Momentum Trading Through Reference Dependent Preferences

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Abstract

The endowment effect is a well-known behavioral regularity in which a person is less likely to trade a good when he is endowed with it. In their generalization of prospect theory to consumption bundles with multiple attributes, Tversky and Kahneman [1991] imply the endowment effect as a consequence of loss aversion and diminishing sensitivity in gains. It has since frequently been presumed that this form of reference dependent preferences will inhibit trade. However, in this paper it is demonstrated that loss aversion and diminishing sensitivity in gains also imply a dynamic momentum trading effect that increases exchange, so the net effect of such preferences on trading volume is ambiguous. In fact, the momentum trading effect is shown to completely cancel out the endowment effect in an important class of examples.

Keywords: Loss aversion, reference dependence, prospect theory, endowment effect, status quo bias, momentum trading

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Introduction

The idea that a person’s expected utility from owning a particular portfolio is conditional on his point of reference has gained much traction since it was introduced in the economics literature as *prospect theory* by Kahneman and Tversky [1979]. While prospect theory specifically characterizes expected utility over lotteries on one dimension (wealth) and remains a very active area of research, a parallel literature has developed concerning utility over deterministic multidimensional portfolios. Designated *reference dependence* by Tversky and Kahneman [1991], this literature had until recently been dominated by laboratory research into a consistent discrepancy between an individual’s elicited willingness to accept (WTA) a good and his willingness to pay (WTP) for it (see Knetsch and Sinden [1984], Knetsch [1989], Kahneman, Knetsch and Thaler [1990], Loewenstein and Adler [1995], Bateman, Munro, Rhodes, Starmer and Sugden [1997], Myagkov and Plott [1997], and List [2004], or Kahneman [2003] for a short review; Plott and Zeiler [2005] examine inconsistencies with eliciting valuations in these studies).

An important conclusion of this literature is that individuals endowed with a good tend to report a higher WTA than the WTP reported by individuals who are not; that is, individuals are biased towards maintaining the status quo. To motivate this result, suppose members of one group of individuals are brought one-at-a-time into a booth, given a coffee mug, and asked if they would like to trade it for a chocolate bar. Another group of individuals is given a chocolate bar and asked if they would like to trade it for a mug. Under standard neoclassical preferences we would assume that about half of all individuals would prefer to trade. However, List [2004] reports that 81% of his subjects keep an endowed chocolate bar and 77% keep an endowed mug\(^1\). In a related set-up where members of one group are endowed with a consumption good and members of another are endowed with fungible tokens, less trade occurs than is predicted by neoclassical theory [Kahneman et al. 1990]. Buttressed by such results, Kahneman [2003] writes, “Loss-aversion [a formulation of reference dependence

\(^{1}\)These were inexperienced subjects. List [2003; 2004] also finds that experienced traders appear to overcome this status quo bias.
which implies the status quo bias contributes to stickiness in markets, because loss-averse agents are much less prone to exchanges than final-states agents.”

However, in these experiments the opportunity to trade was one-shot, so what was observed was a static endowment effect. The goal of the present paper is to demonstrate that loss aversion (LA) and diminishing sensitivity in gains (DSG), behavioral axioms adopted by Tversky and Kahneman [1991], Munro and Sugden [2003], and K˝ oszegi and Rabin [2006] that imply this static endowment effect and which appear to be robust descriptions of behavior, also imply a dynamic momentum effect that actually increases the propensity to trade.

Imagine there are two individuals, Robinson and Crusoe, who exhibit LA and DSG. Suppose these individuals have identical preferences and value a chocolate bar and a coffee mug equally when they own neither, but in fact Robinson owns a chocolate bar and Crusoe owns a coffee mug. LA and DSG imply that Robinson prefers the chocolate bar, Crusoe prefers the mug, and they will not trade; this is the endowment effect. But now imagine that Robinson owns ten chocolate bars and Crusoe owns ten coffee mugs. The endowment effect will still inhibit exchange, but if convexity of preferences is satisfied we might reasonably expect they will trade a little. So suppose they exchange two coffee mugs for two chocolate bars, and for simplicity let this reallocation be Pareto optimal prior to the exchange taking place. It will be demonstrated in this paper that LA and DSG will cause Robinson to increase his relative preference for coffee mugs and Crusoe his relative preference for chocolate, so that they will benefit from trading again, chocolate for mugs, respectively. This “momentum trading” is robust to both myopic and perfectly anticipated reference dependence, and can be quite powerful. In fact, in an important class of examples momentum trading and the endowment effect are equal but opposing influences on trade, so that reference dependent individuals are not necessarily prone to less exchange as Kahneman suggests.

The intuition for reference dependent momentum trading partially extends to more than two goods. For example, if a reference dependent individual owns a portfolio of goods 1, 2, and 3 and then trades some of good 1 to obtain more of goods 2 and 3, under LA and DSG his preferences will adjust in such a way that he has a stronger preference for new net trades with the same sign as the one he just executed. However, the results in this paper are limited
to the case of two goods because LA and DSG do not permit a characterization of preference adjustment in indirectly affected orthonts of net trade. That is to say, in the example above LA and DSG are silent on what happens to the relative preference for portfolios that involve trading good 2 for goods 1 and 3 after the first exchange.

**A Model of Reference Dependent Behavior**

There exist two perfectly divisible goods, 1 and 2, where a *portfolio* of these goods is an element of $\mathbb{R}^2_+$. Portfolios will be denoted $a$, $r$, $s$, $x$, $y$, and $z$. Preferences $\succeq_r$ depend on reference portfolio $r$ and exist for each $r \in \mathbb{R}^2_+$. Let $I_r(y) \equiv \{z : z \sim_r y\}$, $R_r(y) \equiv \{z : z \succeq_r y\}$, and $P_r(y) \equiv \{z : z \succ_r y\}$ be the indifference and preferred-to sets. The following assumptions are reference dependent analogues to standard microeconomic theory.

**A1** Completeness: For all $r$, $\succeq_r$ is complete.

**A2** Transitivity: For all $r$, $\succeq_r$ is transitive.

**A3** Strict monotonicity: For all $r$, $x$, $y$, if $x_i > y_i$ and $x_j \geq y_j$, then $x \succ_r y$.

**A4** Continuity for a given reference point: For all $r$, $x$, $\{y | y \succeq_r x\}$ and $\{z | x \succeq_r z\}$ are closed.

**A5** Continuity for a change in reference point: For all $x$, $y$, $\{r | x \succeq_r y\}$ is closed.

Tversky and Kahneman [1991] develop three axioms to describe how preferences adjust in response to a change in reference point: Loss aversion (LA), diminishing sensitivity in gains (DSG), and diminishing sensitivity in losses (DSL).\(^2\) Munro and Sugden [2003] show that the WTP/WTA discrepancy and many other regularities associated with reference dependence\(^3\) can be explained entirely by LA and DSG (p. 412); in fact DSL explains no

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\(^2\)These assumptions are related to their namesakes in prospect theory. In this paper these terms will always imply reference dependence over multiple goods and not over lotteries.

\(^3\)Including conservatism, the equivalent gain/loss effects, the inner asymmetric dominance effect, the outer asymmetric dominance, the relative closeness effect, and the advantages/disadvantages effect.
regularities reported in the literature. Further DSL imposes non-convexity of preferences when combined with LA and DSG, a major mathematical cost to tolerate for an assumption with no empirical support. Therefore this paper focuses on characterizing the impact of LA and DSG on exchange, and assumes non-diminishing sensitivity in losses (NDSL).

The assumption of NDSL greatly facilitates analysis, because increasing sensitivity in losses (ISL) complements the impact of LA and DSG while DSL works in the opposite direction. Further, it is demonstrated in Appendix III that all of the results in this paper are robust to some DSL, provided its impact on preferences in the region of potential exchange is everywhere smaller than the combined influence of LA and DSG. The reference dependence axiom in Munro and Sugden (which they label A7) implies LA, DSG, and ISL, and the momentum trading characterized in the present paper is also implied by their model of reference dependence. The model of Kőszegi and Rabin [2006] also implies momentum trading if DSL is replaced in their model by NDSL; demonstration of this fact is presented after Lemma 1. It is of interest to note that Masatlioglu and Uler [2008] directly test four theories of reference dependence, including a special case of LA and DSG, and found that about 90% of subjects behaved consistently with this specification; unfortunately DSL was not tested in the paper because it is difficult to enforce losses in laboratory research.4

**Reference Dependence:** Let \( r, s, x, \) and \( y \) be elements of \( \mathbb{R}_+^2 \). Suppose \( x_1 > y_1, y_2 > x_2 \) and \( r_2 = s_2 \). Consider the change in reference point from \( r \) to \( s \).

**A6** Loss Aversion: If \( x_1 \geq s_1 > r_1 \geq y_1 \), then \( x \sim_r y \Rightarrow x \succ_s y \).

**A7** Diminishing Sensitivity in Gains: If \( y_1 \geq r_1 > s_1 \), then \( x \sim_r y \Rightarrow y \succ_s x \).

**A8** Non-diminishing Sensitivity in Losses: If \( s_1 > r_1 \geq x_1 \), then \( x \sim_r y \Rightarrow x \succeq_s y \).

The assumption of DSL would require \( y \succ_s x \) in A8. To motivate A6, notice that from the perspective of reference point \( r \), \( y \) is relatively advantageous to \( x \) in good 2 and \( x \) is

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4A similar proportion of subjects also behaved consistently with the procedural reference dependent choice model [Masatlioglu and Ok 2007]. Additive constant loss aversion [Tversky and Kahneman 1991], the SQB (status quo bias) model [Masatlioglu and Ok 2005], and the general SQB model [Masatlioglu and Ok 2007] did considerably worse.
relatively advantageous to \( y \) in good 1. Moving from \( r \) to \( s \), the relative advantage of \( x \) in good 1 from the new reference point decreases by exactly the same quantity as the relative disadvantage of \( y \) increases in good 1, while the comparison in good 2 remains the same. Loss aversion implies that the effect of a disadvantage on preferences is greater than the effect of an advantage of identical size. DSG operates through a decreasing relative advantage in good 1 of \( x \) over \( y \). For the sake of completeness, DSL (not assumed in this paper) acts on a decreasing relative disadvantage in good 1 of \( y \) over \( x \).

So what are reference points? In this paper it is assumed that an individual’s reference point is simply his current portfolio; thus his point of reference changes immediately with portfolio adjustment. Bateman, Kahneman, Munro, Starmer and Sugden [2005] specifically test this “current endowment hypothesis” and find reasonably strong support, even though the context was one where Kahneman, in particular, believed this hypothesis was likely to fail. The current endowment hypothesis is also consistent with the above-referenced experimental literature, since preferences in these experiments were elicited almost immediately after a change in subject endowments.\(^5\)

Suppose an individual who obeys A1-8 and whose reference point is his current portfolio acquires some quantity of one good in exchange for some quantity of the other. That is, he begins with portfolio \( x \) and acquires portfolio \( y \), such that either \( x_1 > y_1 \) and \( x_2 < y_2 \), or \( x_1 < y_1 \) and \( x_2 > y_2 \). Without loss of generality, the former is assumed to be the case. It is now shown that the indifference curve through any portfolio \( z \in \mathbb{R}^2_+ \) will pivot counterclockwise given the change in reference point from \( x \) to \( y \).

**Lemma 1** Preference rotation under exchange: Assume A1- A8, and let \( a, a', x, y, z \in \mathbb{R}^2_+ \), with \( x_1 > y_1 \) and \( x_2 < y_2 \). Suppose trade occurs from \( x \) to \( y \). For any \( z \) such that \( z_1 > y_1 \) or \( z_2 > x_2 \), consider \( a \in I_x(z) \) and \( a' \in I_y(z) \) such that \( a'_1 = a_1 \). If \( a_1 < z_1 \),

\(^5\)There is also evidence that subjects do not anticipate their own reference dependence, although myopia is neither assumed nor ruled out in this paper. Loewenstein and Adler [1995] report that subjects do not anticipate that their preferences change when new endowments are adopted. Related literature includes Loewenstein, O’Donoghue and Rabin [2003], Van Boven, Lowenstein and Dunning [2003], and Dhar, Huber and Kahn [2007]. Benartzi and Thaler [1995] use a myopic loss aversion model to explain the equity premium puzzle in one-dimensional stochastic setting.
then $a'_2 < a_2$. If $a_1 > z_1$, then $a'_2 > a_2$. Similarly, if trade occurs from $y$ to $x$, then if $a_1 < z_1$, $a'_2 > a_2$, and if $a_1 > z_1$, then $a'_2 < a_2$.

**Proof.** See Appendix I. □

Thus for any $z \in \mathbb{R}_+^2$ not dominated on both dimensions by both $x$ and $y$ (that is, it is not the case that both $z_1 \leq y_1$ and $z_2 \leq x_2$), if exchange of good 1 for good 2 takes place, the indifference curve through $z$ will rotate counter-clockwise about $z$. If exchange of good 2 for good 1 takes place, the indifference curve through $z$ will rotate clockwise. See Figure 1. Under ISL the indifference curve rotation occurs in the dominated region, as well, while under constant sensitivity in losses (CSL) no rotation takes place in this region. Lemma 2 will rule out the possibility that trade ever occurs in the dominated region.

Let $MRS (a|b)$ be the marginal rate of substitution of good 1 for good 2 at portfolio $a$ given reference point $b$. Lemma 1 is obviously equivalent to the condition that $MRS (z|y) < MRS (z|x)$ for permissible $z$. It is now shown that this condition is satisfied in Köszegi and Rabin [2006] if their model is altered by assuming NDSL rather than DSL. Let $u (a|b) = \sum_{j=1}^{2} m_j (a_j) + \mu (m_j (a_j) - m_j (b_j))$ be the utility of portfolio $a$ given reference point $b$. The
functions $m_j$ and $\mu$ are continuous, strictly monotonically increasing, and twice-differentiable. Kőszegi and Rabin assume $\mu''(f) \leq 0$ for $f > 0$ (weak DSG) and $\mu''(f) \geq 0$ for $f < 0$ (weak DSL). But consider for the moment DSG and NDSL as in the present paper. Given $x$, $y$, and $z$ as in Lemma 1 we have

$$MRS(z|x) = \frac{m_1(z_1)'[1 + \mu'(v_1)]}{m_2(z_2)'[1 + \mu'(v_2)]},$$

where $v_j = m_j(z_j) - m_j(x_j)$ for $j = 1, 2$. By strict monotonicity of $m$ it must be the case that

$$MRS(z|y) = \frac{m_1(z_1)'[1 + \mu'(v_1 + \varepsilon_1)]}{m_2(z_2)'[1 + \mu'(v_2 - \varepsilon_2)]},$$

where $\varepsilon_j > 0$ for $j = 1, 2$. NDSL and DSG imply $\mu'(v_1 + \varepsilon_1) \leq \mu'(v_1)$ and $\mu'(v_2 - \varepsilon_2) \geq \mu'(v_2)$, respectively. At least one of these inequalities will be strict unless $z$ is dominated on both dimensions by both $x$ and $y$. Thus the change in reference point from $x$ to $y$ decreases the marginal rate of substitution at $z$ for all allowable $z$ in Lemma 1. Under the assumption of NDSL (or the weaker condition that LA and DSG everywhere dominate DSL, as discussed in Appendix III), Kőszegi and Rabin, Tversky and Kahneman, and Munro and Sugden all imply the indifference curve rotation characterized in Lemma 1. As will be shown later in this paper, it is this rotation of indifference curves through reference dependence which drives momentum trading.

Sagi [2006] recognizes that for $x_1 > y_1$ and $x_2 < y_2$ where $y \succeq^x x$, under A1-A8 there exists $z$ such that $z_1 < y_1$ and $z_2 > y_2$ and $y >^x z$ but $z >^y y$. While he does not specifically provide a characterization of indifference curve rotation under A1-A8, he undoubtedly anticipates such a result. But rather than explore the consequences of indifference curve rotation through reference dependent preferences as in this paper, he objects to the possibility of such preference reversals and rules them out explicitly with his Axiom 1. He provides several justifications for this objection, both normative and behavioral, where the latter is supported by several manifestations of individual regret. Sagi conveys a broad perspective in summarizing his position, stating that “from a modeling perspective, it seems useful to develop an understanding of the types of models that do and do not satisfy ‘no regret’,” (p.
thus placing his Axiom 1 as a benchmark rather than a necessary condition in the study of reference-dependent preferences. Whether individuals actually produce the preferences reversals implied by A1-A8 is an important empirical question that awaits resolution. In this paper I take up Segi’s challenge and further the understanding of a large class of models that expressly do not satisfy ‘no regret’.

Reference Dependent Exchange

More microeconomic structure is required before considering exchange among agents. It is assumed that preferences relative to any reference point are convex, and that individuals do not engage in utility-diminishing transactions.

A9 (Strict) Convexity of preferences: Suppose \( r, x, y, z \in \mathbb{R}_+^2, x \neq z, \) and \( x, z \in R_r(y) \).

Then for \( \alpha \in (0, 1) \), \( \alpha x + (1 - \alpha)z \in P_r(y) \).

A10 (Weak) Utility improvement: For \( x, y \in \mathbb{R}_+^2 \), if \( x \) is the current portfolio then \( y \) will be voluntarily adopted only if \( y \succeq_x x \).

Let \( x_i^t \in \mathbb{R}_+^2 \) denote the portfolio of individual \( i \) at time \( t \), where \( t \in \{1, 2, \ldots\} \) and \( i \in \{1, 2, \ldots, M\} \) with \( M \) finite. In a convenient abuse of notation, \( M \) will also denote the set of all agents when the context is clear. When necessary, \( x_{j,t}^i \) will denote the quantity of a particular good, \( j = 1, 2 \), owned by \( i \) at time \( t \). Subscripts and superscripts will continue to be suppressed for notational convenience when possible.

There are \( M \geq 2 \) individuals in a pure exchange economy. An allocation \( \mathbf{x} \) is defined as an \( M \)-tuple of portfolios \( [x^1, x^2, \ldots, x^M] \). There is a fixed, finite quantity of goods available in the economy, \( q = (q_1, q_2)^T \in \mathbb{R}_+^{2+} \). Denote by \( \Psi \) the set of feasible allocations, or

\[
\Psi = \left\{ \mathbf{x} \in \mathbb{R}_+^{2M} : \sum_{i=1}^{M} x_j^i \leq q_j, \ j = 1, 2 \right\}.
\]
By A3 and A10 feasibility will always be satisfied with equality. An allocation $x$ which is Pareto optimal relative to reference allocation $r$ is denoted $x \in ps(r)$; if $x$ is also individually rational relative to $r$ it may be denoted $x \in cs(r)$ (i.e., $x$ is in the contract set relative to $r$). The special case when $x$ is Pareto optimal relative to itself is called reflexive Pareto optimality. Formally,

**Definition 1** A feasible allocation $x$ is a reflexive Pareto optimum if there does not exist $y \in \Psi$ such that $y^i \succeq_x x^i$ for each $i \in 1, 2, \ldots, N$ and $y^i \succ_x x^i$ for some $i$.

Consider a sequence of feasible allocations $\langle x_t \rangle$ such that for each $t$, the move from allocation $x_t$ to $x_{t+1}$ satisfies A1-A5 and A9-A10 for all agents. Munro and Sugden impose two additional assumptions on this sequence to guarantee it converges to a reflexive Pareto optimum. One is an innocuous non-triviality condition on exchange. Intuitively, realized gains from trade are not permitted to vanish asymptotically unless available gains from trade do so as well. Specifically, for $x, z \in \Psi$ let $\Phi^i(x, z) = \{y \in \Psi | y^i \succeq_x x^i \land y^i < z^i\}$, with $L(\Phi^i[x, z])$ the Lesbesgue measure of this set. Thus, when $z$ is Pareto-improving from $x$, $L(\Phi^i[x, z])$ is a measure of the gains from trade for agent $i$ in moving from $x$ to $z$.

**A11** Limit non-trivial improvement: If $\sum_{i=1}^{M} L(\Phi^i[x_t, x_{t+1}]) \to 0$ as $t \to \infty$, then $\max_{z \in \Psi} \sum_{i=1}^{M} L(\Phi^i[x_t, z]) \to 0$ as $t \to \infty$.

The other assumption is limit acyclicity of portfolios, which rules out the possibility that an agent prefers to return to an open neighborhood of $x^i$. It is difficult to motivate acyclicity without implicitly assuming that a reference dependent individual is sufficiently forward-looking to not engage in a sequence of utility-improving trades that will leave him where he started, but such anticipatory behavior has not been assumed (or ruled out) in this paper. However, the restriction is crucial to rule out money pumps and guarantee the stability of exchange. Fortunately when there are only two types of goods, including the assumptions A6-A8 (loss aversion, DSG, and NDSL) guarantee a version of limit acyclicity sufficient to obtain convergence to a reflexive Pareto optimum.
Lemma 2  Discrete limit acyclicity of portfolios: Assume A1-A10. Let (x^i_t)_{t=1}^T be a sequence of feasible portfolios for agent i. Suppose d(x^i_t, x^i_{t+1}) > 0 for all t ∈ [1, T − 1], where d(·) is the Euclidean metric. For all z there exists ε > 0 such that if d(x^i_1, z) < ε and d(x^i_T, z) < ε, then x^i_{t+1} ⪰ x^i_t for some t ∈ [1, T − 1].

Proof. See Appendix II. ■

This is a discrete trade version of limit acyclicity in Munro and Sugden, assumption C7**. Assuming A1-A5, A9-A11, and C7**, Munro and Sugden prove exchange must converge to a reflexive Pareto optimum. Since the possibility of continuous exchange is not invoked in their convergence proof, A1-A10 (which imply Lemma 2) are sufficient to guarantee convergence to a reflexive Pareto optimum.

Lemma 2 can be extended to accommodate continuous exchange, as well. It is straightforward to modify the proof of Lemma 2 for a weaker version, weak acyclicity, under continuous exchange: If the sequence (x^i_t) respects A1-A5 and A9-A10, it cannot be the case that x^i_1 ⪰ x^i_T. Munro and Sugden argue this condition is sufficient to guarantee the existence of long-run indifference curves. Recall that I_r(r) is the indifference curve through portfolio r at reference point r. Each point r on a long-run indifference curve has the same local properties as I_r(r). Intuitively, if an individual starts with portfolio r and trades along a continuous path of portfolios to which he is indifferent at each instant, he would trace out his long-run indifference curve through r. Note that I_r(r) is strictly contained within the “preferred-to set” of this long-run indifference curve except at r. Thus if trade is discrete, each reallocation resides on a higher long-run indifference curve than the previous one for each individual, which makes it intuitive why discrete limit acyclicity obtains.

However, under continuous exchange an individual who trades along his long-run indifference curve can trade back to his starting portfolio, violating Lemma 2. But if at least one of the trades in the sequence is discrete, or if a subsequence of trades with positive measure is strictly utility-improving, then the individual will necessarily be on a higher long-run indifference curve bounded away from the original one, making the proof of limit acyclicity for a fixed sequence trivial. Thus convergence to a reflexive Pareto optimum when continuous
trade is permitted requires that such “jumps” to higher long-run indifference curves occur occasionally for all individuals until gains from trade have been exhausted.

**Momentum trading with two agents**

Suppose there are two agents in the economy, and a Pareto-improving trade from \( r \) to \( x \) has occurred as in Figure 2, where \( r_1^1 > x_1^1 \) and \( r_2^1 < x_2^1 \). The inner (light) lens-shaped region represents the individually rational set at \( x \) when the reference point is \( r \), while the dark and light lens-shaped regions together represent the individually rational set at \( x \) after trade has occurred. The adjusted individually rational set contains the original by Lemma 1, so more potential trades which satisfy A10 are available after the shift in preferences. Also note that further exchange must necessarily occur in the same direction as before; that is, agent 1 would give good 1 to agent 2 in exchange for good 2. In fact the entire contract set shifts in the direction of exchange under A1-A10 and the current endowment hypothesis, as will be implied by Proposition 1.
Proposition 1 Momentum Trading (Two Agents): Suppose there are two agents and assume A1-A11. Let $\omega \in \Psi$ be a non-optimal initial endowment. Let $x \in cs (\omega)$, and without loss of generality suppose $x_1^1 < \omega_1^1$ and $x_2^1 > \omega_2^1$. If $y \in cs (x)$ then $y_1^1 < x_1^1$ and $y_2^1 > x_2^1$. Further, for any reflexive Pareto optimum $z$ attainable from $\omega$, there exists an allocation $a \in ps (\omega)$ such that $a_1^1 > z_1^1$ and $a_2^1 < z_2^1$.

Proof. Under the standing assumptions on preferences the Pareto set relative to any feasible reference point is a one-dimensional manifold from origin to origin in the Edgeworth box. From agent 1's origin to agent 2's origin the Pareto set is monotonically increasing in both goods for agent 1 by A3. Since the contract set relative to any feasible reference point is non-empty, $ps (\omega)$ partitions the set of non-Pareto optimal allocations into two sets, $se (\omega)$ and $nw (\omega)$ (southeast and northwest). For $a \in se (\omega)$, $MRS^1 (a^1 | \omega^1) < MRS^2 (a^2 | \omega^2)$. For $b \in nw (\omega)$, $MRS^1 (b^1 | \omega^1) > MRS^2 (b^2 | \omega^2)$. By assumption $MRS^1 (y^1 | x^1) = MRS^2 (y^2 | x^2)$. By Lemma 1, $MRS^1 (y^1 | \omega^1) > MRS^1 (y^1 | x^1)$ and $MRS^2 (y^2 | \omega^2) < MRS^2 (y^2 | x^2)$. Thus $y \in nw (\omega)$, proving the first part of the proposition.

For any direct reallocation $b$ from $\omega$, it is trivial that $b_1^1 < \omega_1^1$ and $b_2^1 > \omega_2^1$. By Lemma 1, $b^i \succ_p \omega^i$ for $i \in \{1, 2\}$. By Lemma 2 exchange can never take place in the weakly positive or weakly negative quadrants relative to $\omega$. By A9 and Lemmas 1 and 2, allocations $b$ such that $b_1^1 > \omega_1^1$ and $b_2^1 < \omega_2^1$ are also ruled out. Thus for any reflexive Pareto optimum $z$ attainable through voluntary exchange, $z_1^1 < \omega_1^1$ and $z_2^1 > \omega_2^1$. Now suppose $z \in se (\omega)$. Then $MRS^1 (z^1 | \omega^1) < MRS^2 (z^2 | \omega^2)$ (this is true for all allocations in $se(\omega)$). But by Lemma 1, $MRS^1 (z^1 | z^1) < MRS^1 (z^1 | \omega^1)$ and $MRS^2 (z^2 | z^2) > MRS^2 (z^2 | \omega^2)$. Thus $MRS^1 (z^1 | z^1) < MRS^2 (z^2 | z^2)$, contradicting the assumption that $z$ is a reflexive Pareto optimum. Therefore $z \in nw(\omega)$, so the second part of the proposition holds.

Thus in a two agent economy at a non-Pareto optimal endowment, the final allocation adopted by reference-dependent individuals reflects a greater volume of exchange than would be expected if identical preferences were fixed at the endowment. For an example of such momentum trading, consider the reference dependent CES function from Munro and Sugden:

$$u(x, r) = Q(r) \left[ \sum_j \gamma_j r^\alpha - \beta x_j^\beta \right]^{1/\beta},$$
Figure 3: Contract Curves, Reference Dependent CES Function

where $x$ and $r$ are an individual’s portfolio and reference point, respectively, and $\sum_j \gamma_j = 1$ and $1 > \rho \geq \beta > -\infty$. Let $Q(r) = 1$, $\rho = 0.75$, $\beta = 0.25$, and $\gamma_1 = \gamma_2 = 0.5$. Suppose both individuals have identical preferences and normalize the aggregate quantity of each good to one. Figure 3 illustrates how trade ‘pushes’ the current Pareto set in the same direction as net exchange. For example, a trade from $a$ to $b$ would shift the current Pareto set up and to the left, from $ps(a)$ to $ps(b)$. A trade from $d$ to $a$ would shift the current Pareto set from $ps(d)$ to $ps(a)$, and so forth.

In the figure $c \in ps(c)$, so $c$ is a reflexive Pareto optimum. In fact $ps(c)$ coincides with the entire reflexive Pareto set, a point that will be verified momentarily. So suppose the economy begins at initial endowment $a$. If the reference point is fixed at $a$, under A10-A11 exchange will eventually converge to some allocation in $ps(a)$. However, under the current endowment hypothesis exchange will converge to an allocation in the reflexive Pareto set $ps(c)$. Clearly $ps(c)$ is further from $a$ than $ps(a)$ in any reasonable sense, so that reference dependence has generated a momentum trading effect.

Of course, this example has so far ignored the static endowment effect. Because at allocation $a$ each agent $i$ possesses mostly good $i$, if his reference point is $a$ his preferences are inclined
towards good \(i\), inhibiting exchange; this is why \(ps(a)\) is so close to \(a\). Thus reference dependence pulls the Pareto set closer to the endowment (the endowment effect), but through trade it also pushes it away (the momentum trading effect). Is it possible to determine if one effect dominates the other? The two effects actually cancel each other out for two identical CES individuals. To see why, consider an individual’s marginal rate of substitution:

\[
\frac{\gamma_1 r_i^{\rho - \beta} x_i^{\beta - 1}}{\gamma_2 r_2^{\rho - \beta} x_2^{\beta - 1}}
\]

Under the current endowment hypothesis, \(r_j = x_j\) for \(j = 1, 2\), so the expression simplifies to \(\gamma_1 x_1^{\rho - 1} / \gamma_2 x_2^{\rho - 1}\). For an allocation to be a reflexive Pareto optimum, it must be the case that the marginal rates of substitution of the two agents are equal at the current allocation; thus \(\gamma_1 x_1^{\rho - 1} / \gamma_2 x_2^{\rho - 1} = \gamma_1 (1 - x_1)^{\rho - 1} / \gamma_2 (1 - x_2)^{\rho - 1}\), an expression that simplifies to \(x_1 = x_2\). This is the reason why \(ps(c)\) in Figure 3 represents the entire reflexive Pareto set.

But what if the individuals share an identical reference point which does not adjust with exchange, so that they only differ in their endowments?\(^6\) The Pareto set relative to a common, fixed reference point is, surprisingly, also characterized by the condition \(x_1 = x_2\) regardless of the reference point adopted; when equating the marginal rates of substitution the common reference points cancel out. Thus the reference independent Pareto set (obtained under a common, static reference point) coincides with the reflexive Pareto set (obtained under the current endowment hypothesis) when agents share identical CES preferences. That is, the endowment effect and the momentum trading effect exactly offset.

Returning to Figure 3 and the current endowment hypothesis, if both individuals have (virtually) nothing then \(ps(c)\) is the (almost entirely infeasible) Pareto set. So if an experimenter provided these individuals with an allocation on \(ps(c)\) at the beginning of an experiment, they could not Pareto-improve upon it. If instead the experimenter endowed the economy at \(a\), then the current endowment hypothesis would cause the Pareto set to shift

\(^6\)To be clear, this is to assume that individuals share a common portfolio as a fixed reference point, not a common allocation in which their individual portfolios potentially differ. It would perhaps be most natural to fix the origin as the reference point (this would certainly be the case in experiments where the entire endowment is assigned by the experimenter), although with CES preferences portfolios and reference points may only get arbitrarily close to the origin.
immediately from \(ps(c)\) to \(ps(a)\), an impact consistent with the endowment effect (that is, the individuals have become more biased towards keeping their endowments). But through the process of exchange, the momentum trading effect will eventually push the final Pareto set back to \(ps(c)\). So we have a two agent economy where reference dependent individuals are not “much less prone to exchanges than final states agents” as claimed by Kahneman; they can be expected to generate the same volume of exchange.

**Momentum trading with many agents**

Let \(\omega\) be the initial endowment and suppose \(x \in ps(\omega)\) is strictly interior. Further suppose that in moving from \(\omega\) to \(x\) each agent must give up some quantity of one good in exchange for the other; i.e., there is full participation and no free lunch in moving to \(x\). Let \(B\) be the set of agents such that for all \(i \in B\), \(x_i^1 - \omega_i^1 > 0\) (“buyers”). Let \(S\) be the set of agents such that for all \(j \in S\), \(x_j^1 - \omega_j^1 < 0\) (“sellers”). By assumption, \(B \cup S = M\). It is now shown there exists an open set of reflexive Pareto optima attainable through Pareto-improving trade such that for any allocation in this set, on average buyers remain buyers and sellers remain sellers.

**Proposition 2** Momentum Trading (Existence): Assume A\(_1\)-A11 for all \(k \in M\). Let \(\omega \in \Psi\) be a non-optimal initial endowment. Let \(x \in ps(\omega)\) be a strictly interior feasible Pareto optimum relative to \(\omega\) such that \((x_k^1 - \omega_k^1) \cdot (x_k^2 - \omega_k^2) < 0\) for all \(k \in M\). Then there exists an open set of reflexive Pareto optima \(Z\) such that for each \(z \in Z\), \(\text{sgn} \sum_{k \in A} (z^k - x^k) = \text{sgn} \sum_{k \in A} (x^k - \omega^k)\) for \(A \in \{B, S\}\) (here \(\text{sgn}\) is the signum function).

**Proof.** Let \(x_{s(t)}\) be an allocation at time \(t\) and stage \(s\), where \(x_{0(0)} = \omega\) and \(x_{0(1)} = x\). For each stage \(s\), a sequence \(\langle x_{s(t)}(t)\rangle_{t=0}^\infty\) of Pareto-improving trades is constructed such that \(\lim_{t \to \infty} x_{s(t)} = x_{s+1(0)}\) is a reflexive Pareto optimum relative to a subset of agents, and \(\text{sgn} \sum_{k \in A} (x_{s+1(0)}^k - x^k) = \text{sgn} \sum_{k \in A} (x^k - \omega^k)\) for \(A \in \{B, S\}\). In each successive stage this subset will strictly contain the previous one, and since \(M\) is finite there exists a stage where the limiting allocation is a reflexive Pareto optimum.
Consider the reallocation from \( x_{0(0)} \) to \( x_{0(1)} \). \( MRS_i \left( \frac{x_{0(1)}^i}{x_{0(0)}^i} \right) = MRS_j \left( \frac{x_{0(1)}^j}{x_{0(0)}^j} \right) \) for all \( i, j \in M \) by assumption. By Lemma 1, \( MRS_i \left( \frac{x_{0(1)}^i}{x_{0(0)}^i} \right) > MRS_j \left( \frac{x_{0(1)}^j}{x_{0(0)}^j} \right) \) for all \( i \in B, j \in S \). Let \( p_{s(t)} = \max_{j \in S} MRS^j \left( \frac{x_{s(t)}^j}{x_{s(t)}^j} \right) \) and \( \bar{p}_{s(t)} = \min_{i \in B} MRS^a \left( \frac{x_{s(t)}^i}{x_{s(t)}^i} \right) \).

Letting good 2 be the numéraire, at each time \( t \) of stage \( s \) a price \( p_{s(t)} \) will be chosen such that \( p_{s(t)} \in \left( p_{s(t)}, \bar{p}_{s(t)} \right) \). Let the long-run demand for \( k \in M \) be the set of locally optimal portfolios on the budget line defined by \( x_{s(t)}^k \) and price \( p_{s(t)} \) (if long-run indifference sets are strictly convex this portfolio is unique). Let \( y_{0(1)}^k \) be the element of long-run demand where \( sgn \left( y_{0(1)}^k - x_{0(0)}^k \right) = sgn \left( x_{0(1)}^k - x_{0(0)}^k \right) \) and net trade is minimized; such a portfolio exists trivially by Lemma 1 and the relationship between \( p_{0(1)} \) and \( k \)'s marginal rate of substitution. If \( \sum_{k \in M} y_{0(1)}^k = q \), \( y_{0(1)} \) is a reflexive Pareto optimum and the proposition holds. If not, without loss of generality assume good 1 is in excess demand (by Walras’ Law we need only consider good 1).

For all \( j \in S \), let \( x_{0(2)}^j = y_{0(1)}^j \). For interior \( j \), \( MRS^j \left( \frac{x_{0(2)}^j}{x_{0(0)}^j} \right) = p_{0(1)} \). For \( j \) possessing only good 2, \( MRS^j \left( \frac{x_{0(2)}^j}{x_{0(0)}^j} \right) < p_{0(1)} \). Let \( \alpha_s(t) < 1 \) be the ratio of aggregate supply to demand of good 1 at stage \( s \) and time \( t \). Then let \( x_{0(2)}^i = x_{0(1)}^i + \alpha_{0(1)} \left[ y_{0(1)}^i - x_{0(0)}^i \right] \) for all \( i \in B \), so that \( x_{0(2)} \) is a feasible reallocation. By Lemma 1, \( MRS^i \left( \frac{x_{0(2)}^i}{x_{0(0)}^i} \right) > p_{0(1)} \) for all \( i \in B \). If all \( j \in S \) have corner portfolios the stage is complete. Otherwise, by assumption \( p_{0(2)} \in (p_{0(1)}, \bar{p}_{0(2)}) \). Choose \( p_{0(2)} \) so good 1 remains in excess demand; such a price exists since the aggregate excess demand of good 1 is strictly positive as \( p_{0(2)} \) approaches \( p_{0(1)} \).

Construct the sequence \( \langle x_{0(t)} \rangle \) by iterating this procedure, appropriately restricting prices so that good 1 is in excess demand at every reallocation. One of two things occurs. At some finite time \( T \), all \( j \in S \) only possess good 2, in which case \( x_{1(0)} = x_{0(T)} \). Or at \( x_{1(0)} = \lim_{t \to \infty} x_{0(t)} \), some non-empty subset of agents \( B_1 \subseteq B \) has a marginal rate of substitution equal to the limiting price in stage 0. If the former case is true, let \( B_1 = \arg \min_{i \in B} \left[ MRS^i \left( \frac{x_{1(0)}^i}{x_{1(0)}^i} \right) \right] \).

If \( B_1 = B \), a reflexive Pareto optimum has been reached. If not, stage 1 begins at allocation \( x_{1(0)} \) (the final or limiting allocation of stage 0). The sellers in stage 1 are now \( S \cup B_1 \) (note \( B_1 \subseteq B \) is necessarily non-empty), and \( p_{1(1)} \) is chosen such that there is excess demand at \( y_{1(1)} \). Stage 1 proceeds as stage 0, culminating in \( B_2 \supseteq B_1 \), a subset of buyers through stages.
0 and 1 who have become sellers. This stage process is iterated so that in each stage \(s + 1\), \(x_{s+1}(0)\) is the final or limiting allocation of stage \(s\). All \(i \in S \cup B_{s+1}\) either share the same marginal rate of substitution at \(x_{s+1}(0)\) or have zero of good 1 and a smaller MRS. At some finite stage \(T\), \(B_{T+1} = B\). Thus \(x_{T+1}(0)\) (the final or limiting allocation of stage \(T\)) is by necessity a reflexive Pareto optimum. ■

Therefore A1-A11 guarantee the existence of an open set of attainable reflexive Pareto optima at which momentum trading would be realized on average. However, more structure on preferences is necessary to guarantee momentum trading with more than two agents.

**Proposition 3** Momentum Trading (Sufficient Condition): Assume A1-A11 and further suppose that preferences are homothetic relative to any reference point for all \(k \in M\).

Let \(\omega \in \Psi\) be a non-optimal initial endowment. Let \(x \in ps(\omega)\) be a feasible Pareto optimum relative to \(\omega\) such that \((x^k_1 - \omega^k_1) \times (x^k_2 - \omega^k_2) < 0\) for all \(k \in M\). Then for all \(z \in ps(x)\) where \((z^k_1 - x^k_1) \times (z^k_2 - x^k_2) < 0\) for all \(k \in M\), \(\text{sgn} \sum_{k \in A} (z^k - x^k) = \sum_{k \in A} (x^k - \omega^k)\) for \(A \in \{B, S\}\).

**Proof.** Partition \(i \in B\) into subsets \(B_1\) and \(B_2\), where \(z^i_1 > x^i_1\) for all \(i \in B_1\), and \(z^i_2 < x^i_1\) for all \(i \in B_2\). That is, \(B_1\) is the set of good 1 buyers from \(\omega\) to \(x\) who remain buyers from \(x\) to \(z\), and \(B_2\) is the set of buyers who become sellers. Similarly, partition \(j \in S\) into subsets \(S_1\) and \(S_2\) where \(z^j_1 < z^j_2\) for all \(j \in S_1\), and \(z^j_2 > z^j_1\) for all \(j \in S_2\) (\(S_1\) is the set of sellers who remain sellers, \(S_2\) is the set of sellers who become buyers). Let \(p = MRS^k(x^k|\omega^k)\) for all \(k \in M\).

For all \(i \in B_2\), \(MRS^i(x^i|x^i) > p\) by Lemma 1. By homotheticity, \(MRS^i(z^i|x^i) > MRS^i(x^i|x^i)\), so \(MRS^i(z^i|x^i) > p\) for all \(i \in B_2\). Similarly, for all \(j \in S_2\), \(MRS^j(x^j|x^j) < p\) by Lemma 1. By homotheticity, \(MRS^j(z^j|x^j) < MRS^j(x^j|x^j)\), so \(MRS^j(z^j|x^j) < p\) for all \(j \in S_2\). Thus \(z \notin ps(x)\) unless \(B_2\) or \(S_2\) (or both) are empty. Without loss of generality, suppose \(S_2 = \emptyset\). The proposition trivially holds for \(S\). Since \(\sum_{i \in B_1}(z^i - x^i) = \sum_{i \in B_2}(z^i - x^i) + \sum_{j \in S_1}(z^j - x^j)\), the proposition holds for \(B\) as well. ■

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Corollary 1 Assume A1-A11 and further suppose that long-run preferences are strictly convex and homothetic relative to any reference point for all $k \in M$. Let $\omega \in \Psi$ be a non-optimal initial endowment. Let $x \in ps(\omega)$ be a feasible Pareto optimum relative to $\omega$ such that $(x^k_1 - \omega^k_1) \times (x^k_2 - \omega^k_2) < 0$ for all $k \in M$. Then for all reflexive Pareto optima $z$ where $(z^k_1 - x^k_1) \times (z^k_2 - x^k_2) < 0$ for each $k \in M$, $\text{sgn} \sum_{k \in A} (z^k - x^k) = \text{sgn} \sum_{k \in A} (x^k - \omega^k)$ for $A \in \{B, S\}$.

Proof. Obvious from the proof of Proposition 3. ■

It is worth noting that Munro and Sugden show long-run preferences are strictly convex and homothetic for the reference dependent CES function. Therefore, for this class of preferences we know by Corollary 1 that momentum trading must necessarily occur.

Conclusion

This paper delivers an unexpected result. In addition to implying the familiar endowment effect, loss aversion and diminishing sensitivity in gains also imply a competing momentum effect which can be large enough to erase the bias against exchange associated with the endowment effect. This momentum effect is derived from assumptions with strong support in the literature, and does not conflict with any of the observed reference dependent empirical regularities. It is important to emphasize that myopia was neither assumed nor ruled out to obtain the results. Provided that agents do not trade outside of their current individually rational sets and eventually exhaust all gains from trade, any degree of anticipatory rationality is permitted.

Why has momentum trading not been observed in the experimental economics literature? One reason is that goods are insufficiently divisible in most of these experiments, but more importantly preferences have always been elicited before and after one change in subject endowments, rather than during several changes in sequence. Interesting paths for future laboratory research include eliciting preferences over sequences of endowments, and testing the diminishing versus increasing sensitivity in losses hypotheses.
It is tempting to relate the momentum trading effect studied in this paper to the growing literature on momentum trading in finance. For example, Barber and Odean [2000] document that individual investors tend to lose money by over-trading in equity markets. However, it is not clear the mechanisms that generate these two different momentum trading results are at all the same; for example, over-confidence and herding are often expected to play a large role in the finance literature on momentum trading.

Much closer to the momentum trading story in this paper, Dhar et al. [2007] study the ‘shopping momentum effect.’ Consumers who choose to purchase an item in a department store are shown to be more likely to purchase a second item, after controlling for desirable aggregation of purchases to offset time and travel costs. “Shopping momentum arises from the idea that shopping has an inertial quality . . . which once crossed makes further purchases more likely.” (p. 3) Potentially benefitting from such shopping momentum, retailers use loss leaders to attract customers to a store in the expectation that sales of other goods will increase [Mulhern and Padgett 1995]. If one were to treat department store purchases as one type of good and cash as a composite second good, this observed shopping momentum effect is consistent with the momentum effect characterized in this paper. In any event, momentum exchange in a multi-attribute deterministic setting has been empirically observed and may potentially imply real economic consequences. The characterization of momentum trading in the present paper may usefully inform the exploration of this and potentially other such behavior.
Appendix I - Proof of Lemma 1

First note that if Lemma 1 is established for a change in reference point from \(x\) to \(y\), then the consequences of a change from \(y\) to \(x\) are trivial, since the two sets \(I_x(z)\) and \(I_y(z)\) will already have been sufficiently characterized. It will be useful to introduce the portfolio \(s = (y_1, x_2)\).

\[ 
\begin{array}{c|c|c}
\text{Good 1} & 1 & 2 \\
\hline
4 & 5 & 6 \\
\hline
7 & 8 & 9 \\
\end{array}
\]

\[ 
\begin{array}{c|c|c}
\text{Good 2} & 1 & 2 \\
\hline
3 & 4 & 5 \\
\hline
6 & 7 & 8 \\
\end{array}
\]

Figure 4: Distinct Regions of Reference Dependence

The strategy for the proof is as follows. In \textbf{Step A} it will be shown that for a change in reference point from \(x\) to \(s\), for any \(z\) such that \(z_1 > y_1\) the indifference curve through \(z\) will rotate counter-clockwise; this corresponds to \(z\) in regions 2, 3, 5, 6, 8, and 9 in Figure 4. It will also be shown that under ISL the proposition holds for a change in reference point from \(x\) to \(s\) for any \(z\) such that \(z_1 \leq y_1\) (regions 1, 4, and 7), while under CSL the preference relation between any two portfolios in these regions is unaffected. Now fix the indifference map from reference point \(s\).

In \textbf{Step B} it will be shown that for the new indifference map defined in \textbf{Step A}, a change in reference point from \(s\) to \(y\) will rotate the indifference curve through \(z\) counter-clockwise for any \(z\) such that \(z_2 > x_2\); this corresponds to \(z\) in regions 1, 2, 3, 4, 5, and 6 in Figure 4.
Under ISL this rotation also occurs in regions 7, 8, and 9, while under CSL the preference relation between any two portfolios in these regions is unaffected. It will then have been demonstrated that the proposition holds for \( z \) in any region but 7 in Figure 4, finishing the proof. If ISL is assumed, the proposition holds for \( z \) in region 7, as well. Under very mild rationality restrictions no sequence of trades will ever take the individual into region 7, so the fact that Lemma 1 does not hold in this region under CSL is of little consequence.

**Step A: Change in reference point from \( x \) to \( s \)**

There are three cases to consider, \( z_1 \geq x_1, z_1 \in (y_1, x_1) \), and \( z_1 \leq y_1 \).

**Case A1: \( z_1 \geq x_1 \).** This corresponds to regions 3, 6, and 9 in Figure 4.

**Subcase A1a: \( a_1 < z_1 \).**

- Suppose \( a_1 \geq x_1 \). A7 implies \( a \succ_s z \).
- Suppose \( a_1 \in (y_1, x_1) \). A6 implies \( a \succ_{(a_1, x_2)} z \). By A3, A4 and IVT, \( \exists \alpha > 0 : z \sim_{(a_1, x_2)} (a_1, a_2 - \alpha) \). By A7, \( (a_1, a_2 - \alpha) \succ_s z \).
- Suppose \( a_1 \leq y_1 \). A6 implies \( a \succ_s z \).

Thus, for all \( a_1 < z_1 \), by A3, A4 and IVT, \( \exists \varepsilon > 0 : z \sim_s (a_1, a_2 - \varepsilon) \).

**Subcase A1b: \( a_1 > z_1 \).** Since \( a_1 > z_1 \geq x_1 \), A7 implies \( z \succ_s a \). By A3, A4 and IVT, \( \exists \delta > 0 : z \sim_s (a_1, a_2 + \delta) \).

**Case A2: \( z_1 \in (y_1, x_1) \).** This corresponds to regions 2, 5, and 8 in Figure 4.

**Subcase A2a: \( a_1 < z_1 \).** A8 implies \( a \succeq_{(z_1, x_2)} z \). By A3, A4 and the intermediate value theorem (IVT), \( \exists \alpha > 0 : z \sim_{(z_1, x_2)} (a_1, a_2 - \alpha) \).

- Suppose \( a_1 \in (y_1, z_1) \). A6 implies \( (a_1, a_2 - \alpha) \succ_{(a_1, x_2)} z \). By A3, A4 and IVT, \( \exists \beta > \alpha : z \sim_{(a_1, x_2)} (a_1, a_2 - \beta) \). A7 implies \( (a_1, a_2 - \beta) \succ_s z \).

---

7Actually, A7 implies that a change in reference point from \( s \) to \( x \) strictly favors \( z \) relative to \( a \). Since reference dependence is assumed to be fixed rather than path dependent, then the change in reference point from \( x \) to \( s \) must strictly favor \( a \) relative to \( z \).
Suppose $a_1 \leq y_1$. A6 implies $(a_1, a_2 - \alpha) \succ_s z$.

Thus, for all $a_1 < z_1$, by A3, A4 and IVT, $\exists \varepsilon > 0 : z \sim_s (a_1, a_2 - \varepsilon)$.

**Subcase A2b: $a_1 > z_1$.**

- Suppose $a_1 \geq x_1$. A6 implies $z \succ_{(z_1, x_2)} a$. By A3, A4 and IVT, $\exists \alpha > 0 : z \sim_{(z_1, x_2)} (a_1, a_2 + \alpha)$. By A7, $z \succ_s (a_1, a_2 + \alpha)$.

- Suppose $a_1 \in (z_1, x_1)$. A8 implies $z \succeq_{(a_1, x_2)} a$. By A3, A4 and IVT, $\exists \beta \geq 0 : z \sim_{(a_1, x_2)} (a_1, a_2 + \beta)$. A6 implies $z \succ_{(z_1, x_2)} (a_1, a_2 + \beta)$. By A3, A4 and IVT, $\exists \gamma > \beta : z \sim_{(z_1, x_2)} (a_1, a_2 + \gamma)$. A7 implies $z \succ_s (a_1, a_2 + \gamma)$.

Thus, for all $a_1 > z_1$, by A3, A4 and IVT, $\exists \delta > 0 : z \sim_s (a_1, a_2 + \delta)$.

**Case A3: $z_1 \leq y_1$.** This corresponds to regions 1, 4, and 7 in Figure 4.

**Subcase A3a: $a_1 < z_1$.** A8 implies $a \succeq_s z$. By A3, A4 and IVT, $\exists \varepsilon \geq 0 : z \sim_s (a_1, a_2 - \varepsilon)$. Note $\varepsilon > 0$ under ISL, and $\varepsilon = 0$ under CSL.

**Subcase A3b: $a_1 > z_1$.**

- Suppose $a_1 \leq y_1$. A8 implies $z \succeq_s a$. By A3, A4 and IVT, $\exists \delta \geq 0 : z \sim_s (a_1, a_2 + \delta)$. Note $\delta > 0$ under ISL, and $\delta = 0$ under CSL.

- Suppose $a_1 \in (y_1, x_1)$. A8 implies $z \succeq_{(a_1, x_2)} a$. By A3, A4 and IVT, $\exists \alpha \geq 0 : z \sim_{(a_1, x_2)} (a_1, a_2 + \alpha)$. A6 implies $z \succ_{s} (a_1, a_2 + \alpha)$.

- Suppose $a_1 \geq x_1$. A6 implies $z \succ_s a$. By A3, A4 and IVT, $\exists \delta > 0 : z \sim_s (a_1, a_2 + \delta)$.

To summarize, take any $z \in \mathbb{R}_+^2$, and suppose $z \sim_x a$. It was demonstrated in Step A that for $a_1 < z_1$, there exists $\varepsilon \geq 0$ such that $z \sim_s (a_1, a_2 - \varepsilon)$, and for $a_1 > z_1$, there exists $\delta \geq 0$ such that $z \sim_s (a_1, a_2 + \delta)$. Further, if $z_1 \leq y_1$, $a_1 \leq y_1$, and CSL is assumed, then $\varepsilon = \delta = 0$. If at least one of these conditions is not met, then $\varepsilon > 0$ and $\delta > 0$.  

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Step B: Change in reference point from $s$ to $y$

**Case B1**: $z_2 \geq y_2$. This corresponds to regions 1, 2, and 3 in Figure 4.

**Subcase B1a**: $a_1 < z_1$. By A3, $a_2 - \varepsilon > z_2 \geq y_2$. Therefore, A7 implies $(a_1, a_2 - \varepsilon) \succ_y z$. Since $\varepsilon \geq 0$, by A3, A4, and IVT, $\exists a' \in I_y(z)$ such that $a'_1 = a_1$ and $a'_2 < a_2$.

**Subcase B1b**: $a_1 > z_1$.

- Suppose $a_2 + \delta \leq x_2$. A6 implies $z \succ_y (a_1, a_2 + \delta)$.
- Suppose $a_2 + \delta \in (x_2, y_2)$. A7 implies $z \succ_{(y_1, a_2 + \delta)} (a_1, a_2 + \delta)$. By A3, A4 and IVT, $\exists \alpha > 0 : z' = (z_1, z_2 - \alpha) \sim_{(y_1, a_2 + \delta)} (a_1, a_2 + \delta)$.
  - If $z'_2 \geq y_2$, A6 implies $z' \succ_y (a_1, a_2 + \delta)$, and thus by A3, $z \succ_y (a_1, a_2 + \delta)$.
  - If $z'_2 < y_2$, A6 implies $z' \succ_{(y_1, z'_2)} (a_1, a_2 + \delta)$. By A3, A4 and IVT, $\exists \beta > \alpha : z'' = (z_1, z_2 - \beta) \sim_{(y_1, z'_2)} (a_1, a_2 + \delta)$. By A8, $z'' \succeq_y (a_1, a_2 + \delta)$. Thus by A3, $z \succ_y (a_1, a_2 + \delta)$.

- Suppose $a_2 + \delta \geq y_2$. A7 implies $z \succ_y (a_1, a_2 + \delta)$.

Since $\delta \geq 0$, by A3, A4, and IVT, $\exists a' \in I_y(z)$ such that $a'_1 = a_1$ and $a'_2 > a_2$.

**Case B2**: $z_2 \in (x_2, y_2)$. This corresponds to regions 4, 5, and 6 in Figure 4.

**Subcase B2a**: $a_1 < z_1$. A7 implies $(a_1, a_2 - \varepsilon) \succ_{(y_1, z_2)} z$. By A3, A4 and IVT, $\exists \alpha > 0 : (a_1, a_2 - \varepsilon) \sim_{(y_1, z_2)} z' = (z_1, z_2 + \alpha)$.

- If $a_2 - \varepsilon \geq y_2$, then A6 implies $(a_1, a_2 - \varepsilon) \succ_y z'$, and by A3, $(a_1, a_2 - \varepsilon) \succ_y z$.
- If $a_2 - \varepsilon < y_2$, then A6 implies $(a_1, a_2 - \varepsilon) \succ_{(y_1, a_2 - \varepsilon)} z'$. By A3, A4 and IVT, $\exists \beta > \alpha : (a_1, a_2 - \varepsilon) \sim_{(y_1, a_2 - \varepsilon)} z'' = (z_1, z_2 + \beta)$. A8 implies $(a_1, a_2 - \varepsilon) \succeq_y z''$. Thus by A3, $(a_1, a_2 - \varepsilon) \succ_y z$.

Since $\varepsilon \geq 0$, by A3, A4, and IVT, $\exists a' \in I_y(z)$ such that $a'_1 = a_1$ and $a'_2 < a_2$.

**Subcase B2b**: $a_1 > z_1$.
Case B3: $z_2 \leq x_2$ and $z_1 > y_1$. This corresponds to regions 8 and 9 in Figure 4. Since $z_1 > y_1$, it was established in Step A that $\varepsilon > 0$ and $\delta > 0$.

Subcase B3a: $a_1 < z_1$.

- Suppose $a_2 + \delta \leq x_2$. A6 implies $z \succ_{(y_1, x_2)} (a_1, a_2 + \delta)$. By A3, A4 and IVT, $\exists \alpha > 0 : z' = (z_1, z_2 - \alpha) \sim_{(y_1, z_2)} (a_1, a_2 + \delta)$. By A8, $z' \succeq_y (a_1, a_2 + \delta)$. Then A3 implies $z \succ_y (a_1, a_2 + \delta)$.

- Suppose $a_2 + \delta \in (x_2, z_2)$. A7 implies $z \succ_{(y_1, a_2 + \delta)} (a_1, a_2 + \delta)$. By A3, A4 and IVT, $\exists \beta > 0 : (a_1, a_2 + \delta) \sim_{(y_1, a_2 + \delta)} z'' = (z_1, z_2 - \beta)$. By A6, $z'' \succ_{(y_1, z_2)} (a_1, a_2 + \delta)$. By A3, A4 and IVT, $\exists \gamma > \beta : (z_1, z_2 + \gamma) \sim_{(y_1, z_2)} (a_1, a_2 + \delta)$. A8 implies $(z_1, z_2 + \gamma) \succ_y (a_1, a_2 + \delta)$. Thus by A3, $z \succ_y (a_1, a_2 + \delta)$.

Since $\delta > 0$, by A3, A4, and IVT, $\exists a' \in I_y(z)$ such that $a'_1 = a_1$ and $a'_2 > a_2$.

Subcase B3b: $a_1 > z_1$. By A8, $z \succeq_y (a_1, a_2 + \delta)$. Since $\delta > 0$, by A3, A4, and IVT, $\exists a' \in I_y(z)$ such that $a'_1 = a_1$ and $a'_2 > a_2$.

Case B4: $z_2 \leq x_2$ and $z_1 \leq y_1$. This corresponds to region 7 in Figure 4.

Subcase B4a: $a_1 < z_1$. Note from Step A that $\varepsilon = 0$ in this circumstance.

- Suppose $a_2 \geq y_2$. By A6, $a \succ_y z$. By A3, A4, and IVT, $\exists a' \in I_y(z)$ such that $a'_1 = a_1$ and $a'_2 < a_2$.
• Suppose \( a_2 \in (x_2, y_2) \). By A6, \( a \succeq (y_1, a_2) \). By A3, A4, and IVT, \( \exists \alpha > 0 \) such that \( a \sim_{(y_1, a_2)} z' = (z_1, z_2 + \alpha) \). Note by A3 \( z'_2 < a_2 \). By A8, \( a \succeq y z' \), so by A3, \( a \succ y z \). By A3, A4, and IVT, \( \exists a' \in I_y(z) \) such that \( a'_1 = a_1 \) and \( a'_2 < a_2 \).

• Suppose \( a_2 \leq x_2 \). By A8, \( a \succeq y z \). Under ISL, \( a \succ y z \), so by A3, A4, and IVT, \( \exists a' \in I_y(z) \) such that \( a'_1 = a_1 \) and \( a'_2 < a_2 \). Under CSL, the individual is indifferent between \( z \) and \( a \).

**Subcase B4a: \( a_1 > z_1 \).**

• Suppose \( a_1 > y_1 \). By A8, \( z \succeq a \). From **Step A** \( \delta > 0 \), so by A3, \( z \succeq y a \). By A3, A4, and IVT, \( \exists a' \in I_y(z) \) such that \( a'_1 = a_1 \) and \( a'_2 > a_2 \).

• Suppose \( a_1 \leq y_1 \). From **Step A** \( \delta = 0 \). Under ISL \( z \succ y a \), so by A3, A4, and IVT, \( \exists a' \in I_y(z) \) such that \( a'_1 = a_1 \) and \( a'_2 > a_2 \). Under CSL, the individual is indifferent between \( z \) and \( a \) (subject to the caveat noted in the previous footnote).

\[\square\]

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\(^8\)In fact to maintain A1-5, for some \( a \) it is necessary for \( a \succ y z \). This characterization is not relevant to the proposition, so it is not presented in more detail.
Agent superscripts have been suppressed. The contrapositive of the proposition will be proven; that is, for any sequence \( \langle x_t \rangle \), if \( x_{t+1} \geq x_t \) for all \( t \in [1, 2, \ldots, T - 1] \) then for all \( y \) there exists \( \varepsilon > 0 \) such that \( d(x_1, y) > \varepsilon \) or \( d(x_T, y) > \varepsilon \). Without loss of generality assume \( x_{t+1} \neq x_t \) for \( t \in [1, T - 1] \), to avoid introducing notation for the maximal subsequence of \( \langle x_t \rangle_{t=1}^T \) such that \( x_{t+1} \neq x_t \). Further suppose \( x_{1,2} < x_{1,1} \) and \( x_{2,2} > x_{2,1} \). This assumption is also without loss of generality; if the individual trades good 2 for good 1 in the first period the argument is symmetric. If instead \( x_{1,2} \geq x_{1,1} \) and \( x_{2,2} \geq x_{2,1} \) (at least one inequality must be strict by assumption), if there exists a neighborhood of \( x_2 \) to which the subsequence \( \langle x \rangle_{t=2}^T \) cannot return, then it is trivial there exists a neighborhood of \( x_1 \) to which the full sequence cannot return.

First consider a one-directional sequence of exchange, so that \( x_{1,t+1} < x_{1,t} \) and \( x_{2,t+1} > x_{2,t} \) for \( t \in [1, T - 1] \). Since exchange is discrete, \( d(x_1, x_T) > 0 \) by assumption. It will be useful later to show here that the minimum distance between \( x_1 \) and \( R_{x_T}(x_T) \), the weakly preferred-to set of \( x_T \) from \( x_T \), is also greater than zero. By A10, \( x_{t+1} \succeq x_t \) for all \( t \). By Lemma 1, \( x_{t+1} \succ_{x_{t+1}+m} x_t \) for \( m \geq 0 \). Then since \( x_{t+1} \succ_{x_T} x_t \) for \( t \in [1, T - 1] \), by A2 \( x_T \succ_{x_T} x_1 \). By A9, \( R_{x_T}(x_T) \) is convex. Let \( y(a) = (1 - a)x_1 + ax_T \) for \( a \in [0, 1] \). By Lemma 1, A4, and A5 it is straightforward to show that \( d\left(x_1, R_{y(a)}(y(a))\right) \) is strictly and continuously increasing in \( a \), so \( d(x_1, R_{x_T}(x_T)) > 0 \).

Now consider one-directional exchange through period \( k \in [2, T - 1] \), but in period \( k + 1 \) exchange is only restricted by A1-A10. There are four possibilities to consider. Case 1: \( x_{1,k+1} < x_{1,k} \); Case 2: \( x_{1,k+1} \in [x_{1,k}, x_{1,1}] \); Case 3: \( x_{1,k+1} \geq x_{1,1} \) and \( x_{2,k+1} \geq x_{2,1} \); Case 4: \( x_{1,k+1} > x_{1,1} \) and \( x_{2,k+1} < x_{2,1} \). In each case it is demonstrated that there exists a one-directional subsequence of the portfolios, starting with \( t = 1 \) and ending with \( t = k + 1 \) such that each portfolio is preferred to its predecessor (from its predecessor) and \( x_{k+1} \) is strictly preferred to \( x_1 \). Further, it will be shown that \( d\left(x_1, R_{x_{k+1}}(x_{k+1})\right) > 0 \). Since this process can be repeated for all \( t \), then \( d(x_T, x_1) > 0 \). Thus for all \( z \) there exists \( \varepsilon > 0 \) such that an \( \varepsilon \)-ball centered on \( z \) does not contain both \( x_1 \) and \( x_T \), proving the lemma.
Case 1: $x_{1,k+1} < x_{1,k}$. By A3 and A10 it must be the case that $x_{2,k+1} > x_{2,k}$. But then it has already been established that $(x)_{t=1}^{k+1}$ is one-directional and $d(x_1, R_{x_{k+1}}(x_{k+1})) > 0$.

Case 2: $x_{1,k+1} \in [x_{1,k}, x_{1,1})$. Let $m$ be the minimum $m \in [1, k - 1]$ such that $x_{k+1} \succeq x_m$. That $m$ exists is trivial, since $x_{k+1} \succ x_{k-1}$ by Lemma 1. Claim: It must be the case that $x_{1,k+1} < x_{1,m}$. If $m = 1$, this is true by assumption. So suppose $m > 1$ and $x_{1,k+1} \geq x_{1,m}$. By A10, $x_m \succeq x_m^{-1}$, $x_{m-1}$. By Lemma 1, $x_{k+1} \succ x_{m-1}$, $x_m$. Then by A2, $x_{k+1} \succ x_{m-1}$, $x_{m-1}$, violating the assumption that $m$ is a minimum. Thus $x_{1,k+1} < x_{1,m}$ and $x_{2,k+1} > x_{2,m}$ by A3, and $x_{k+1} \succ x_{k+1}$, $x_m$. Since $d(x_{m}, x_{k+1}) > 0$, by Lemma 1, A4, and A5, $d(x_m, R_{x_{k+1}}(x_{k+1})) > 0$. But then it is without loss of generality to prove the proposition for the subsequence $(x_1, x_m, x_{k+1}, x_{k+2}, \ldots, x_T)$.

Case 3: $x_{1,k+1} \geq x_{1,1}$ and $x_{2,k+1} \geq x_{2,1}$. One of these conditions must hold with strict inequality, since $x_k \succ x_1$. Then it suffices to restrict attention to the subsequence $(x_t)_{t=k+1}^{T}$, because if there exists a neighborhood of $x_{k+1}$ to which this subsequence cannot return, then trivially there exists a neighborhood of $x_1$ to which the full sequence does not return. But then we are back to consideration of an initially one-directional sequence.

Case 4: $x_{1,k+1} > x_{1,1}$ and $x_{2,k+1} < x_{2,1}$. By A3, A4, and IVT, there exists $\alpha > 0$ such that $x_1 \sim x_k$ $(x_{1,k+1}, x_{2,k+1} - \alpha)$. By Lemma 1, $(x_{1,k+1}, x_{2,k+1} - \alpha) \succ x_1$, $x_1$. Thus by A3, $x_{k+1} \succ x_1$, $x_1$, and by Lemma 1, A4, and A5, $d(x_1, R_{x_{k+1}}(x_{k+1})) > 0$. But then it is without generality to prove the proposition for the subsequence $(x_1, x_{k+1}, x_{k+2}, \ldots, x_T)$.
Appendix III - Introducing Diminishing Sensitivity in Losses

If NDSL is replaced by DSL, the rotation of indifference curves characterized by Lemma 1 no longer necessarily applies. To demonstrate, consider how the indifference curve through $y$ adjusts under DSL as trade takes place from $x$ to $y$. Again suppose $x_1 > y_1$, $y_2 > x_2$, and $y \succ_x x$, and now assume $z \in I_x(y)$ such that $z_2 > y_2$. As in the proof of Lemma 1, the change in reference point from $x$ to $y$ is broken into two parts, from $x$ to $s = (y_1, x_2)$, and then from $s$ to $y$. DSL implies that $y \succ_s z$, so by A3, A4 and the intermediate value theorem $(z_1 + \alpha, z_2) \sim_s y$ for some $\alpha > 0$. However, the move from $s$ to $y$ favors $(z_1 + \alpha, z_2)$ relative to $y$ by DSG, shifting the portion of $I_y(y)$ above $y$ back in the direction of $I_x(y)$. In Figure 5 the portion of the light solid curve above $y$ represents DSL dominating DSG, and the portion of the small-dashed curve above $y$ represents DSG dominating DSL. The flattest curve (large dashes) maintains the assumption of NDSL.

Figure 5: Ambiguous Indifference Curve Rotation under ISL
On the other hand, DSL does not impact the indifference curve through \( y \) to the right of \( x \) given a change in reference point from \( x \) to \( y \), so we have the result from Lemma 1 that \( I_y(y) \) is located above \( I_x(y) \) in this region. The indifference curve through \( y \) in the region between \( x \) and \( y \) after the change in reference point is ambiguous due to the competing influences of DSL and the combination of DSG and LA, but continuity and monotonicity must be preserved as in Figure 5. Apparently when DSL is a dominant influence on preferences, the effect of exchange on preferences is to make goods more complementary, which is itself a testable hypothesis.

Lemma 1 can be preserved by replacing NDSL with DSL, provided that LA and DSG jointly dominate DSL everywhere. From the proof of Lemma 1 it is readily verified that when \( z_1 > y_1 \) or \( z_2 > x_2 \), DSL influences how the indifference curve through \( z \) adjusts only in conjunction with the competing influences of LA or DSG, or both. When \( z_1 \leq y_1 \) and \( z_2 \leq x_2 \) (region 7 in Figure 4), DSL exerts exclusive influence on any point on the indifference curve through \( z \) in the same region. By convexity and continuity, it is therefore necessary for DSL to dominate LA and DSG in some portion of regions 4 and 8 in Figure 4. However, if we suspend continuity on the boundary of region 7, it is possible to have LA and DSG everywhere dominating DSL except in region 7. Since Lemma 2 guarantees trade will never occur in this region relative to any previously adopted allocation, this relaxation of continuity comes at little cost. Then Propositions 1, 2, and 3 are robust to the assumption of “dominated” DSL; i.e., DSL whose influence outside of region 7 is always dominated by the joint influence of LA and DSG.
References


