Estimating Fiscal Limits: The Case of Greece *

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Abstract

This paper uses Bayesian methods to estimate the ‘fiscal limit’ distribution implied by a rational expectations framework for Greece. We build a real business cycle model that allows for interactions among fiscal policy instruments, the stochastic ‘fiscal limit,’ and sovereign default risk. The fiscal policy specification takes into account government spending, lump-sum transfers, and distortionary taxation. A fiscal limit measures the debt level beyond which the government is no longer willing to finance, causing a partial default to occur. Using the particle filter to perform likelihood-based inference, we estimate the full nonlinear model with post-EMU data until 2010Q4. We find that the probability of default on Greek debt remained close to zero from 2001 until 2009, when it began to rise sharply to the range of 5% to 10% by 2010Q4. The model also predicts a probability of default between 60-80% by 2011Q4, consistent with the debt restructuring arrangements that took place at the beginning of 2012. In addition, the surge in the real interest rate in Greece in 2011 is within forecast confidence bands of our rational expectations model. Finally, model comparisons based on Bayes factors strongly favor the nonlinear model specification with an endogenous probability of default over linearized specifications with an ad-hoc risk premium.

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1. Introduction

During the past five years, there has been growing concern over the fiscal positions of several Eurozone nations. The long term interest rate spread in the secondary market between German government bonds and several European countries’ government bonds has risen markedly. The spread between Greek bonds and German bonds rose from 2.35 percentage points in 2009 to nearly 30 by the end of 2011. The aim of this paper is to discern the extent to which a rational expectations model can explain the rising interest rates and probabilities of default on sovereign debt in Greece.

Bi and Traum (2012) shows how to use Bayesian methods and likelihood-based inference to estimate a real business cycle (RBC) model that allows for sovereign default. While Bi and Traum (2012) provides a coherent framework for estimating forward-looking ‘fiscal limits,’ their estimated model is unable to forecast the sharp rise in long term interest rates in 2011, in part because their estimates imply low historical probabilities of default. This paper extends the framework of Bi and Traum (2012) to a more realistic set-up and estimates a full nonlinear model using post-EMU data for Greece.

We consider a closed economy in which the government finances transfers and expenditures by collecting distortionary income taxes and issuing bonds. The bond contract is not enforceable and depends on the maximum level of debt that the government is politically able to service, a so-called ‘fiscal limit.’ We assume that the fiscal limit is stochastic and its distribution follows a logistical function. At each period, an effective fiscal limit is drawn from the distribution. If the level of government debt surpasses the effective limit, then the government reneges on a fraction of its debt. Given the fiscal limit distribution, households can decide the quantity of government debt that they are willing to purchase and the price they are willing to pay.

The economy switches between the default and no-default regimes endogenously, depending upon the level of government debt and the fiscal limit distribution. Therefore, the model cannot be solved using a first-order approximation; instead, it is solved using the monotone mapping method and estimated using Bayesian inference methods and a sequential Monte Carlo approximation of the likelihood [similar estimation methods are used in Fernandez-Villaverde and Rubio-Ramirez (2007), Doh (2011), and Armisano and Tristani (2010)].

We estimate the model for Greece during the post-EMU period until 2010Q4. Using the estimated structural parameters, we compute model-implied default probabilities for Greece’s historical debt-to-GDP ratios. We find that the probability of default on Greek debt remained close to zero from 2001 until 2009, when it began to rise sharply to the range of 5% to 10% by the fourth quarter of 2010. By the end of 2011, the model predicts a
probability of default between 60-80%, consistent with the debt restructuring arrangements that took place at the beginning of 2012. In addition, the surge in the real interest rate observed in Greece from 2011Q1 to Q4 is within the forecast confidence bands of our rational expectations model. This suggests that the recent interest rate surge can be explained by macroeconomics fundamentals in a rational expectation framework.

Finally, model comparisons based on Bayes factors strongly favor the nonlinear model specification with an endogenous probability of default over linearized specifications with an ad-hoc risk premium.

This paper is related to the large empirical literature that studies the determinants of sovereign default risk premia through reduced-form regressions. Recent examples include Lonning (2000), Lemmen and Goodhart (1999), Codogno et al. (2003), Alesina et al. (1992), Bernoth et al. (2006), Haugh et al. (2009), Bernoth and Erdogan (2011), Abmann and Hogrefe (2009), and Maltritz (2011). This literature has found differences in sovereign risks across time and countries, suggesting the importance of country specific macroeconomic fundamentals to explain sovereign risk premia. In addition, related work by Ostry et al. (2010) estimate historical fiscal responses to construct ‘debt limits,’ although these limits are backward-looking by construction.

2. Model

Following Bi (2012), our model is a closed economy with linear production technology, whereby output depends on the level of productivity ($A_t$) and the labor supply ($n_t$).\footnote{Ceteris paribus, the assumption that the economy is open and all debt is held by foreigners raises the observed risk premium relative to the closed economy environment, as foreigners do not experience the negative wealth effects of debt and, in turn, have less incentive to hold debt. Thus, the estimates from our closed economy framework can be thought as the lower bound to estimates of the fiscal limit associated with certain probabilities of default.} Household consumption ($c_t$) and government purchases ($g_t$) satisfy the aggregate resource constraint,

$$c_t + g_t = A_t n_t. \tag{1}$$

Technological productivity $A_t$ follows the $AR(1)$ process

$$A_t - A = \rho^A(A_{t-1} - A) + \varepsilon^A_t \quad \varepsilon^A_t \sim \mathcal{N}(0, \sigma_A^2). \tag{2}$$
2.1 Government

The government finances lump-sum transfers to households \((z_t)\) and exogenous and unproductive purchases by levying a tax \((\tau_t)\) on labor income and issuing one-period bonds \((b_t)\). Let \(q_t\) be the price of the bond in units of consumption at time \(t\). For each unit of the bond, the government promises to pay the household one unit of consumption in the next period. However, the bond contract is not enforceable. At each period, a stochastic fiscal limit, which is specified in terms of the debt-to-GDP ratio and denoted as \(s_t^*\), is drawn from an exogenous distribution, \(s_t^* \sim \mathcal{S}^*\). We assume that the cumulative density function of the fiscal limit distribution is a logistical function with parameters \(\eta_1\) and \(\eta_2\) dictating its shape.

\[
p^* \equiv P(s_{t-1} \geq s_t^*) = \frac{\exp(\eta_1 + \eta_2 s_{t-1})}{1 + \exp(\eta_1 + \eta_2 s_{t-1})} \tag{3}
\]

where \(s_t\) is defined as \(b_t/y_t\). If the debt-to-GDP ratio surpasses the fiscal limit, then the government partially defaults on \(\delta\) percent of its obligations. The default scheme is summarized as

\[
\Delta_t = \begin{cases} 
0 & \text{if } s_{t-1} < s_t^* \\
\delta & \text{if } s_{t-1} \geq s_t^* 
\end{cases}
\]

The government’s budget constraint is given by

\[
\tau_t A_t n_t + b_t q_t = (1 - \Delta_t) b_{t-1} + g_t + z_t. \tag{4}
\]

The tax rate and government spending evolve according to the rules,

\[
\tau_t = (1 - \rho^\tau) \tau + \rho^\tau \tau_{t-1} + \varepsilon^\tau_t + \gamma^\tau (b^d_t - b) \quad \varepsilon^\tau_t \sim \mathcal{N}(0, \sigma^2_{\tau}) \tag{5}
\]

\[
g_t = (1 - \rho^g) g + \rho^g g_{t-1} + \varepsilon^g_t + \gamma^g (b^d_t - b) \quad \varepsilon^g_t \sim \mathcal{N}(0, \sigma^2_{g}) \tag{6}
\]

with AR(1) components being denoted as \(u^\tau_t\) and \(u^g_t\). The non-distortionary transfers are modeled as a residual in the government budget constraint, exogenously determined by the AR(1) process,

\[
z_t - z = \rho^z (z_{t-1} - z) + \varepsilon^z_t \quad \varepsilon^z_t \sim \mathcal{N}(0, \sigma^2_{z}) \tag{7}
\]

Since transfers are not an observable in our estimation, \(z_t\) can be thought of as capturing all movements in government debt that are not explained by the model.
2.2 Household

With access to the sovereign bond market, a representative household chooses consumption \((c_t)\), labor supply \((n_t)\), and bond purchases \((b_t)\) by solving,

\[
\max E_0 \sum_{t=0}^{\infty} \beta^t \left( \log (c_t - h\tilde{c}_{t-1}) + \phi \log(1 - n_t) \right)
\]

\[
s.t. \quad A_t n_t(1 - \tau_t) + z_t - c_t = b_t q_t - (1 - \Delta_t) b_{t-1}
\]

The household’s first-order conditions are,

\[
\phi \frac{c_t - h\tilde{c}_{t-1}}{1 - n_t} = A_t (1 - \tau_t)
\]

\[
q_t = \beta E_t \left( (1 - \Delta_{t+1}) \frac{c_t - h\tilde{c}_t}{c_{t+1} - h\tilde{c}_t} \right)
\]

The bond price reflects the household’s expectation about the probability and magnitude of sovereign default in the next period. The optimal solution to the household’s maximization problem must also satisfy the following transversality condition,

\[
\lim_{j \to \infty} E_t \beta^{j+1} \frac{u_c(t + j + 1)}{u_c(t)} (1 - \Delta_{t+j+1}) b_{t+j} = 0.
\]

2.3 Model Solution

Other than the specifications for exogenous state variables, the core equilibrium equations are,

\[
q_t = \frac{b_t^d + z_t + g_t - \tau_t A_t n_t}{b_t}
\]

\[
q_t = \beta (c_t - h\tilde{c}_{t-1}) E_t \frac{1 - \Delta_{t+1}}{c_{t+1} - h\tilde{c}_t}
\]

The first equation is derived from the government budget constraint, while the second is from the household’s first-order conditions. We use the monotone mapping method (Coleman (1991), Davig (2004)) to solve for the decision rule of the bond price in terms of the state vector. At time \(t\), the state vector is \((b_t^d, c_{t-1}, A_t, u_t^q, z_t, u_t^r)\), and the decision rule of the bond price can be written as \(q_t = q(b_t^d, c_{t-1}, A_t, u_t^q, z_t, u_t^r)\). Appendix A discusses the solution procedure in details.
3. Estimation

The model is estimated for Greece over the period 2001Q1-2010Q4. The estimation is over the post-EMU period, as interest rates during the pre-Euro period are susceptible to exchange rate risk from which our model abstracts. Five observables are used for the estimation: real output, the government spending to GDP ratio, the tax revenue to GDP ratio, the government debt to GDP ratio, and a 10-year real interest rate. Appendix B.1 provides a detailed description of the data.

3.1 Methodology

We estimate the model using Bayesian methods. The equilibrium system is written in the nonlinear state-space form, linking observables $v_t$ to model variables $x_t$:

\begin{align}
  x_t & = f(x_{t-1}, \epsilon_t, \theta) \\
  v_t & = Ax_t + \xi_t,
\end{align}

where $\theta$ denotes model parameters and $\xi_t$ is a vector of measurement error distributed $N(0, \Sigma)$. We assume that $\Sigma$ is a diagonal matrix and calibrate the standard deviation of each measurement error to be 20% of the standard deviation of the corresponding observable variable.\(^2\)

We use a particle filter to approximate the likelihood function. For a given sequence of observations up to time $t$, $v^t = [v_1, ..., v_t]$, the particle filter approximates the density $p(x_t|v^t, \theta)$ with a swarm of particles $x^t_i$ ($i = 1, ..., N$) (see appendix B.2 for more details). The particle filter is applicable for nonlinear and non-Gaussian distributions, and it is increasingly used to estimate nonlinear DSGE models, to which class our model belongs. Recent examples include An and Schorfheide (2007), Fernandez-Villaverde and Rubio-Ramirez (2007), Armisano and Tristani (2010), Fernandez-Villaverde et al. (2011), and Doh (2011). Doucet et al. (2001) provides a textbook treatment.

We combine the likelihood $L(\theta|v^T)$ with a prior density $p(\theta)$ to obtain the posterior density kernel, which is proportional to the posterior density. We assume that parameters are independent a priori. However, we discard any prior draws that do not deliver a unique rational expectations equilibrium, as we restrict the analysis to the determinacy parameter subspace.\(^3\) We construct the posterior distribution of the parameters using the random walk

\(^2\)Estimating measurement errors provides complications with nonlinear estimation techniques. See Doh (2011) for more discussion on the role of measurement error in nonlinear DSGE model estimation.

\(^3\)A technical appendix of the authors provides more discussion on this point. Only 0.38% of the prior
Metropolis-Hastings algorithm (see appendix B.3 for more details). In each estimation, we sample 120,000 draws from the posterior distribution and discard the first 50,000 draws.4 The likelihood is computed using 60,000 particles, and posterior analysis is conducted with every 25th draw.

3.2 Prior Distributions

We impose dogmatic priors over some parameters, which are listed in table C. The discount rate is 0.99, so that the deterministic net interest rate is 1%.5 We calibrate the household’s leisure preference parameter φ such that a household spends 25% of its time working at the steady state. We calibrate the deterministic debt to GDP ratio, government spending to GDP ratio, and tax rate to the mean values of the data sample.

The priors for the remaining parameters are listed in table C. The prior for habit persistence h is similar to those in the linear DSGE estimation literature, for instance Smets and Wouters (2007). For the remaining parameters, we first estimate using ordinary least squares an AR(1) process for GDP and processes for government spending, the tax rate, and transfers given by equations (5)-(7).6 The results are used as general guidance for the region of the parameter space for the ρ, σ, and γ parameters.

For the responses of government spending and taxes to debt, we form priors for the long run responses in terms of percentage deviations from steady state, that is

\[ \gamma^g,L = \frac{\bar{b}\gamma^g}{\bar{g}(1 - \rho^g)}, \quad \gamma^\tau,L = \frac{\bar{b}\gamma^\tau}{\bar{\tau}(1 - \rho^\tau)} \]

These values are more comparable to estimates in the literature. Since determinacy is sensitive to the combination of the \( \gamma^\tau,L \) and \( \gamma^g,L \) parameters, we restrict the lower bound of the \( \gamma^\tau,L \) (\( \gamma^g,L \)) prior to a value that ensures determinacy when only \( \gamma^\tau,L \) (\( \gamma^g,L \)) finances debt.

For the standard deviations of shocks, we form priors for the standard deviations relative to relevant steady state variables: \( \sigma_{k,p} \equiv \sigma_k/J \) for \( J = \{ A, g, \tau, z \} \) and \( k = \{ a, g, \tau, z \} \). This gives standard deviations as percentage deviations, which provides more intuitive comparisons across values.

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4 We use Fortran MPI code compiled in Intel Visual Fortran for the estimation. We use the computer server system at the Bank of Canada, each CPU of which uses Xeon CPU X5680 at 3.33GHz and has 23 processors with 64G RAM. One evaluation using the particle filter takes 5 seconds. These computational constraints limit the number of draws from the Metropolis-Hastings algorithm.

5 The mean of our data is 0.8% for Greece.

6 We back out the model-consistent tax rate and transfers series implied by our observables for this exercise.
3.2.1 Fiscal Limit

We estimate one parameter from the fiscal limit distribution, given by equation (3). Given two points on the distribution, \((\tilde{s}, \tilde{p})\) and \((\hat{s}, \hat{p})\), the parameters \(\eta_1\) and \(\eta_2\) can be uniquely determined by

\[
\eta_2 = \frac{1}{\tilde{s} - \hat{s}} \log \left( \frac{\tilde{p}}{\hat{p}} \frac{1 - \hat{p}}{1 - \tilde{p}} \right), \quad \eta_1 = \log \frac{\tilde{p}}{1 - \tilde{p}} - \eta_2 \tilde{s}. \tag{17}
\]

Since \((\tilde{s}, \tilde{p})\) and \((\hat{s}, \hat{p})\) provide a more intuitive description about the fiscal limit distribution than \(\eta_1\) and \(\eta_2\), we fix \(\tilde{p}\) and \(\hat{p}\) at certain levels and estimate the corresponding \(\tilde{s}\) and \(\hat{s}\), instead of estimating \(\eta_1\) and \(\eta_2\) directly. We choose \(\tilde{p} = 0.3\) and \(\hat{p} = 0.999\). Unfortunately, given that defaults are not observed in our data sample, the data is unlikely to be informative about the upper bound of the distribution. Therefore, we estimate \(\tilde{s}\) and fix the difference between \(\tilde{s}\) and \(\hat{s}\) to be 60% of steady-state output. This difference is chosen to capture the observation that once risk premia begin to rise, they do so rapidly.\(^7\) Given the lack of guidance for the parameter \(\tilde{s}\), we adopt a diffuse uniform prior over the interval 1.4 to 1.8, implying that the debt level associated with a 30% probability of default ranges from 140 to 180% of GDP.

3.2.2 \(\delta\) Identification

To our knowledge, this paper is the first attempt to estimate a DSGE model of sovereign default. Thus, prior to estimating the model with real data, we performed several estimations with simulated data.\(^8\) Unfortunately, the results revealed that we cannot jointly identify the rate of partial default \(\delta\) and the fiscal limit parameter \(\tilde{s}\) when the data excludes observed defaults. Parameters related to default affect observable variables through their influence on the risk premium. Since various combinations of \(\delta\) and \(\tilde{s}\) are consistent with the same risk premium, we cannot jointly identify the parameters. Given this limitation, we estimate our model for two different calibrations of \(\delta\): 0.05 and 0.075. These calibrations imply annualized rates of default \(\delta_A\) of 20% and 30% respectively, which falls within the range of actual default rates in emerging market economies over the period 1983 to 2005, as documented by?.

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\(^7\)The difference of 60% of output, albeit ad-hoc, should not change the key estimation results as the data is unlikely to be informative about the upper bound of distribution.

\(^8\)The results are available in a technical appendix from the authors.
3.3 Posterior Estimates

Table 2 compares the medians and 90% credible intervals of the posterior distributions estimated under the two alternative partial default rate specifications. The data appear informative for all of the parameters, as the 90% credible intervals are smaller than those from the prior distributions.

The estimates of $\tilde{s}$ are similar for the two specifications. If the partial default rate, $\delta^A$, is 30%, the debt to GDP ratio that is associated with a 30% probability of default is between 1.53-1.59, with the median being 1.56. If the partial default rate is 20%, the debt to GDP ratio that is associated with a 30% probability of default is between 1.5-1.55, with the median being 1.52. The lower $\tilde{s}^*$ estimates for a lower $\delta^A$ calibration are consistent with theory. The model tries to match the risk premium in the data through the values of $\delta^A$ and $\tilde{s}^*$. For higher values of $\delta^A$, agents expect to lose more of the face-value of debt following a default. Thus, households demand a higher interest rate to compensate for this risk. Thus, for the given risk premium implied by the data, a slightly higher $\tilde{s}^*$ is needed to offset a higher $\delta^A$ value.

For comparison, we also list the estimates implied by a log-linearized version of our model without default and a log-linearized version of the model with an ad-hoc risk premium, similar to Garcia-Cicco et al. (2010). The system of equations for the log-linearized models are listed in appendix C. We use the Kalman filter to calculate the likelihood function and initialize the Metropolis-Hastings algorithm using the posterior mode and inverse Hessian at the posterior mode.

Interestingly, the estimates from the linear models suggest that the data is not informative about $\gamma^{g,L}$ and $\sigma^z$, as the 90% credible interval from the posterior distribution mirrors that from the prior distribution, shown in table (2). Based on the estimates, the ad hoc risk premium parameter, $\psi_b$, appears to be zero, as the 90% posterior interval encompasses zero. Comparing the results to the nonlinear model, it appears that allowing default in the standard RBC model may help to identify the fiscal policy responses in Greece.

4. Analysis

4.1 Model Fit

To examine how well the model fits the data, we compute smoothed estimates of model variables using the sequential monte carlo approximation of the forward-backward smoothing recursion. Figure 4 compares the smoothed values from the various model specifications to
the observable variables. For each specification, the fitted values are computed using the corresponding posterior median. The fit for most variables is accurate, with output and the real interest rate being the least precise.

We also compute smoothed estimates of the measurement errors $E(\xi_t | v^T, \theta)$ and report their mean absolute values and relative standard deviations in Table 3. The standard deviation of each measurement error was fixed to be 20% of the standard deviation of the respective observable variable. However, for most observables, the estimated relative standard deviation is less than 20%, suggesting that the measurement error did not introduce many constraints for the model fit. The exception is the measurement error for the real interest rate in the nonlinear model and for output in the linear models. Table 3 also shows that mean absolute values of measurement error are close to zero.

Given that the different model specifications imply similar smoothed estimates, we perform posterior odds comparisons to determine which model is favored by the data. Bayes factors are used to evaluate the relative model fit of the two nonlinear and two linear specifications. Table 4 presents the results. Bayes factors are based on log-marginal data densities calculated using Geweke’s (1999) modified harmonic mean estimator with a truncation parameter of 0.5. The results demonstrate that the data strongly prefer the nonlinear model specifications. In addition, it appears the data cannot distinguish between the two calibrated partial default rates, as the log Bayes factor is only 3.5.9

### 4.2 Default Probability and Interest Rate Dynamics

#### 4.2.1 Default Risk

Figure 5 shows model-implied sovereign default probabilities for Greece, based upon the posterior estimates for the fiscal limit distribution when $\delta^A = 0.3$. Solid lines show the median and 90% confidence interval for historical debt-to-GDP ratios calculated from our debt and output observables for the estimated sample period.10 Dashed lines denote the median and 90% confidence interval default probabilities for the out-of-sample debt-to-GDP ratio in 2011Q4.

Figure 5 shows that Greek debt had virtually zero probability of default from 2001 until 2009. Starting in 2009 the probability of default rose steadily, and ranged from 5-10% in the end of the estimated period 2010Q4. The model-implied probability of default that

9This value is close to Jeffreys’ (1961) limiting difference of 3 that gives inconclusive evidence in favor of one model over the other.

10For the estimation we use data in terms of percentage deviations from the sample average. In contrast, we need level variables to back out model-implied probabilities of default. For model consistency, we convert the percentage deviations of the data to level variables using the steady state model variables.
is consistent with the observed debt-GDP ratio shoots up in 2011 and reaches 60-80% by 2011Q4, which is consistent with the debt restructuring arrangements that took place at the beginning of 2012. Estimates from the model’s fiscal limit distribution thus suggest the unsustainability of the Greek fiscal position and reflect large deterioration in confidence in Greek debt over 2010.

4.2.2 Out-of-Sample Interest Rate Forecasts

Over the course of 2011, the long term nominal interest rate in the secondary market for Greek government bonds rose from 9.1 percentage points in December 2010 to 21.2 in December 2011 based on BIS data, and to 31.2 based on Bloomberg data. Can the estimated model account for this sharp increase in the Greek interest rate?

To examine this issue, we simulate four quarters of time series 10,000 times starting from the fitted values for model variables in 2010Q4,\textsuperscript{11} which gives a distribution for the forecasted path of the real interest rate in 2011. Figure 6 displays the median (blue, dotted line) and 90% interval (blue, dashed lines) of these model-implied interest rate forecasts for 2011, calculated using the posterior median parameter estimates. The figure also plots the path of the real interest rate implied from the data (black solid line for BIS data and dotted red line for Bloomberg data).

Figure 6 shows that the surge in the real interest rate in Greece is within forecast confidence bands of our rational expectations model. The figure suggests that it is possible for model forecasts to be consistent with the 2011 interest rate path at all horizons, conditional on the posterior median parameter estimates (note that this case is constructed based upon the posterior median estimates of \textit{each} parameter). It suggests that the interest rate surge can be explained by macroeconomic fundamentals in the rational expectation framework.

5. Conclusion

This paper uses Bayesian methods to estimate the probability of sovereign default for Greece. We build a real business cycle model that allows for the interactions among fiscal policy instruments, the stochastic fiscal limit, and sovereign default risks. The fiscal policy specification takes into account government spending, lump-sum transfers, and distortionary taxation. We model the fiscal limit distribution with a logistical function, which illustrates the market’s belief about the government’s ability to service its debt at various debt levels.

\textsuperscript{11}See section ?? for details on the construction of fitted values.
Using the particle filter to perform likelihood-based inference, we estimate the full non-linear model with post-EMU data. We find that the probability of default on Greek debt remained close to zero from 2001 until 2009, when it began to rise sharply to the range of 5% to 10% by the fourth quarter of 2010. By the end of 2011, the model predicts a probability of default between 60-80%, consistent with the debt restructuring arrangements that took place at the beginning of 2012. In addition, the surge in the real interest rate in Greece in 2011 is within forecast confidence bands of our rational expectations model. Finally, model comparisons based on Bayes factors strongly favor the nonlinear model specification with an endogenous probability of default over linearized specifications with an ad-hoc risk premium.

In current ongoing research, we are estimating the model for other European countries, so as to allow cross-country comparisons of default probabilities and political risk factors. Although our nonlinear model allows complex interactions among fiscal policy instruments and the fiscal limit, it is only a first step to understanding and estimating probabilities of default for developed countries. To understand fully the complexities associated with default risk, several other features are worthy of modeling attention, including the interaction of monetary and fiscal policies; the interaction of the financial sector and the government; and open economy issues including foreign holdings of debt and risks of contagion.

References


A Solving the Nonlinear Model

Other than the end-of-period government debt, all other variables are either exogenous or can be computed in terms of the current state $\psi_t = (b^d_t, c^d_{t-1}, A_t, u^g_t, z_t, u^\tau_t)$.

\begin{align*}
\tau_t &= u^\tau_t + \gamma^\tau (b^d_t - b) & \text{(A.1)} \\
g_t &= u^g_t + \gamma^g (b^d_t - b) & \text{(A.2)} \\
z_t &= (1 - \rho^z)z + \rho^z z_{t-1} + \varepsilon^z_t & \text{(A.3)} \\
A_t &= (1 - \rho^A)A + \rho^A A_{t-1} + \varepsilon^A_t & \text{(A.4)} \\
c_t &= \frac{\phi h c_{t-1} + (A_t - g_t)(1 - \tau_t)}{1 + \phi - \tau_t} & \text{(A.5)}
\end{align*}

$$
\Delta_t = \begin{cases} 
0 & \text{if } b_{t-1} < b^*_t \\
\delta & \text{if } b_{t-1} \geq b^*_t 
\end{cases}
$$

We use the monotone mapping method to solve for the decision rule of the bond price in terms of the state vector. In terms of computation, the most time-consuming part is the loop iterations of the numerical integration in equation (14) of the text.

\begin{align*}
E_t \frac{1 - \Delta_{t+1}}{c_{t+1} - h c_t} &= \int \int \int \int \frac{1 - \Delta_{t+1}}{c_{t+1} - h c_t} \\
&= (1 - \Phi(s_t \geq s^*_t)) \int \int \int \frac{1}{c_{t+1} - h c_t} | \text{no default} \\
&\quad + \Phi(s_t \geq s^*_t) \int \int \int \frac{1 - \delta}{c_{t+1} - h c_t} | \text{default}
\end{align*}

(A.6)

Given the utility function, consumption is determined by,

$$
c_t = \frac{\phi h c_{t-1} + (A_t - g_t)(1 - \tau_t)}{1 + \phi - \tau_t} \quad \text{(A.7)}
$$

Thus, the integration in Equation (A.6) can be re-written as

\begin{align*}
\int \int \int \int \frac{1}{c_{t+1} - h c_t} &= \int \int \int \int \frac{1 + \phi - \tau_{t+1}}{(1 - \tau_{t+1})(A_{t+1} - g_{t+1} - h c_t)} \\
&= \int \int \int \frac{1 + \phi - \tau_{t+1}}{1 - \tau_{t+1}} \int \int \int \frac{1}{A_{t+1} - g_{t+1} - h c_t} \quad \text{(A.8)}
\end{align*}

(A.9)

The logarithmal utility function helps to reduce the 4-dimension integration into 1- and 2-dimension integrations. The decision rule for government debt, $b_t = f^b(\psi_t)$, is solved in the
following steps:

- Step 1: Define the grid points by discretize the state space $\psi_t$. Make an initial guess of the decision rule $f^b_0$ over the state space.

- Step 2: At each grid point, solve the following core equation and obtain the updated rule $f^b_i$ using the given rule $f^b_{i-1}$. The integral in the right-hand side is evaluated as described above using numerical quadrature.

$$\frac{b^d_t + z_t + g_t - \tau_t A_t n(\psi_t)}{f^b_t(\psi_t)} = \beta(1 - \Delta t^{t+1}) E_t \frac{c(\psi_t) - h c_{t-1}}{c(\psi_{t+1}) - h c(\psi_t)}$$  \hspace{1cm} (A.10)

where $\psi_{t+1} = \left( (f^b_{i-1}(\psi_t), \Delta t^{t+1}), c_t, A_{t+1}, u^g_{t+1}, z_{t+1}, u^r_{t+1} \right)$.

- Step 3: Check the convergence of the decision rule. If $|f^b_i - f^b_{i-1}|$ is above the desired tolerance (set to $1e-5$), go back to step 2; otherwise, $f^b_i$ is the decision rule and used to evaluate the particle filter as described below.

**B Estimation**

**B.1 Data Description**

Five observables for Greece over the period 2001Q1-2010Q4 are used for the estimation: real output, the government spending to GDP ratio, the tax revenue to GDP ratio, the government debt to GDP ratio, and a 10-year real interest rate. This appendix provides documentation for the construction of these series. First, data for real GDP, government spending, tax revenue, government debt, and the 10-year interest rate are constructed as follows.

**Real GDP.** Constructed by dividing the nominal quarterly gross domestic product from the OECD quarterly National Accounts (using the expenditure approach, series B1_GE) by the gross domestic product deflator (constructed using the expenditure approach, series B13).

**Real Gov. Spending.** Constructed using general government final consumption expenditure from the OECD quarterly National Accounts (series P3S13) divided by the gross domestic product deflator (constructed using the expenditure approach, series B13).

**Real Tax Revenue.** A quarterly tax revenue series is not provided by the OECD. Thus, we construct a quarterly measure in the following way. First, we construct a measure of
total tax revenue by combining Eurostat quarterly series for tax receipts on income/wealth, production and imports, capital taxes, and social contributions. We seasonal adjustment this series using Demetra+ and the tramo-seat RSA4 specification. Next, using an annual nominal tax revenue series from the OECD volume 90 (consisting of indirect and direct taxes and social security contributions, TIND + TY + SSRG), we interpolate a quarterly frequency series using the method of Chow and Lin (1971)\textsuperscript{12} and the seasonally adjusted quarterly Eurostat tax revenue series for the interpolation. Finally, we construct a quarterly real tax revenue series by dividing the interpolated series by the OECD’s gross domestic product deflator (constructed using the expenditure approach, series B13).

**Real Gov. Debt.** A quarterly government debt series is not provided by the OECD. Thus, we construct a quarterly measure in the following way. First, we seasonally adjust using Demetra+ and the tramo-seat RSA4 specification the Eurostat quarterly series for nominal gross government consolidated debt. Next, using the annual nominal gross public debt series (under the Maastricht criterion) from the OECD volume 90, we interpolate a quarterly frequency series using the method of Chow and Lin (1971) and the seasonally adjusted quarterly Eurostat tax revenue series for the interpolation. Finally, we construct a quarterly real debt series by dividing the interpolated series by the OECD’s gross domestic product deflator (constructed using the expenditure approach, series B13).

**Real Interest Rate.** To construct a 10-year real interest rate measure, we use data for the nominal interest rate, \( i_t \) (taken from the BIS) and the expected inflation rate, \( \pi_t^e \). Our measure of expected inflation for Greece is the expected inflation series from the Survey of Professional Forecasts EU-area five year ahead expected inflation series. The gross real interest rate is constructed using the relation

\[
R_t = \frac{1 + i_t}{1 + \pi_t^e}
\]

**Data for Estimation.** We calculate the government spending to GDP ratio, tax revenue to GDP ratio, and government debt to GDP ratio by taking each real fiscal series described above and dividing by our real GDP series. The estimation uses these three series, along with the real GDP and real interest rate series described above. For each series, we transform the series into percentage deviations from its mean value over the period 2001Q1-2010Q4. In addition, the real GDP series is linearly detrended. The black solid lines of figure 4 graphs the observables.

\textsuperscript{12}Forni et al. (2009) use a similar approach.
B.2 Particle Filter Algorithm

Let $v^T$ denote $\{\hat{v}_t\}_{t=1}^T$, which evolves according to equations (15) and (16) in the text. To evaluate the likelihood function $L(\theta|v^T)$, we use a sequential Monte Carlo filter (specifically, the sequential importance resampling filter of Kitagawa (1996)). The algorithm is as follows:

- Step 1. Initialize the state variable $x_0$ by generating 60,000 values from the unconditional distribution $p(x_0|\theta)$. Denote these particles by $x_0^i$ for $i = 1, \ldots, 60,000$. Draw 40,000 values from standard normal distributions for each of the structural shocks ($\epsilon^A, \epsilon^g, \epsilon^t, \epsilon^z$) and 40,000 values from a standard uniform distribution for fiscal limit probabilities. Denote the vector of these particles by $u^i$. By induction, in period $t$ these are particles $u^{|t-1,i}$.

- Step 2. Construct $x^{|t-1,i}$ using equation (15) in the text. Assign to each draw $(u^{|t-1,i}, x^{|t-1,i})$ a weight defined as:

$$w_i^t = \frac{1}{(2\pi)^{5/2}|\Sigma|^{1/2}} \exp \left\{-\frac{1}{2} \left(y_t - Ax^{|t-1,i}\right)'\Sigma \left(y_t - Ax^{|t-1,i}\right)\right\} \quad (B.1)$$

- Step 3. Normalize the weights:

$$\tilde{w}_i^t = \frac{w_i^t}{\sum_{i=1}^N w_i^t}$$

Update the values of $x^{|t-1,i}$ by sampling with replacement 60,000 values of $x^{|t-1,i}$ using the relative weights $\tilde{w}_i^t$ and the residual resampling algorithm.

- Repeat steps 2-3 for $t \leq T$.

The log-likelihood function is approximated by

$$L(\theta|v^T) \simeq \sum_{t=1}^T \ln \left(\frac{1}{60,000} \sum_{i=1}^{60,000} w_i^t\right) \quad (B.2)$$

B.3 MCMC Algorithm

The random walk Metropolis-Hastings algorithm used for estimation works as follows:

- Step 1. Compute the posterior log-likelihood for 500 draws from the priors. Call the draw with the highest posterior log-likelihood value $\theta^*$.  

- Step 2. Starting from $\theta^*$, generate a MCMC chain using the following random-walk proposal density

$$\theta^\text{prop}_{j+1} = \theta^\text{prop}_j + c\mathcal{N}(0, \Lambda), \quad j = 1, \ldots, 100,000$$
where $\Lambda$ is the covariance matrix of 500 draws from the priors and $c > 0$ is a tuning parameter set to determine the acceptance ratio.

- Step 3. Compute the acceptance ratio $\varphi = \min \left\{ \frac{p(\theta_{j+1}^{\text{prop}}|v^{T})}{p(\theta_{j}|u^{T})} , 1 \right\}$. Given a draw $u$ from the standard uniform distribution. Then $\theta_{j+1} = \theta_{j}^{\text{prop}}$ if $u < \varphi$ and $\theta_{j+1} = \theta_{j}$ otherwise. Repeat for $j = 1, ..., 100,000$.

- Step 4. Update the random walk proposal density in the following way. Update $\Lambda$ to be the covariance matrix from the previous draws $\{\theta_{j}\}_{1}^{10,000}$. Update $\theta^{*}$ to be the mean of previous draws $\{\theta_{j}\}_{1}^{100,000}$. Starting from the new $\theta^{*}$, proceed through steps 2 and 3 for 120,000 draws from the new MCMC chain.

We burn the first 50,000 draws from the final MCMC chain and thin every 25 draws.

## C Log-Linearized Model Equations

The log-linearized system of equations for the basic variant of the model without default are:

$$
\dot{c}_t - \frac{1}{1 + h} \hat{E}_t \hat{c}_{t+1} + \frac{1 - h}{1 + h} \hat{R}_t = \frac{h}{1 + h} \hat{c}_{t-1}
$$

$$
\frac{1}{1 - h} \hat{c}_t + \frac{n}{1 - n} \hat{n}_t - \hat{A}_t + \frac{\tau}{1 - \tau} \hat{\tau}_t = \frac{h}{1 - h} \hat{c}_{t-1}
$$

$$
\frac{c}{y} \dot{b}_t - \frac{g}{y} \dot{g}_t - \frac{z}{y} \dot{z}_t + \tau (\hat{\tau}_t + \hat{A}_t + \hat{n}_t) = R * \frac{b}{y} (\hat{R}_{t-1} + \hat{b}_{t-1})
$$

$$
\dot{g}_t = (1 - \rho^g) \hat{g}_{t-1} - \gamma^G L (1 - \rho^g) b_{t-1} + \sigma_{g,p} \epsilon^g_t, \quad \epsilon^g_t \sim N(0,1)
$$

$$
\dot{\tau}_t = (1 - \rho^\tau) \hat{\tau}_{t-1} + \gamma^\tau L (1 - \rho^\tau) b_{t-1} + \sigma_{\tau,p} \epsilon^\tau_t, \quad \epsilon^\tau_t \sim N(0,1)
$$

$$
\dot{z}_t = (1 - \rho^z) \hat{z}_{t-1} + \sigma_{z,p} \epsilon^z_t, \quad \epsilon^z_t \sim N(0,1)
$$

$$
\dot{\tau}_t = (1 - \rho^a) \hat{\tau}_{t-1} + \sigma_{a,p} \epsilon^\tau_t, \quad \epsilon^\tau_t \sim N(0,1)
$$

In addition, we consider a log-linearized version of the model that incorporates an ad-hoc risk premium, similar to Garcia-Cicco et al. (2010). In this case, the real interest rate for government debt, $R^g_t$, is given by

$$
R^g_t = R_t \exp \psi_b \left( \frac{b_t}{y_t} - \frac{b}{y} \right)
$$
In this case, the log-linearized Euler equation and government budget constraint are given by

\[
\hat{c}_t - \frac{1}{1+h}E_t\hat{c}_{t+1} + \frac{1-h}{1+h}\hat{R}_t + \frac{\psi_b(1-h)}{1+h}\hat{b}_t - \frac{\psi_b(1-h)}{1+h}\hat{y}_t = \frac{h}{1+h}\hat{c}_{t-1} \\
\frac{\hat{b}}{y}\hat{b}_t - \frac{g}{y}\hat{g}_t - \frac{z}{y}\hat{z}_t + \tau(\hat{r}_t + \hat{A}_t + \hat{n}_t) = R \frac{\hat{b}}{y}(\hat{R}_{t-1} + (\psi_b + 1)\hat{b}_{t-1} - \psi_b\hat{y}_{t-1})
\]

We assume a normal prior for $\psi_b$ centered at 0.05 and with a standard deviation of 0.02.
Figure 1: Prior distributions (black solid lines) versus posterior distributions (blue dashed lines) for the nonlinear model.
Figure 2: Prior distributions (black solid lines) versus posterior distributions (blue dashed lines) for the linear model.

Figure 3: Prior distributions (black solid lines) versus posterior distributions (blue dashed lines) for the linear model with adhoc sovereign risk premium.
Figure 4: Fitted values for various estimations for Greece. Black, solid lines: data. Blue, dashed lines: Nonlinear model with $\delta^A = 0.3$. Red, dotted-dashed lines: Linear model.

Figure 5: Model-implied sovereign default probabilities for Greece. Solid lines denote the median and 90% confidence interval probabilities for in-sample debt-to-GDP ratios. Dashed lines denote the median and 90% confidence interval probabilities for out-of-sample debt-to-GDP ratios.
Figure 6: Data versus fitted and forecast values for the Greek interest rate. The median (blue, dotted line) and 90% interval (blue, dashed lines) of model-implied interest rate forecasts for 2011 are calculated based on the posterior median parameter estimates (top panel) and 5 percentile parameter estimates (bottom panel). The black solid line shows BIS data, and red dotted line shows Bloomberg data.
Table 1: Calibration and Priors for Greece

<table>
<thead>
<tr>
<th>Calibration</th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$\beta$</td>
<td>0.99</td>
<td></td>
</tr>
<tr>
<td>$\bar{n}$</td>
<td>0.75</td>
<td></td>
</tr>
<tr>
<td>$\bar{g}/\bar{y}$</td>
<td>0.181</td>
<td></td>
</tr>
<tr>
<td>$\bar{b}/\bar{y}$</td>
<td>1.095*4</td>
<td></td>
</tr>
<tr>
<td>$\tau$</td>
<td>0.333</td>
<td></td>
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</table>

<table>
<thead>
<tr>
<th>Priors</th>
<th>Function</th>
<th>Mean</th>
<th>St. Dev.</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h$</td>
<td>Beta</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>$\tilde{s}^*$</td>
<td>Uniform</td>
<td>1.6</td>
<td>0.013</td>
</tr>
<tr>
<td>$\gamma^\tau, L$</td>
<td>Gamma</td>
<td>0.4</td>
<td>0.2</td>
</tr>
<tr>
<td>$\gamma^g, L$</td>
<td>Gamma</td>
<td>1.1</td>
<td>0.3</td>
</tr>
<tr>
<td>$\rho^a$</td>
<td>Beta</td>
<td>0.8</td>
<td>0.1</td>
</tr>
<tr>
<td>$\rho^g$</td>
<td>Beta</td>
<td>0.8</td>
<td>0.1</td>
</tr>
<tr>
<td>$\rho^\tau$</td>
<td>Beta</td>
<td>0.5</td>
<td>0.2</td>
</tr>
<tr>
<td>$\rho^z$</td>
<td>Beta</td>
<td>0.5</td>
<td>0.2</td>
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<tr>
<td>$\sigma_{a,p}$</td>
<td>Gamma</td>
<td>0.005</td>
<td>0.01</td>
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<tr>
<td>$\sigma_{g,p}$</td>
<td>Gamma</td>
<td>0.02</td>
<td>0.015</td>
</tr>
<tr>
<td>$\sigma_{\tau,p}$</td>
<td>Gamma</td>
<td>0.005</td>
<td>0.01</td>
</tr>
<tr>
<td>$\sigma_{z,p}$</td>
<td>Gamma</td>
<td>0.5</td>
<td>0.1</td>
</tr>
</tbody>
</table>
### Table 2: Greece Estimates.

<table>
<thead>
<tr>
<th>Prior</th>
<th>Posterior: $\delta^A = 0.3$</th>
<th>Posterior: $\delta^A = 0.2$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$h^*$</td>
<td>0.5 [0.17, 0.83]</td>
<td>0.14 [0.04, 0.33]</td>
</tr>
<tr>
<td>$\tilde{s}^*$</td>
<td>1.6 [1.42, 1.78]</td>
<td>1.56 [1.53, 1.59]</td>
</tr>
<tr>
<td>$\gamma_{\tau,L}$</td>
<td>0.4 [0.14, 0.78]</td>
<td>0.18 [0.07, 0.39]</td>
</tr>
<tr>
<td>$\gamma_{g,L}$</td>
<td>1.1 [0.66, 1.64]</td>
<td>1.44 [0.96, 1.94]</td>
</tr>
<tr>
<td>$\rho^a$</td>
<td>0.8 [0.61, 0.94]</td>
<td>0.97 [0.96, 0.98]</td>
</tr>
<tr>
<td>$\rho^g$</td>
<td>0.8 [0.61, 0.94]</td>
<td>0.95 [0.91, 0.97]</td>
</tr>
<tr>
<td>$\rho^*$</td>
<td>0.5 [0.17, 0.83]</td>
<td>0.57 [0.25, 0.77]</td>
</tr>
<tr>
<td>$\rho^t$</td>
<td>0.5 [0.17, 0.83]</td>
<td>0.93 [0.91, 0.96]</td>
</tr>
<tr>
<td>$\sigma_{a,p}$</td>
<td>0.01 [0.003, 0.02]</td>
<td>0.022 [0.019, 0.027]</td>
</tr>
<tr>
<td>$\sigma_{g,p}$</td>
<td>0.02 [0.003, 0.05]</td>
<td>0.04 [0.034, 0.048]</td>
</tr>
<tr>
<td>$\sigma_{z,p}$</td>
<td>0.5 [0.35, 0.68]</td>
<td>0.47 [0.35, 0.55]</td>
</tr>
<tr>
<td>$\sigma_{\tau,p}$</td>
<td>0.01 [0.003, 0.02]</td>
<td>0.025 [0.021, 0.031]</td>
</tr>
</tbody>
</table>

### Table 3: Smoothed estimates of measurement error.

<table>
<thead>
<tr>
<th>Greece</th>
<th>$b_t$</th>
<th>$q_t$</th>
<th>$T_t$</th>
<th>$y_t$</th>
<th>$R_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Nonlinear $\delta^A = 0.3$</td>
<td>mean absolute value</td>
<td>0.012</td>
<td>0.004</td>
<td>0.002</td>
<td>0.003</td>
</tr>
<tr>
<td></td>
<td>relative standard deviation</td>
<td>0.116</td>
<td>0.099</td>
<td>0.110</td>
<td>0.150</td>
</tr>
<tr>
<td>Linear</td>
<td>mean absolute value</td>
<td>0.01</td>
<td>0.006</td>
<td>0.002</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>relative standard deviation</td>
<td>0.091</td>
<td>0.128</td>
<td>0.077</td>
<td>0.292</td>
</tr>
<tr>
<td>Linear, Risk Prem.</td>
<td>mean absolute value</td>
<td>0.011</td>
<td>0.006</td>
<td>0.002</td>
<td>0.01</td>
</tr>
<tr>
<td></td>
<td>relative standard deviation</td>
<td>0.099</td>
<td>0.128</td>
<td>0.073</td>
<td>0.289</td>
</tr>
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</table>

### Table 4: Model Fit Comparisons

<table>
<thead>
<tr>
<th>Model Specification</th>
<th>Bayes Factor</th>
<th>Rel. to M1</th>
</tr>
</thead>
<tbody>
<tr>
<td>M1: Nonlinear Model w/ $\delta^A = 0.3$</td>
<td>1</td>
<td></td>
</tr>
<tr>
<td>M2: Nonlinear Model w/ $\delta^A = 0.2$</td>
<td>exp[3.5]</td>
<td></td>
</tr>
<tr>
<td>M3: Linear</td>
<td>exp[52]</td>
<td></td>
</tr>
<tr>
<td>M4: Linear, Risk Prem.</td>
<td>exp[55]</td>
<td></td>
</tr>
</tbody>
</table>

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