3) is said to satisfy the no-bubble condition (NBC) if the expectational term involving future endogenous variables converges to zero in the forward representation of the model, (21), for every $s_t$ when expectations are formed with that solution:

$$\lim_{k \to \infty} E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] = 0_{n \times 1}. \quad (26)$$
The forward solution must satisfy the NBC by definition. That is, when expectations are formed with this solution, the expectations of endogenous variables far in the future should not affect the current endogenous variables. The NBC in the context of linear RE models has been typically assumed to hold in the literature. But such an assumption can lead to a misjudgment of a RE solution as an equilibrium whenever it differs from the forward solution. Cho and Moreno (2011) show that any fundamental solution to a linear RE model, if different from the forward solution, must violate the NBC\textsuperscript{16} and it is preserved for MSRE models as the following proposition shows:

**Proposition 6** The forward solution (25) to the MSRE model (3) is the unique fundamental solution that satisfies the no-bubble condition (26).

**Proof.** See Appendix H. \(\blacksquare\)

The NBC eliminates two kinds of RE solutions. First, the NBC eliminates all of the RE solutions to non-FC models. For this reason, Cho and McCallum (2012) refer to the FCC as a model refinement. Recall that any fundamental solution must satisfy the forward representation of the model, (21), at all \(k \geq 1\). Hence, that a fundamental solution to a non-FC model satisfies the NBC implies the convergence of \(\Omega_k(s_t)x_{t-1} + \Gamma_k(s_t)z_t\) to that solution, which contradicts the fact that at least one of \(\Omega_k(s_t)\) and \(\Gamma_k(s_t)\) is not convergent. As a concrete example, suppose that \(\Omega_k(s_t)\) converges\textsuperscript{17}. We can solve the expectational term in (22b) and get a closed-form expression for \(\Gamma_k(s_t)\):

\[
\begin{bmatrix}
\Gamma_k(1) \\
\vdots \\
\Gamma_k(S)
\end{bmatrix} = \begin{bmatrix}
\{I_n - E_t[A(1, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}C(1) \\
\vdots \\
\{I_n - E_t[A(S, s_{t+1})\Omega_{k-1}(s_{t+1})]\}^{-1}C(S)
\end{bmatrix} + \Psi_{R^*F_{k-1}} \begin{bmatrix}
\Gamma_{k-1}(1) \\
\vdots \\
\Gamma_{k-1}(S)
\end{bmatrix}
\]

This equation in the absence of the subscripts yields the formula for computing \(\Gamma(s_t)\) in (6b). It can be shown that \(r_\sigma(\Psi_{R^*F^*}) = r_\sigma(\Psi_{F^*})r_\sigma(R)\) as \(R\) is regime-independent. We have assumed that \(r_\sigma(R) < 1\). However, if \(r_\sigma(\Psi_{F^*}) > 1\) and \(r_\sigma(R)\) is sufficiently high, then the FCC fails to hold. Thus, a RE solution to a non-FC model can be mistakenly accepted as an equilibrium if the FCC is not examined\textsuperscript{18}. This is a non-trivial case that

\textsuperscript{16}For further discussion and concrete economic examples, refer to Cho and Moreno (2011) and Cho and McCallum (2012).

\textsuperscript{17}\(\Omega_k(s_t)\) seems to converge in any case under determinacy, or indeterminacy, although there is no closed-form expression for this condition.

\textsuperscript{18}This claim, highlighting the essential feature of the forward method, may be better understood by recalling an elementary computation of \(1 + r + r^2 + \ldots\) where \(r\) is a real number. The forward method
can actually happen. We will show why such a RE solution does not make economic sense using the Fisherian and New-Keynesian models in Section 7.

Second, for FC-MSRE models, all of the fundamental solutions but the forward solution violate the NBC. By a similar logic, that a fundamental solution different from the forward solution satisfies the NBC cannot be true because \( \Omega_k (s_t) x_{t-1} + \Gamma_k (s_t) z_t \) converges to the forward solution.

Note that we have not discussed the stability of RE solutions. Nor did we use any information about determinacy throughout the forward method in this section. The NBC is completely an independent economic solution refinement for the class of fundamental solutions. Note also that one does not need to solve for other fundamental solutions and examine the NBC for them. The forward method automatically produces a fundamental solution that is refined by the NBC only, which is nothing but the forward solution.

6 Rational Expectations Equilibria to MSRE models

We have proposed two independent refinement schemes for RE solutions to MSRE models: mean-square stability and the no-bubble condition. And we refer to a solution passing both refinements as a rational expectations equilibrium (REE). Let FC-(in)determinacy denote the case of the FC model that is (in)determinate. Now our task is to identify conditions for FC-determinate and FC-indeterminate models and classify the REEs for each of those models. By now it is clear that we need to consider neither non-FC models nor the models that have no stable RE solution.

6.1 Classification of RE Equilibria

Since the forward method yields just one fundamental solution, the forward solution, the condition \( r_a (\Psi_{\Omega^*}) < 1 \) alone does not establish that the forward solution is the only
stable fundamental solution. However, if $r_{\sigma}(\Psi F^* \otimes F^*) \leq 1$ as well where $F^*(s_t, s_{t+1})$ is associated with $\Omega^*(s_t)$, then there is no other stable fundamental solution. It is with this observation that we establish FC-determinacy for MSRE models as follows.

**Proposition 7** Suppose that the forward solution (25) exists for the MSRE model (3). Then, the model is determinate in the mean-square stability sense if

$$r_{\sigma}(\bar{\Psi} \Omega^* \otimes \Omega^*) < 1 \text{ and } r_{\sigma}(\Psi F^* \otimes F^*) \leq 1,$$

and the determinate equilibrium is the forward solution (25).

**Proof.** See Appendix I. ■

Our Proposition 7, the most important result of this paper, is very compact and easy to apply to any MSRE model of the form (3). One has only to solve a given model forward, examine the FCC, compute the forward solution and check the maximum eigenvalues of the two matrices in (28) for determinacy.

While the formal proof of Proposition 7 is given in the appendix, it is instructive to think about why the conditions (28) warrant that all other fundamental solutions, if any, are unstable. For simplicity, assume away the exogenous variables $z_t$. The expectational term in the forward representation (21) can be recursively defined as $L_k(s_t; \Omega)x_t$, where $L_k(\cdot)$ is a function of a particular fundamental solution $\Omega(s_t)$ at time $t$ for every $k \geq 1$: $L_k(s_t; \Omega) = E_t[F_k(s_t, s_{t+1})L_{k-1}(s_{t+1}; \Omega)\Omega(s_{t+1})]$ with $L_0(s_t; \Omega) = I_n$. Then for any fundamental solution, say, $x_t = \tilde{\Omega}(s_t)x_{t-1}$ different from the forward solution, $L_k(s_t; \tilde{\Omega})$ should converge to a non-zero matrix $L(s_t; \tilde{\Omega})$ as $k$ goes to infinity.\(^{19}\) Therefore, the recursion formula in the limit becomes:

$$L(s_t; \tilde{\Omega}) = E_t[F^*(s_t, s_{t+1})L(s_{t+1}; \tilde{\Omega})\tilde{\Omega}(s_{t+1})],$$

which restricts $\tilde{\Omega}(s_{t+1})$ in a way similar to how (17) restricts $\Lambda(s_t, s_{t+1})$: it can be shown that the matrix $\Psi_{\tilde{\Omega} \otimes (F^*)'}$ contains a unit root, implying that $r_{\sigma}(\Psi_{\tilde{\Omega} \otimes (F^*)'}) \geq 1$. Therefore, as in Proposition 3, if $r_{\sigma}(\Psi F^* \otimes F^*) \leq 1$, then $r_{\sigma}(\bar{\Psi} \tilde{\Omega} \otimes \tilde{\Omega}) \geq 1$ for all $\tilde{\Omega}(s_{t+1})$ subject to (29), implying that $x_t = \tilde{\Omega}(s_t)x_{t-1}$ is unstable in the MSS sense.

\(^{19}\) $L(s_t; \Omega)$ cannot be zero because $\tilde{\Omega}(s_t)$ differs from the forward solution, nor can it explode because the solution should solve the forward representation of the model. In fact, it can be shown that for any fundamental solution, $\tilde{\Omega}(s_t)$, $(I_n - L(s_t; \Omega))\tilde{\Omega}(s_t) = \Omega^*(s_t)$ when the FCC holds.
Next, suppose that \( r_\sigma(\Psi_{\Omega^*}) < 1 \) and \( r_\sigma(\Psi_{F^*}) > 1 \). Then, we need to apply the search method and the following proposition establishes the FC-indeterminacy results.

**Proposition 8** Suppose that the forward solution (25) exists for the MSRE model (3). Then, the model is indeterminate in the mean-square stability sense if

\[
\begin{align*}
  r_\sigma(\Psi_{\Omega^*}) &< 1, \\
  r_\sigma(\Psi_{F^*}) &> 1 \\
  \tau_2^* &< 1,
\end{align*}
\]

where \( \tau_2^* = \min_{A(s_t, s_{t+1})} r_\sigma(\Psi_{\Lambda\Lambda}) \) subject to (17) with \( F(s_t, s_{t+1}) = F^*(s_t, s_{t+1}) \). The indeterminate equilibria are the mean-square stable RE solutions associated with the forward solution and they are given by:

\[
x_t = \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t + w_t^*,
\]

where \( w_t^* \) is either \( 0_{n \times 1} \) or has the form of (8) subject to (9) such that \( r_\sigma(\Psi_{\Lambda\Lambda}) \in [\tau_2^*, 1) \).

**Proof.** Since the forward solution is stable and the supposition \( \tau_2^* < 1 \) implies the existence of stable bubble components, the model is indeterminate. Therefore, any non-fundamental component with \( \Lambda(s_t, s_{t+1}) \) is mean-square stable if \( r_\sigma(\Psi_{\Lambda\Lambda}) \in [\tau_2^*, 1) \). All other solutions are not admissible as REEs because either they are unstable or their fundamental components violate the NBC or both. Q.E.D.

We have shown that the condition \( \tau_2^* = 1/r_\sigma(\Psi_{F^*}) \) would prevail for most MSRE models. If so, the condition \( \tau_2^* < 1 \) becomes redundant. Nevertheless, it is necessary to employ our search method to check this condition, which is straightforward to implement.

The remaining case is that \( r_\sigma(\Psi_{\Omega^*}) < 1 \) and \( r_\sigma(\Psi_{F^*}) > 1 \), but \( \tau_2^* \geq 1 \). As we have argued, it is highly unlikely that this case will arise. However, what matters is that if such a case arises, the forward solution is the unique stable REE because all other fundamental solutions, if they exist, must violate the NBC, regardless of determinacy.

This last case, together with FC-determinacy from Proposition 7 and FC-indeterminacy from Proposition 8, completes our classification of REEs to MSRE models. All other solutions are rejected as REEs on the grounds of the NBC from Proposition 6. In any case, the forward method explained in Proposition 5 is the prerequisite to our judgment. Without it, we cannot characterize the REEs. Without it, we may fail to reject the additional stable fundamental solutions, but if they exist, they must violate the NBC.

\[\text{\footnotesize{20}}\text{This case corresponds to a mixture of Cases 1 and 2 in Table 2 because there may or may not be additional stable fundamental solutions, but if they exist, they must violate the NBC.}\]
solutions in the case of non-FC models or the solutions other than the forward solution for the FC-indeterminate models.

**Remark** What would be the most plausible equilibrium to FC-indeterminate models? For linear models, the roots of $\Lambda$ are the reciprocals of some or all of the eigenvalues of $F^*$. So there is at most a finite number of $\Lambda$ and one may choose a particular one to construct a non-fundamental REE. (Of course the bubble shocks associated with a $\Lambda$ are uncountably many.) For MSRE models, there are uncountably many $\Lambda(s_t, s_{t+1})$: even for a choice of a particular value of $r$ such as any value arbitrarily close to one, the model has a continuum of $\Lambda(s_t, s_{t+1})$. For this reason, even under indeterminacy, the forward solution would be much more relevant as a representative equilibrium than other bubble solutions. Then, if one seeks a REE within the class of fundamental solutions, she or he may read the main message of this paper as follows: the unique REE to a MSRE model is the mean-square stable forward solution, independent of determinacy.

### 6.2 Other Refinement Schemes

Can mean stability or boundedness be a viable alternative to mean-square stability in defining determinacy? Our answer to this question is negative on the following grounds. First, it turns out that determinacy generally requires much stronger conditions under mean stability than under MSS. Under the condition $r_\sigma(\Psi F^* F^*) \leq 1$, no MSS bubble exists from Proposition 7, but mean stable bubbles may well exist because $r_\sigma(\bar{\Psi}_A) \geq 1$ does not imply $r_\sigma(\bar{\Psi}) \geq 1$. Appendix J shows that there is no mean stable bubble if $r_\sigma(\Psi F^* F^*) \leq 1$ and $r_\sigma(\Psi F^* F^*) \leq 1$ and $\tau_2^* = 1/r_\sigma(\Psi F^* F^*)$. By the same token, the uniqueness of the mean stable forward solution also requires a stronger condition than $r_\sigma(\Psi F^* F^*) < 1$. Second, little is known about determinacy under boundedness for general MSRE models. The condition $r_\sigma(\Psi F^*) \leq 1$ for the models without predetermined variables appears to be related to the LRTP of Davie and Leeper (2007), but it is not the determinacy condition under boundedness for our class of MSRE models as we show in the following section.

Apart from stability, Farmer et al. (2011) have proposed a likelihood-based selection criterion for fundamental solutions. Their simulation exercise shows that a likelihood approach may well choose a less stationary solution with high probabilities. If such a phenomenon prevails in linear models as well, then their criterion would conflict with most of the existing solution refinements for linear models, which almost always pick up the fundamental solution with $\Omega^{MOD}$. Therefore, it is highly likely that the selected
solution differs from the forward solution and hence violates the NBC for both linear and MSRE models. The likelihood-based criterion is a relative measure: it always selects the fundamental solution yielding the maximum likelihood value as an equilibrium, and it appears to be silent about the economic relevance of the selected solution.

7 Applications

We apply our methodology to several economic examples including those introduced in Section 2. All models belong to one of the following three cases: FC-determinacy, FC-indeterminacy and non-FC models. While identifying the first two cases and computing the REEs is the most important contribution of this paper, it is almost equally important to detect the third case where no economically relevant REEs exist. For all of the models we examine, we find \( \Lambda(s_t, s_{t+1}) \) such that \( \tau^*_2 = 1/[r_\sigma(\Psi_{F^* \otimes F^*})] \). Therefore, within FC models, the conditions in Proposition 7 and 8 become necessary and sufficient for determinacy and indeterminacy, respectively. We first consider the Fisherian model, followed by New-Keynesian models with regime-switching monetary policy and the models with regime-switching in the private sector.

7.1 The Fisherian Model

We reexamine a simple Fisherian model with a regime-switching monetary policy considered by Davig and Leeper (2007) and Farmer et al. (2011). The model consists of the Fisher equation \( i_t = E_t \pi_{t+1} + r_t \), a regime-switching policy rule \( i_t = \alpha(s_t) \pi_t \) over two regimes and the exogenous real interest rate \( r_t \). Substituting out the nominal interest rate \( i_t \) yields an equation for inflation, \( \pi_t \):

\[
\pi_t = A(s_t)E_t \pi_{t+1} + A(s_t)r_t, \quad r_t = Rr_{t-1} + \epsilon_t,
\]

where \( A(s_t) = 1/\alpha(s_t) \) and \( 0 < R < 1 \). \( \epsilon_t \) is assumed to be white noise with mean zero. Since there is no predetermined variable, \( \Omega(s_t) = 0, F(s_t) = A(s_t) \) and the fundamental solution has the form of \( \pi_t = \Gamma(s_t)r_t \) from Proposition 1. Hence, for determinacy, we have only to consider \( r_\sigma(\Psi_{F \otimes F}) < 1 \) from Proposition 4. Now we apply the forward method. In this simple model \( \Omega^*(s_t) = 0 \) and \( F(s_t)^* = A(s_t) \), but the existence of \( \Gamma^*(s_t) \) depends on the parameter values. We take the parameter values from Farmer et al. (2011): \( p_{11} = 0.8, p_{22} = 0.9, R = 0.9 \) and \( \alpha_2 = 1.5 \). Depending on the value of \( \alpha_1 \), the
model has the three cases reported in Table 3.

<table>
<thead>
<tr>
<th>Case</th>
<th>FC-Determinacy</th>
<th>FC-Indeterminacy</th>
<th>Non-FC</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\alpha_1)</td>
<td>0.95</td>
<td>0.9</td>
<td>0.7</td>
</tr>
<tr>
<td>(r_\sigma(\Psi_{F^<em>\otimes F^</em>}))</td>
<td>0.91</td>
<td>1.01</td>
<td>1.65</td>
</tr>
<tr>
<td>Fundamental solution</td>
<td>(\Gamma^*(1) = 6.11)</td>
<td>(\Gamma^*(1) = 8.01)</td>
<td>(\Gamma(1) = -29.0)</td>
</tr>
<tr>
<td>(r_\sigma(\Psi_{R^<em>\otimes F^</em>}))</td>
<td>0.80</td>
<td>0.84</td>
<td>1.06</td>
</tr>
</tbody>
</table>

When \(\alpha_1 = 0.95\), the model is FC-determinate from Proposition 7 because \(r_\sigma(\Psi_{F^*\otimes F^*}) = 0.91 < 1\) and the determinate equilibrium is the forward solution \(x_t = \Gamma^*(s_t)z_t\). Indeed, a slightly passive policy is admissible as part of a determinate equilibrium as the transition probability from the passive to active regime is sufficiently high. If \(\alpha_1 = 0.9\), however, \(r_\sigma(\Psi_{F^*\otimes F^*}) = 1.01\) and the model becomes FC-indeterminate from Proposition 8. The class of indeterminate equilibria is given by \(x_t = \Gamma^*(s_t)z_t + w_t^*\) where \(w_t^*\) has the form of (8) with a continuum of \(\Lambda(s_t, s_{t+1})\) such that \(1/1.01 \leq r_\sigma(\Psi_{\Lambda\otimes\Lambda}) < 1\).

Consider the third case with \(\alpha_1 = 0.7\). From \(r_\sigma(\Psi_{F^*\otimes F^*}) = 1.65\), one might conclude that this is yet another indeterminate model, qualitatively indistinguishable from the second case. Even a well-defined fundamental solution \(\Gamma(s_t)\) can be obtained by the formula (27) without the subscripts. But this is a non-FC model and highlights the importance of the NBC as a solution refinement. Since \(\Gamma_k(s_t)\) goes to positive infinity as \(k \to \infty\), the forward solution does not exist; thus, no fundamental solution satisfies the NBC. The explosion of \(\Gamma_k(s_t)\) can be detected by \(r_\sigma(\Psi_{R^*\otimes F^*}) = 1.06\) in equation (27).

But what is exactly problematic with this solution economically? In the first two cases, inflation rises following a positive real rate shock, as it should. The fundamental solution with \(\Gamma(s_t)\) to the non-FC model predicts exactly the opposite dynamics: inflation *falls* following the positive real rate shock as above in both regimes! This model with \(\alpha_1 = 0.5\) and \(\alpha_2 = 0.8\) is the one considered by Farmer et al. (2011) and the fundamental solution exists: \(\Gamma(1) = -12.14\) and \(\Gamma(2) = 9.29\). But the policy stance is passive in both regimes and the model fails to satisfy the FCC as \(r_\sigma(\Psi_{R^*\otimes F^*}) = 1.52 > 1\); hence, the fundamental solution violates the NBC.

---

Footnote 21: Solving for \(\Gamma(s_t)\) using the formula (27) when \(r_\sigma(\Psi_{R^*\otimes F^*}) > 1\) is analogous to solving the summation problem in footnote 18 by the formula \(S\) when \(r > 1\) that erroneously yields a negative number.
7.2 Regime-Switching Monetary Policy

Here we analyze a regime-switching monetary policy in a standard New-Keynesian framework. The main results are virtually the same as those of the Fisherian model above. The Markov-switching monetary policy model (1) can be cast into the form of (3) with

\[ x_t = [\pi_t \ y_t \ i_t]', \quad z_t = [z_{S,t} \ z_{D,t} \ z_{MP,t}]', \quad \epsilon_t = [\epsilon_t^S \ \epsilon_t^D \ \epsilon_t^{MP}]', \]

and,

\[ B_1(s_t)x_t = A_1(s_t)E_t[x_{t+1}] + B_2x_{t-1} + z_t, \quad z_t = Rz_{t-1} + \epsilon_t, \]

\[ B_1(s_t) = \begin{bmatrix} 1 & -\kappa & 0 \\ 0 & 1 & 1/\sigma \\ -(1-\rho)\phi_\pi(s_t) & 0 & 1 \end{bmatrix}, \quad A_1(s_t) = \begin{bmatrix} \beta & 0 & 0 \\ 1/\sigma & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \quad B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}. \]

The matrix \( R \) is a diagonal matrix with the diagonal elements, \( \rho_S, \rho_D \) and \( \rho_{MP} \). For simplicity, \( \phi_y \) is assumed to be zero. Then the model is given by:

\[ x_t = A(s_t)E_t[x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t, \]

where \( A(s_t) = B_1(s_t)^{-1}A_1(s_t) \), \( B(s_t) = B_1(s_t)^{-1}B_2 \) and \( C(s_t) = B_1(s_t)^{-1} \).

7.2.1 A Model Without Predetermined Variables

We first consider the model without a predetermined variable (\( \rho = 0 \)). For simplicity, we set \( \rho_S = \rho_D = 0 \) but \( \rho_{MP} = 0.95 \). The other parameter values are set similar to those in \cite{Farmer2009}: \( \beta = 0.99, \kappa = 0.132, \sigma = 1, \ p_{11} = 0.85 \) and \( p_{22} = 0.95 \). We identify again the three regions by the policy parameters \( \phi_\pi(1) \) and \( \phi_\pi(2) \). These three regions are plotted in Figure \[1\]. It is clear that the FC-determinacy region is larger than the region where the policy is active in both regimes. The complement of this FC-determinacy region is what would be described as an indeterminacy region based on a stability criterion only. However, the criterion of the NBC separates FC-indeterminacy (both A and B regions) from the non-FC region where no economically relevant equilibrium exists because no fundamental solution satisfies the NBC. As in the Fisherian model, \( r_\sigma(\Psi_{R^c}^{\sigma,F'}) \geq 1 \) characterizes the non-FC region as \( \Gamma_k(s_t) \) fails to converge. The 45-degree line with \( \phi_\pi(1) = \phi_\pi(2) \) is the special case of the MSRE model, namely, a linear RE model. This line also has the FC-determinacy, FC-indeterminacy and non-FC regions, which are the classifications of linear RE models proposed by \cite{ChoMcCallum2012}.

When \( \phi_\pi(1) = 0.9 \) and \( \phi_\pi(2) = 1.5 \), the model is FC-determinate. A reduction of
This figure depicts the FC-determinacy ($r_\sigma(\Psi_{F^*\otimes F^*}) \leq 1$), FC-indeterminacy ($r_\sigma(\Psi_{F^*\otimes F^*}) > 1$ and $r_\sigma(\Psi_{R^*\otimes F^*}) < 1$) and non-FC ($r_\sigma(\Psi_{F^*\otimes F^*}) > 1$ and $r_\sigma(\Psi_{R^*\otimes F^*}) \geq 1$) regions in terms of $\phi_\pi(1)$ and $\phi_\pi(2)$ for a Markov-switching monetary policy model without predetermined variables. The sub-region A (B) in the FC-indeterminacy area represents the case $r_\sigma(\Psi_{F^*}) \leq 1$ ($> 1$). The thick blue 45-degree line passing through the point (1,1) represents a fixed regime linear RE model.

$\phi_\pi(1)$ to 0.8 leads to FC-indeterminacy. The forward solution in this case is qualitatively similar to the determinate solution. When it is further lowered to 0.5, the model enters into the non-FC region, but again, a fundamental solution exists. This fundamental solution and the FC-determinate forward solution are reported below for contrast.

<table>
<thead>
<tr>
<th>Forward Solution (FC-Determinacy)</th>
<th>Fundamental Solution (Non-FC)</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Gamma^*(1) = \begin{bmatrix} 0.89 &amp; 0.12 &amp; -4.85 \ -0.80 &amp; 0.89 &amp; -4.80 \ 0.80 &amp; 0.11 &amp; -3.37 \end{bmatrix}$</td>
<td>$\Gamma(1) = \begin{bmatrix} 0.94 &amp; 0.11 &amp; 26.14 \ -0.47 &amp; 0.94 &amp; 35.26 \ 0.47 &amp; 0.06 &amp; 14.07 \end{bmatrix}$</td>
</tr>
<tr>
<td>$\Gamma^*(2) = \begin{bmatrix} 0.84 &amp; 0.11 &amp; -2.41 \ -1.25 &amp; 0.84 &amp; -0.21 \ 1.25 &amp; 0.17 &amp; -2.61 \end{bmatrix}$</td>
<td>$\Gamma(2) = \begin{bmatrix} 0.84 &amp; 0.11 &amp; 4.18 \ -1.25 &amp; 0.84 &amp; -5.94 \ 1.25 &amp; 0.17 &amp; 7.27 \end{bmatrix}$</td>
</tr>
</tbody>
</table>
The third column of $\Gamma^*(s_t)$ for each regime represents the initial responses of inflation, the output gap and the interest rate to a contractionary monetary policy shock of size 1. Regardless of the initial regime, a contractionary monetary policy lowers inflation and the output gap. However, the fundamental solution in the non-FC case implies the opposite dynamics except for the output gap in regime 2. Again, we reject this solution as an equilibrium by the criterion of the NBC.

For region A in FC-indeterminacy and FC-determinacy, $r_\sigma(\Psi_{F^*}) \leq 1$, whereas $r_\sigma(\Psi_{F^*}) > 1$ for region B. Therefore, the value of $r_\sigma(\Psi_{F^*})$ is not related to our determinacy in mean-square stability. Nor is it related to the determinacy region in mean stability, which is strictly smaller than the FC-determinacy region (see Section 6.2). The only exception is the linear RE model in which these two stability concepts are identical, as shown in Figure 1.

Interestingly, $r_\sigma(\Psi_{F^*}) \leq 1$ appears to be akin to the area of a unique bounded equilibrium, expressed as the long-run Taylor principle (LRTP) of Davig and Leeper (2007). In their Proposition 2, the LRTP is equivalent to $r_\sigma(M) < 1$ with some positivity assumptions, and the matrix $M$ in their equation (10) coincides with $\Psi_{F^*}$. However, the LRTP does not apply to our model because it is the determinacy condition under boundedness that applies to the linear version of our model, as Davig and Leeper (2009) show.

### 7.2.2 A Model with Predetermined Variables

While any predetermined variable in any equation can be easily accommodated, we focus on the role of interest rate smoothing by setting $\rho = 0.95$. Instead, we shut down the persistence of the monetary policy shock with $\rho_{MP} = 0$. All other parameters remain unchanged. Figure 2 depicts the FC-determinacy and FC-indeterminacy regions with a special case of a linear model. Since there is no persistent shock process, the FCC holds as long as $\phi_\pi(s_t) > 0$ in both regimes. The first thing to note is that determinacy for the regime-switching model is neither necessary nor sufficient for active policy in both regimes. On the one hand, there exists a FC-determinacy region where one regime is passive, just as in the previous example. For instance, when $\phi_\pi(1) = 0.9$ and $\phi_\pi(2) = 1.5$, we have $r_\sigma(\bar{\Psi}_{\Omega^* \otimes \Omega^*}) = 0.458 < 1$ and $r_\sigma(\Psi_{F^* \otimes F^*}) = 0.99 < 1$. In this case, the

\footnote{A non-FC region may well emerge if a shock is allowed to be persistent. For example, the economy belongs to this region if $\phi_\pi(1) = 0.25$, $\phi_\pi(2) = 1.25$ and $\rho_D = 0.98$. In this case, $\Gamma(k(s_t))$ fails to converge as $r_\sigma(\Psi_{D^* \otimes F^*}) = 1.002$.}
Figure 2: Markov-Switching Monetary Policy Model With Predetermined Variables.

This figure depicts the regions of FC-determinacy \( r_{\sigma}(\bar{\Psi}\Omega^{*}\otimes\bar{\Omega}^{*}) < 1 \) and \( r_{\sigma}(\Psi^{*}\otimes F^{*}) \leq 1 \) and FC-indeterminacy \( r_{\sigma}(\bar{\Psi}\Omega^{*}\otimes\bar{\Omega}^{*}) < 1 \) and \( r_{\sigma}(\Psi^{*}\otimes F^{*}) > 1 \) in terms \( \phi_{\pi}(1) \) and \( \phi_{\pi}(2) \) for a Markov-switching monetary policy model with a predetermined variable. The sub-region A (B) in the FC-indeterminacy area represents the case \( r_{\sigma}(\Psi^{*}) \leq 1 \) (\( >1 \)). The thick blue 45-degree line passing through the point (1,1) represents a fixed regime linear RE model.

The determinate forward solution is given by:

\[
\begin{bmatrix}
\pi_t \\
y_t \\
i_t
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & -5.42 \\
0 & 0 & -13.56 \\
0 & 0 & 0.71
\end{bmatrix}
\begin{bmatrix}
\pi_{t-1} \\
y_{t-1} \\
i_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
0.74 & 0.10 & -5.70 \\
-0.64 & 0.92 & -14.27 \\
0.03 & 0.00 & 0.74
\end{bmatrix}
\begin{bmatrix}
z_{S,t} \\
z_{D,t} \\
z_{MP,t}
\end{bmatrix}, \text{ if } s_t = 1,
\]

\[
\begin{bmatrix}
\pi_t \\
y_t \\
i_t
\end{bmatrix}
= \begin{bmatrix}
0 & 0 & -3.89 \\
0 & 0 & -9.91 \\
0 & 0 & 0.66
\end{bmatrix}
\begin{bmatrix}
\pi_{t-1} \\
y_{t-1} \\
i_{t-1}
\end{bmatrix}
+ \begin{bmatrix}
0.69 & 0.09 & -4.10 \\
-0.78 & 0.90 & -10.43 \\
0.05 & 0.01 & 0.69
\end{bmatrix}
\begin{bmatrix}
z_{S,t} \\
z_{D,t} \\
z_{MP,t}
\end{bmatrix}, \text{ if } s_t = 2.
\]

On the other hand, there is a region where both regimes are active but the model is FC-indeterminate: for a given weakly active policy stance at regime 1, the economy can enter this region if the policy becomes more and more active at regime 2. For example, with \( \phi_{\pi}(1) = 1.05 \) and \( \phi_{\pi}(2) = 3.5 \), we have \( r_{\sigma}(\bar{\Psi}\Omega^{*}\otimes\bar{\Omega}^{*}) = 0.464 \) but \( r_{\sigma}(\Psi^{*}\otimes F^{*}) = 1.003 > 30 \).
1. A large swing of the policy stance between two active but distant regimes injects additional volatility into the model through a fluctuation in $\Omega^*(s_{t+1})$, which depends on the parameter $\rho$. An immediate implication is that in order to ensure a determinate equilibrium under regime-switching, one regime should not be too active relative to the other, even if the policy is active in both regimes.

Technically, this can happen because $F^*(s_t)$ is larger than 1 in one regime and $\Psi_{F^*\otimes F^*}$ is a non-linear weighting matrix. A final remark is that the active monetary policy can never be indeterminate for linear models, as the 45 degree line in Figure 2 indicates.

### 7.3 Markov-Switching Elasticity of Intertemporal Substitution

Now we analyze another type of model with Markov-switching elasticity of intertemporal substitution (EIS) introduced in Section 2, which can be written in matrix form as:

$$B_1(s_t)x_t = E_t[A_1(s_t, s_{t+1})x_{t+1}] + B_2x_{t-1} + C_1(s_t)z_t,$$

$$z_t = Rz_{t-1} + \epsilon_t, \quad E_{t-1}(\epsilon_t) = 0_{n\times1},$$

where the coefficient matrices are defined as:

$$B_1(s_t) = \begin{bmatrix} 1 & -\kappa & 0 \\ 0 & 1 & \frac{1}{\sigma(s_t)} \\ -(1-\rho)\phi & 0 & 1 \end{bmatrix}, \quad A_1(s_t, s_{t+1}) = \begin{bmatrix} \beta & 0 & 0 \\ \frac{1}{\sigma(s_t)} & \frac{\sigma(s_{t+1})}{\sigma(s_t)} & 0 \\ 0 & 0 & 0 \end{bmatrix},$$

$$B_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \rho \end{bmatrix}, \quad C_1(s_t) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sigma(s_t)} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad R = \begin{bmatrix} \rho_S & 0 & 0 \\ 0 & \rho_D & 0 \\ 0 & 0 & \rho_{MP} \end{bmatrix}.$$

The model can be written in the form of (3) with $A(s_t, s_{t+1}) = B_1^{-1}(s_t)A_1(s_t, s_{t+1})$, $B(s_t) = B_1^{-1}(s_t)B_2$ and $C(s_t) = B_1^{-1}(s_t)C_1(s_t)$. The following numerical example is designed to show the quantitative and qualitative differences of the impulse response

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23Suppose that the policy is active in both regimes. For linear models, $r_\sigma(F^*) \leq 1$ under determinacy. Under Markov-switching, however, $F^*(1)$ can have a root greater than 1, while $r_\sigma(F^*(2)) < 1$. Specifically, note that $r_\sigma(E[\Omega^*(s_{t+1})|s_t = 1]) > r_\sigma(E[\Omega^*(s_{t+1})|s_t = 2])$ and the gap between these two gets larger the more active regime 2 is relative to regime 1. Hence, it is possible to have $r_\sigma(F^*(1)) > 1$ if the gap becomes larger from $F^*(s_i) = (I_n - A(s_i)E[\Omega^*(s_{t+1})])^{-1}A(s_i)$, equation (22c) in the limit. $\Psi_{F^*}$ is a kind of weighted average of $r_\sigma(F^*(1))$ and $r_\sigma(F^*(2))$, and it is still true that $r_\sigma(\Psi_{F^*}) \leq 1$. However, $r_\sigma(\Psi_{F^*})$ amounts to the average of squares of $r_\sigma(F^*(i))$ for $i = 1, 2$, ultimately leading to $r_\sigma(\Psi_{F^*}) > 1$. 

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31
analysis between MSRE and linear models. Consider the parameter values $\sigma(1) = 1$, $\sigma(2) = 5$. In the asset pricing literature, an even higher value of $\sigma$ is often required to account for the so-called equity premium puzzle. We specify $p_{11} = 0.95$ and $p_{22} = 0.875$. This means that the average durations of the regimes of high and low EIS are 5 years and 2 year, respectively. The remaining parameter values are specified as $\beta = 0.99$, $\kappa = 0.132$, $\phi_\pi = 1.5$, $\phi_y = 0$, $\rho_D = \rho_S = 0.95$, $\rho_{MP} = 0$ and $\rho = 0.95$. Since the policy stance against expected inflation is active and the same in both regimes, the model would be FC-determinate. Indeed, we confirm that $r_{\sigma}(\tilde{\Psi}_{\Omega^c\otimes\Omega^c}) = 0.561$ and $r_{\sigma}(\Psi_{F^c\otimes F^c}) = 0.952$.

Figure 3 plots the impulse response functions to each shock of size 1 starting at different regimes. The regime-switching EIS leads to quantitatively sizable differences in the responses of the variables across the initial regimes. With a small EIS (high $\sigma$) at the initial regime, the responses of all variables to a demand shock are much smaller than those under a high EIS. This is because the immediate response of the output gap

Farmer et al. (2011) provide four examples similar to our New-Keynesian models. We find that the first two examples of their model based on a regime-switching monetary policy are indeterminate and the last two based on regime-switching in the private sector or shock variance are determinate. Indeed, they find two MSS fundamental solutions for the first two examples, and thus they are indeterminate as well, but they did not clearly show whether the last two examples are determinate.
is small. A high $\sigma$ also makes the stabilizing effect of monetary policy quite small. In contrast, inflation and the interest rate react more and output less to a supply shock. Intuitively, a smaller response of the output gap makes inflation more responsive to a supply shock.

The responses under different initial regimes converge to each other, and the rate of convergence becomes faster as the probability of regime shift becomes higher. This regime-switching feature can also generate more pronounced hump-shaped responses. For instance, when $\sigma$ is high at the initial regime, the initial response of the output gap to a supply shock is small. But the probability that $\sigma$ can be smaller in the future raises the response of the output gap to a higher level in size for a prolonged period.

8 Conclusion

It is surprising that, until very recently, Markov-switching has not been actively applied to dynamic stochastic general equilibrium models under rational expectations, a workhorse of modern macroeconomics. This paper brings together mean-square stability and the no-bubble condition in order to solve for rational expectations equilibria in Markov-switching rational expectations models under determinacy and indeterminacy. The forward solution is the unique stable REE under determinacy and the REEs under indeterminacy are the stable solutions associated with the forward solution.

While exogenous Markov-switching is not entirely satisfactory, the virtue of MSRE models should be evaluated relative to the existing class of linear RE models, models of single or multiple structural breaks. Ultimately, it would be desirable to have models featuring endogenous regime-switching, but our technical foundation for MSRE models may also help identify the equilibrium paths of those models around the steady states.

MSRE models can provide a flexible and promising way to theoretically model the optimal behavior of economic agents and the central bank and to examine those models empirically. Many researchers cited in this paper have remarked upon and predicted these potentially salient features of MSRE models. We believe that our work provides a technical foundation and an economic judgment about REEs needed for future research in the field of MSRE models.
Appendix

A Proof of Proposition 2

The proof closely follows that of Theorem 1 in Farmer et al. (2009). Let \( V_t \) be an \( n \times k_t \) matrix where the columns are orthonormal, spanning the support of \( w_t 1_{\{s_t=i\}} \) for all \( t \) and \( i \in \{1,2,...,S\} \). \( 1_{\{s_t=i\}} = 1 \) when \( s_t = i \) and 0 otherwise. Thus, for any \( w_t = w \) and \( s_t = i \), it must be true that \( w \in \text{Col}(V_i) \) and \( E[w_{t+1}|w_t = w, s_t = i, s_{t+1} = j] \in \text{Col}(V_j) \) almost surely. Since \( w \) solves Equation (5), it must be true that \( E_t[F(s_t=i, s_{t+1}=j)w_t = w, s_t = i] = \delta_{ij} \). Hence, there must exist a \( k_t \times k_t \) matrix \( \Phi_{ij} \) such that \( \sum_{j=1}^{S} p_{ij} F_{ij} \Phi_{ij} = V_i \). Since \( w_{t+1} = \text{Col}(V(s_{t+1})) \), \( V(s_{t+1})^\prime \eta_{t+1} = \text{Col}(V(s_{t+1})) \) almost surely from Equation (8). Finally, \( E_t[F(s_t=i, s_{t+1})w_{t+1} - F(s_t=i, s_{t+1}) \Lambda(s_t, s_{t+1})w_t] = w_t - E_t[F(s_t=i, s_{t+1})w_{t+1}] = w_t - V_i^\prime w_t = 0_{n \times 1} \) almost surely as \( w_t \in \text{Col}(V_i) \). Q.E.D.

B Extended Regime Variables

We rewrite equation (10), \( y_t = G(s_{t-1}, s_t)y_{t-1} + H(s_t)\eta_t \) in terms of an extended state vector \( \hat{s}_t = (s_{t-1}, s_t) \). The rule of indexation for \( \hat{s}_t \) is given by \( \hat{s}_t = l = S(i-1) + j \in \{1,2,...,S^2\} \) for \( s_{t-1} = i, s_t = j \) and \( i,j \in \{1,...,S\} \). The corresponding transition matrix is defined as \( \hat{P} = (i_S \otimes I_S \otimes i_S) \text{diag} (\text{vec}(P')) \), where \( i_S \) is an \( S \times 1 \) column vector of ones. For instance, when \( S = 2 \), \( \hat{P} \) is given by:

\[
\hat{P} = [\hat{p}_{ij}] = \begin{bmatrix}
p_{11} & p_{12} & 0 & 0 \\
0 & 0 & p_{21} & p_{22} \\
p_{11} & p_{12} & 0 & 0 \\
0 & 0 & p_{21} & p_{22}
\end{bmatrix} \quad (32)
\]

Now define \( \hat{y}_{t+1} = y_t, \hat{G}(\hat{s}_t) = G(s_{t-1}, s_t), \hat{H}(\hat{s}_t) = H(., s_t) = H(s_t) \). Then, (10) can be written in a canonical form of (3.27) in Costa et al. (2005) as:

\[
\hat{y}_{t+1} = \hat{G}(\hat{s}_t)\hat{y}_t + \hat{H}(\hat{s}_t)\eta_t. \quad (33)
\]
It is straightforward to show that the non-zero eigenvalues of \( \Psi_G (\Psi_{G\otimes G}) \) and \( \Psi_G (\Psi_{G\otimes G}) \) are equivalent. The homogeneous part of Equation (33) is identical to equation (3.1) in Costa et al. (2005). Hence, we can apply their results directly to the process (10).

C Proof of Lemma 2

Proof ofAssertion 1. Consider the homogeneous part of two processes \( w_t \) and \( u_t \) of the form (10) at time \( t \) given by \( w_t = \Lambda(s_{t-1}, s_t)w_{t-1} \) and \( u_t = F'(s_{t-1}, s_t)u_{t-1} \). Let a variable or matrix with a hat denote that associated with the extended regime variable \( \hat{s}_t \) and the corresponding \( \hat{P} \) defined in Appendix B. So we can rewrite \( w_t \) and \( u_t \) as \( \hat{w}_t = \hat{\Lambda}(\hat{s}_t)\hat{w}_{t-1} \) and \( \hat{u}_t = \hat{F}'(\hat{s}_t)\hat{u}_{t-1} \). Define \( \Psi_{\hat{F}'\otimes \hat{F}'} = [\hat{p}_{ji}\hat{F}'_j \otimes \hat{F}'_j] \) for \( \hat{s}_{t-1} = i, \hat{s}_t = j \) and \( i, j \in \{1, ..., S^2\} \). Then, it can be easily verified that \( r_\sigma(\Psi_{\hat{F}'\otimes \hat{F}'}) = r_\sigma(\Psi_{\hat{F}'\otimes \hat{F}'}) \), and thus, \( w_t \) is MSS if and only \( \hat{w}_t \) is MSS. Similarly, we have \( r_\sigma(\Psi_{\hat{A}\otimes \hat{A}}) = r_\sigma(\Psi_{\hat{A}\otimes \hat{A}}) \). Therefore, we need to prove that if \( r_\sigma(\Psi_{\hat{F}'\otimes \hat{F}'}) < 1 \) and \( r_\sigma(\Psi_{\hat{A}\otimes \hat{A}}) < 1 \), then \( r_\sigma(\Psi_{\hat{F}'\otimes \hat{F}'}) < 1 \).

Define the sum of \( \hat{u}_t \) and \( \hat{w}_t \) as:

\[
\hat{z}_t = \hat{u}_t + \hat{w}_t = \hat{F}'(\hat{s}_t)\hat{u}_{t-1} + \hat{\Lambda}(\hat{s}_t)\hat{w}_{t-1}.
\]

From theorem 3.9 in Costa et al. (2005), \( r_\sigma(\Psi_{\hat{F}'\otimes \hat{F}'}) < 1 \) if and only if \( \sum_{t=0}^{\infty} E(||\hat{u}_t||^2) < \infty \) for all \( \hat{u}_0 \) and \( \hat{s}_0 \) where \( || \cdot || \) is a standard vector norm such that \( ||\hat{u}_t||^2 = \text{trace}(\hat{u}_t\hat{u}_t') \). Similarly, \( r_\sigma(\Psi_{\hat{A}\otimes \hat{A}}) < 1 \) if and only if \( \sum_{t=0}^{\infty} E(||\hat{w}_t||^2) < \infty \). Then, \( 0 \leq ||\hat{z}_t||^2 = ||\hat{u}_t + \hat{w}_t||^2 \leq 2(||\hat{u}_t||^2 + ||\hat{w}_t||^2) \). Hence,

\[
0 \leq \sum_{t=0}^{\infty} E(||\hat{z}_t||^2) \leq 2 \left( \sum_{t=0}^{\infty} E(||\hat{u}_t||^2) + \sum_{t=0}^{\infty} E(||\hat{w}_t||^2) \right) < \infty.
\]

This implies that \( \hat{z}_t \) is also mean-square stable. To proceed, define \( Q_{\hat{u},\hat{u}} = E[\hat{u}_t\hat{u}_t'|_{\hat{s}(t)=j}] \), \( Q_{\hat{w},\hat{w}} \) and \( Q_{\hat{z},\hat{z}} \) are defined analogously. Define \( Q_{\hat{u},\hat{w}} = E[\hat{u}_t\hat{w}_t'|_{\hat{s}(t)=j}] \) and \( Q_{\hat{z},\hat{w}} = E[\hat{u}_t\hat{w}_t'|_{\hat{s}(t)=j}] \). We additionally vectorize these square matrices: \( v_{\hat{u},\hat{u}} = vec(Q_{\hat{u},\hat{u}}) \), \( v_{\hat{w},\hat{w}} = vec(Q_{\hat{w},\hat{w}}) \) and \( v_{\hat{z},\hat{w}} = vec(Q_{\hat{z},\hat{w}}) \). Then,

\[
Q_{\hat{z},\hat{z}} = Q_{\hat{u},\hat{u}} + Q_{\hat{w},\hat{w}} + Q_{\hat{u},\hat{w}} + Q_{\hat{w},\hat{w}} = \sum_{i=1}^{S^2} \hat{p}_{ij} \left\{ \hat{F}'_j Q_{\hat{u},\hat{u}}_{i,j} + \hat{\Lambda}_j Q_{\hat{w},\hat{w}}_{i,j} + \hat{F}'_j Q_{\hat{u},\hat{w}}_{i,j} + \hat{\Lambda}_j Q_{\hat{w},\hat{w}}_{i,j} \right\}.
\]
\[ v_{j,t}^\hat{z} = v_{j,t}^\hat{u} + v_{j,t}^\hat{w} + v_{j,t}^{\hat{u}\hat{w}} + v_{j,t}^{\hat{w}\hat{u}} \]
\[ = \sum_{i=1}^{S^2} \hat{p}_{ij} \left\{ \hat{F}_j' \otimes \hat{F}_j' \hat{v}_{j,t-1}^\hat{u} + \hat{\Lambda}_j \otimes \hat{\Lambda}_j \hat{v}_{j,t-1}^\hat{w} + \hat{\Lambda}_j \otimes \hat{\Lambda}_j \hat{v}_{j,t-1}^{\hat{u}\hat{w}} + \hat{\Lambda}_j \otimes \hat{\Lambda}_j \hat{v}_{j,t-1}^{\hat{w}\hat{u}} \right\}. \]

By stacking the vectors for all \( \hat{s}_t = 1, ..., S^2 \), we have the following:

\[ v_t^\hat{z} = v_t^\hat{u} + v_t^\hat{w} + v_t^{\hat{u}\hat{w}} + v_t^{\hat{w}\hat{u}} = \Psi_{F' \otimes F'} v_{t-1} + \Psi_{\Lambda \otimes \Lambda} v_{t-1} + \Psi_{F' \otimes \Lambda} v_{t-1} + \Psi_{\Lambda \otimes F'} v_{t-1}. \]

Since \( \hat{z}_t \) is MSS and homogeneous, \( v_t^\hat{z} \) must converge to the vector of zeros as \( t \) goes to infinity. Since \( r_\sigma(\Psi_{\Lambda \otimes \Lambda}) < 1 \) and \( r_\sigma(\Psi_{F' \otimes F'}) < 1 \), \( v_t^\hat{u} \) and \( v_t^\hat{w} \) converge to zero as well. This implies that \( v_t^{\hat{u}\hat{w}} + v_t^{\hat{w}\hat{u}} \) must converge to zero. Note that \( v_t^{\hat{u}\hat{w}} \) is the vectorized version of a positive semi-definite matrix, \( \sum_{j=1}^{S^2} Q_{j,t}^{\hat{u}\hat{w}} \). The same is true for \( v_t^{\hat{w}\hat{u}} \). Therefore, for \( v_t^{\hat{u}\hat{w}} + v_t^{\hat{w}\hat{u}} \) to converge to zero, both vectors must converge to zero individually, implying \( r_\sigma(\Psi_{F' \otimes \Lambda}) < 1 \) and \( r_\sigma(\Psi_{\Lambda \otimes F'}) < 1 \). In fact, the eigenvalues of \( \Psi_{F' \otimes \Lambda} \) are the same as those of \( \Psi_{\Lambda \otimes F'} \). Finally, it is easy to see that \( r_\sigma(\Psi_{\Lambda \otimes F'}) = r_\sigma(\Psi_{F' \otimes F'}) \). Q.E.D.

**Proof of Assertion 2.** Assertion 1 implies that if \( r_\sigma(\Psi_{F' \otimes F'}) < 1 \) and \( r_\sigma(\Psi_{\Lambda \otimes F'}) \geq 1 \), then \( r_\sigma(\Psi_{\Lambda \otimes \Lambda}) \geq 1 \). Now, we need to prove that the first condition can be replaced with equality, \( r_\sigma(\Psi_{\Lambda \otimes \Lambda}) = 1 \). Suppose that \( r_\sigma(\Psi_{F' \otimes F'}) = 1 \) and \( r_\sigma(\Psi_{\Lambda \otimes F'}) \geq 1 \), but \( \tau_2 = r_\sigma(\Psi_{\Lambda \otimes \Lambda}) < 1 \). Now consider two new processes, \( \hat{u}_{t+1} = \sqrt{\alpha} \hat{F}'(\hat{s}_{t+1}) \hat{u}_t \) and \( \hat{w}_{t+1} = (\hat{\Lambda}(\hat{s}_{t+1}) / \sqrt{\alpha}) \hat{w}_t \) for any \( \alpha \in (\tau_2, 1) \). Then, \( r_\sigma(\Psi_{\sqrt{\alpha}F' \otimes \sqrt{\alpha}F'}) = r_\sigma(\Psi_{\Lambda \otimes \Lambda}) / \alpha = \tau_2 / \alpha < 1 \). But \( r_\sigma(\Psi_{\sqrt{\alpha}F' \otimes \sqrt{\alpha}F'}) \geq 1 \). This is a contradiction to Assertion 1. Q.E.D.

**D Proof of Lemma 3**

Multiply \( V_i' \) to both sides of (9) to replace \( V_j \Phi_{ij} V_i' \) with \( \Lambda_{ij} \). Since \( \Lambda_{ij} = V_j V_i' \Lambda_{ij} \), (9) can be written as:

\[ \sum_{j=1}^{S} p_{ij} F_{ij} \Lambda_{ij} = \sum_{j=1}^{S} p_{ij} F_{ij} V_j V_i' \Lambda_{ij} = V_i V_i'. \]  

(34)

We first vectorize equation (34) for each \( i \) and stack them over \( i = 1 \) through \( S \) to yield:

\[
\begin{bmatrix}
   p_{11} \Lambda_{11} \otimes F_{11} & \cdots & p_{1S} \Lambda_{1S} \otimes F_{1S} \\
   \cdots & \cdots & \cdots \\
   p_{S1} \Lambda_{S1} \otimes F_{S1} & \cdots & p_{SS} \Lambda_{SS} \otimes F_{SS}
\end{bmatrix}
\begin{bmatrix}
   \text{vec}(V_i V_i') \\
   \cdots \\
   \text{vec}(V_S V_S')
\end{bmatrix}
= \begin{bmatrix}
   \text{vec}(V_1 V_1') \\
   \cdots \\
   \text{vec}(V_S V_S')
\end{bmatrix}.
\]

36
Note that the matrix of the left-hand side is $\Psi_{\Lambda \otimes F}$ and the vector in this equation, if normalized, becomes an eigenvector. This implies that $\Psi_{\Lambda \otimes F}$ has an eigenvalue $1$ for any $\Lambda$ subject to (34). Note that $\bar{\Psi}_{\Lambda \otimes F'} = \bar{\Psi}_{\Lambda \otimes F}$. Therefore, $r_\sigma(\bar{\Psi}_{\Lambda \otimes F'}) \geq 1$. Q.E.D.

E Proof of Lemma 4

Equation (34) is invariant to multiplying an arbitrary positive scalar $\sqrt{\alpha}$ to $F_{ij}$ and its reciprocal to $\Lambda_{ij}$:

$$\sum_{j=1}^{S} p_{ij}(\sqrt{\alpha}F_{ij})\left(\frac{1}{\sqrt{\alpha}}\Lambda_{ij}\right) = V_i V'_i.$$  (35)

Let $\tilde{F}_{ij} = \sqrt{\alpha}F_{ij}$ and $\tilde{\Lambda}_{ij} = \Lambda_{ij}/\sqrt{\alpha}$ for all $i, j \in \{1, ..., S\}$. From Lemma 3, $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}}) \geq 1$ for $\tilde{F}$ and $\tilde{\Lambda}$. Moreover, $r_\sigma(\bar{\Psi}_{\tilde{F} \otimes \tilde{F}}) = \alpha r_\sigma(\bar{\Psi}_{\tilde{F} \otimes \tilde{F}})$, $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{\Lambda}}) = (\frac{1}{\alpha}) r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{\Lambda}})$, and $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}}) = r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}})$. $\alpha$ can be arbitrarily chosen such that $r_\sigma(\bar{\Psi}_{\tilde{F} \otimes \tilde{F}}) < 1$, and for such $\tilde{F}$, $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}}) \geq 1$ from Assertion 2 of Lemma 2. Now, for any $r_\sigma(\bar{\Psi}_{\tilde{F} \otimes \tilde{F}}) = \xi_2$, suppose that there exists $\Lambda(s_t, s_{t+1})$ subject to (34) such that $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}}) = \tau_2$ and $\xi_2 \tau_2 < 1$. Then by setting $\alpha = \sqrt{\tau_2/\xi_2}$, we have $\tilde{F}$ and $\tilde{\Lambda}$ such that $r_\sigma(\bar{\Psi}_{\tilde{F} \otimes \tilde{F}}) = r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}}) = \sqrt{\xi_2 \tau_2} < 1$. This implies that $r_\sigma(\bar{\Psi}_{\tilde{\Lambda} \otimes \tilde{F}}) < 1$ from Assertion 1 of Lemma 2, which contradicts Lemma 3. Therefore, $\xi_2 \tau_2 \geq 1$ for all $\Lambda(s_t, s_{t+1})$ subject to (34). Q.E.D.

F Searching for Stable Non-fundamental Components

The task here is to identify the existence of $\Lambda(s_t, s_{t+1})$ such that $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda}) < 1$ in the case of $r_\sigma(\bar{\Psi}_{\tilde{F} \otimes \tilde{F}}) > 1$. To do so, we transform the problem as finding $\tau_2$, the minimum value of $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$ with respect to a set of arbitrary $n \times k_i$ matrices $V_i$ of which columns are orthonormal and a set of $k_j$ by $k_i$ arbitrary matrices $\Phi_{ij}$ for $0 \leq k_i \leq n$, subject to (9) and $\Lambda_{ij} = V_j^\prime \Phi_{ij} V_i^\prime$ for all $i, j \in \{1, ..., S\}$. For each $i$, it is sufficient to set $k_i$ to be the maximum of the rank of $F_{ij}$ for all $j$. The procedure starts with randomized initial values; thus, each minimization problem will produce a different $\Lambda(s_t, s_{t+1})$ in general but yield the same $\tau_2$. This is because there are many more free parameters than are required to construct $\Lambda(s_t, s_{t+1})$ as we discussed earlier following Proposition 2 and thus the function $r_\sigma(\bar{\Psi}_{\Lambda \otimes \Lambda})$ has a "flat bottom" over the domain of those parameters. Put differently, the problem is similar to minimizing $(x + y)^2$ with respect to real numbers $x$ and $y$. The fact that $1/[r_\sigma(\bar{\Psi}_{\tilde{F} \otimes \tilde{F}})]$ is the lower bound of $\tau_2$ from Lemma 4 makes the minimization
problem particularly simple: once a \( \Lambda(s_t, s_{t+1}) \) leading to \( \tau_2 = 1/[r_\sigma(\Psi_{F \otimes F})] \) is found, the search process stops, and this would be true for almost all MSRE models without absorbing states. To summarize, if \( 1/[r_\sigma(\Psi_{F \otimes F})] \leq \tau_2 < 1 \), then there is a continuum of mean-square bubbles and if \( 1 \leq \tau_2 \), there is no stable bubble.

**G Proof of Proposition 5**

Let \( M_t(s_t, s_{t+1}) = A(s_t, s_{t+1}), \Omega_1(s_t) = B(s_t), \Gamma_1(s_t) = C(s_t) \) for the MSRE model \( x_t = E_t[A(s_t, s_{t+1})x_{t+1}] + B(s_t)x_{t-1} + C(s_t)z_t \). Suppose that there exists a set of sequences of matrices \( \{M_{k-1}(s_t, s_{t+1}, ..., s_{t+k-1}), \Omega_{k-1}(s_t), \Gamma_{k-1}(s_t)\} \) for \( k > 1 \) such that:

\[
x_t = E_t[M_{k-1}(s_t, s_{t+1}, ..., s_{t+k-1})x_{t+k-1}] + \Omega_{k-1}(s_t)x_{t-1} + \Gamma_{k-1}(s_t)z_t.
\]

Shift this equation forward one period and pre-multiply \( A(s_t, s_{t+1}) \) to both sides. By taking conditional expectations of \( A(s_t, s_{t+1})x_{t+1} \) and using the law of iterative expectations, we have:

\[
E_t[A(s_t, s_{t+1})x_{t+1}] = E_t[A(s_t, s_{t+1})M_{k-1}(s_{t+1}, s_{t+2}, ..., s_{t+k})x_{t+k}]
+ E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})]x_t + E_t[A(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})R]z_t.
\]

Substitute this into the model and rearrange it to yield:

\[
x_t = E_t[\Xi_{k-1}(s_t)^{-1}A(s_t, s_{t+1})M_{k-1}(s_{t+1}, s_{t+2}, ..., s_{t+k})x_{t+k}]
+ \Xi_{k-1}(s_t)^{-1}B(s_t)x_{t-1}
+ (\Xi_{k-1}(s_t)^{-1}C(s_t) + E_t[\Xi_{k-1}(s_t)^{-1}A(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})R])z_t,
\]

where \( \Xi_{k-1}(s_t) = I_n - E_t[A(s_t, s_{t+1})\Omega_{k-1}(s_{t+1})] \), provided that \( \Xi_{k-1}(s_t) \) is invertible for all \( s_t \). Note that \( \Xi_{k-1} \) depends only on \( s_t \) as \( s_{t+1} \) is integrated out in the expectational term. Thus it can enter inside the expectations on the right-hand side. Therefore, there exists a sequence of matrices \( M_k(s_t, s_{t+1}, ..., s_{t+k}), \Omega_k(s_t) \) and \( \Gamma_k(s_t) \) such that:

\[
x_t = E_t[M_k(s_t, s_{t+1}, ..., s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1} + \Gamma_k(s_t)z_t,
\]

(36)
where \( F_1(s_t, s_{t+1}) = A(s_t, s_{t+1}) \) and for \( k \geq 2, \)

\[
M_k(s_t, s_{t+1}, \ldots, s_{t+k}) = F_k(s_t, s_{t+1})M_{k-1}(s_{t+1}, s_{t+2}, \ldots, s_{t+k}), \quad (37a)
\]

\[
\Omega_k(s_t) = \Xi_{k-1}(s_t)^{-1}B(s_t), \quad (37b)
\]

\[
\Gamma_k(s_t) = \Xi_{k-1}(s_t)^{-1}C(s_t) + E_t[F_k(s_t, s_{t+1})\Gamma_{k-1}(s_{t+1})R], \quad (37c)
\]

\[
F_k(s_t, s_{t+1}) = \Xi_{k-1}(s_t)^{-1}A(s_t, s_{t+1}). \quad (37d)
\]

If \( \Xi_{k-1}(s_t) \) is invertible for all \( k = 2, 3, \ldots \) and for all \( s_t = 1, 2, \ldots, S \), the sequences, \( \{M_k(s_t, s_{t+1}, \ldots, s_{t+k}), \Omega_k(s_t), \Gamma_k(s_t), F_k(s_t, s_{t+1})\} \) are well-defined. Since the initial values of these sequences are real-valued and given by the model for all \( s_t \), they are unique and real-valued if they exist. \textit{Q.E.D.}

\section*{H Proof of Proposition 6}

When the FCC does not hold, the forward solution does not exist by definition. Therefore, at least one of \( (\Omega_k(s_t), \Gamma_k(s_t)) \) in (21) is either not well-defined if the regularity condition is violated or not convergent. Now suppose that there exists a fundamental solution. If this satisfies the NBC, it is a contradiction to the supposition that \( \Omega_k(s_t) \) or \( \Gamma_k(s_t) \) or both are not convergent. Suppose now that the FCC holds. From Proposition 5, since \( (\Omega_k(s_t), \Gamma_k(s_t)) \) is unique and real-valued given the initial state \( s_t \), the limiting values \( (\Omega^*(s_t), \Gamma^*(s_t)) \) are also unique and real-valued. Since \( (\Omega^*(s_t), \Gamma^*(s_t)) \) solves equation (6a) and (6b), the forward solution (25) is a fundamental solution to the model by Proposition 1, and therefore, it must solve equation (24), the forward representation of the model as \( k \) goes to infinity:

\[
x_t = \lim_{k \to \infty} E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] + \Omega^*(s_t)x_{t-1} + \Gamma^*(s_t)z_t. \quad (38)
\]

Therefore, it must be true that \( \lim_{k \to \infty} E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] = 0_{n \times 1} \) when expectations are formed with the forward solution, implying that the forward solution satisfies the NBC. Now suppose that the NBC holds for a fundamental solution, different from the forward solution. Since the solution must solve (38), (38) becomes the forward solution under the NBC, which is a contradiction to the supposition that this solution differs from the forward solution. \textit{Q.E.D.}
I Proof of Proposition 7

Suppose that the forward solution is MSS, i.e., \( r_\sigma(\tilde{\Psi}_{\Omega^r \otimes \Omega^r}) < 1 \), and \( r_\sigma(\Psi_{F^r \otimes F^r}) \leq 1 \). The task is to show that there is no other MSS fundamental solution. Since the proof does not hinge on the presence of exogenous variables, we assume them away for simplicity. Any fundamental solution must satisfy the forward representation of the model:

\[
x_t = E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] + \Omega_k(s_t)x_{t-1},
\]

(39)

at all \( k = 1, 2, \ldots \). First, we show that \( E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] \) evaluated with a particular fundamental solution \( x_t = \Omega(s_t)x_{t-1} \) can be expressed as \( L_k(s_t; \Omega)x_t \). For \( k = 1 \), \( M_1(s_t, s_{t+1}) = A(s_t, s_{t+1}) = F_1(s_t, s_{t+1}) \). Therefore,

\[
E_t[M_1(s_t, s_{t+1})x_{t+1}] = L_1(s_t; \Omega)x_t,
\]

where \( L_1(s_t; \Omega) = E_t[F_1(s_t, s_{t+1})\Omega(s_{t+1})] \). Suppose that for \( k > 1 \), there exist \( L_{k-1}(s_t; \Omega) \) such that \( E_t[M_{k-1}(s_t, s_{t+1}, \ldots, s_{t+k-1})x_{t+k-1}] = L_{k-1}(s_t; \Omega)x_t \). Then, equation (37a) and the law of iterative expectations imply that:

\[
E_t[M_k(s_t, s_{t+1}, \ldots, s_{t+k})x_{t+k}] = E_t[F_k(s_t, s_{t+1})M_{k-1}(s_{t+1}, s_{t+2}, \ldots, s_{t+k})x_{t+k}]
= E_t[F_k(s_t, s_{t+1})E_{t+1}[M_{k-1}(s_{t+1}, s_{t+2}, \ldots, s_{t+k})x_{t+k}]]
= E_t[F_k(s_t, s_{t+1})L_{k-1}(s_{t+1}; \Omega)x_{t+1}]
= E_t[F_k(s_t, s_{t+1})L_{k-1}(s_{t+1}; \Omega)\Omega(s_{t+1})x_t].
\]

Therefore, \( L_k(s_t; \Omega) \) is well-defined as:

\[
L_k(s_t; \Omega) = E_t[F_k(s_t, s_{t+1})L_{k-1}(s_{t+1}; \Omega)\Omega(s_{t+1})].
\]

(40)

Note that using (40), the forward representation of the model, (39), can be written as \( (I - L_k(s_t; \Omega))x_t = \Omega_k(s_t)x_{t-1} \). Hence, it must be true that:

\[
(I - L_k(s_t; \Omega))\Omega(s_t) = \Omega_k(s_t),
\]

for any fundamental solution. Since the model satisfies the FCC, \( \lim_{k \to \infty} \Omega_k(s_t) = \Omega^*(s_t) \) and \( \lim_{k \to \infty} F_k(s_t, s_{t+1}) = F^*(s_t, s_{t+1}) \). Therefore, \( L_k(s_t; \Omega) \) must converge to a solution-dependent matrix, \( L(s_t; \Omega) \). From Proposition 6, the forward solution satisfies the NBC
and all the other solutions violate it. Hence, when Ω = Ω∗, \( L(s_t; Ω^*) = 0_{n \times n} \). Now suppose that there exists another MSS solution \( x_t = \tilde{Ω}(s_t)x_{t-1} \) different from the forward solution. Then \( L(s_t; \tilde{Ω}) \neq 0_{n \times n} \).

Next, we express equation (40) as:

\[
L_k(i; Ω) = \sum_{j=1}^{S} p_{ij} F_k(i, j) L_{k-1}(j; Ω) Ω(j).
\]

By vectorizing this equation and stacking them over \( i = 1, 2, ..., S \), we have the following:

\[
\begin{bmatrix}
\text{vec}(L_k(1; Ω)) \\
\vdots \\
\text{vec}(L_k(S; Ω))
\end{bmatrix} = \Psi_{Ω' \otimes F_k} \begin{bmatrix}
\text{vec}(L_{k-1}(1; Ω)) \\
\vdots \\
\text{vec}(L_{k-1}(S; Ω))
\end{bmatrix}.
\]

Since \( \lim_{k \to \infty} L_k(s_t; \tilde{Ω}) = L(s_t; \tilde{Ω}) \neq 0_{n \times n} \), the vector on both sides becomes an eigenvector if normalized. Henceforth, \( r_σ(\Psi_{Ω' \otimes F_k}) \) has a root of 1. And \( r_σ(\Psi_{Ω' \otimes F_k}) \geq 1 \) as in the proof of Lemma 3. From the definition of \( Ψ \) and \( \Psi \), \( Ψ_A = (Ψ_A')' \) so \( r_σ(\Psi_{Ω' \otimes F_k}) = r_σ(\bar{Ψ}_{Ω' \otimes (F^*)'}) \). Finally, if \( r_σ(\bar{Ψ}_{Ω' \otimes (F^*)'}) \geq 1 \) and \( r_σ(\bar{Ψ}_{F^* \otimes (F^*)'}) = r_σ(Ψ_{F^* \otimes F^*}) \leq 1 \), then \( r_σ(\bar{Ψ}_{Ω' \otimes Ω}) \geq 1 \) by Assertion 2 of Lemma 2. \( Q.E.D. \)

### J On Determinacy in the Mean Stability Sense

Here we show that the conditions for determinacy in the mean stability sense are stronger than those for determinacy under MSS. We first need a condition for \( τ_1^* \geq 1 \), where \( τ_1^* = \min_{Λ(s_t, s_{t+1})} r_σ(Λ_A) \) subject to (17) with \( F(\cdot) = F^*(\cdot) \). For simplicity, suppose that \( τ_2^* = 1/r_σ(Ψ_{F^* \otimes F^*}) \), which would be true for most of the models without absorbing states. Then it can be shown that \( (τ_1^*)^2 \leq τ_2^* \) and \( τ_1^* = r_σ(Ψ_{F^*})/r_σ(Ψ_{F^* \otimes F^*}) = r_σ(Ψ_{F^*})τ_2^* \) using Theorem 1 and our lemmas. Hence, \( τ_1^* \geq 1 \) holds if \( r_σ(Ψ_{F^* \otimes F^*}) \leq r_σ(Ψ_{F^*}) \leq 1 \), which is much stronger than \( r_σ(Ψ_{F^* \otimes F^*}) \leq 1 \). The condition for uniqueness of the mean stable forward solution is also much stronger than \( r_σ(Ψ_{Ω^* \otimes Ω^*}) < 1 \), which is even stronger than \( r_σ(Ψ_{Ω^*}) < 1 \). To see this, note that the condition \( r_σ(Ψ_{Ω^* \otimes Ω^*}) < 1 \), together with \( r_σ(Ψ_{F^* \otimes F^*}) \leq 1 \), implies \( r_σ(Ψ_{Ω^*}) \geq 1 \) for all \( Ω \). But this does not rule out the possibility that \( r_σ(Ψ_{Ω^*}) < 1 \).
References


