Set-valued functions, Lebesgue extensions and saturated probability spaces

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Abstract
Recent advances in the theory of distributions of set-valued functions have been shaped by counterexamples which hinge on the non-existence of measurable selections with requisite properties. These examples, all based on the Lebesgue interval, and initially circumvented by Sun in the context of Loeb spaces, have now led Keisler and Sun (KS) to establish a comprehensive theory of the distributions of set-valued functions on saturated probability spaces (introduced by Hoover and Keisler). In contrast, we show that a countably-generated extension of the Lebesgue interval suffices for an explicit resolution of these examples; and furthermore, that it does not contradict the KS necessity results. We draw the fuller implications of our theorems for integration of set-valued functions, for Lyapunov’s result on the range of vector measures and for the theory of large non-anonymous games.

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1. Introduction

The theory of integration of set-valued functions on the Lebesgue interval, and taking values in a finite-dimensional Euclidean space $\mathbb{R}^n$, or the theory of distribution of set-valued functions taking values in a countably infinite set, has found extensive application in optimal control theory and in mathematical economics; in particular, see the textbooks [10,20,1,9,27], [28, Chapters 6–7] and their references. As is well known, these resulting theories revolve around the properties of convexity, closedness, compactness and upper semicontinuity. In the last ten years, motivated by the study of “perfect competition,” and of “large games,” there has been a need to consider set-valued functions with more general range spaces, and here the Lebesgue interval has been a rather severe limitation. In a series of decisive counterexamples, it has been shown that the theory does not generalize when modeled on the Lebesgue interval; see [32,33,23–25] and the references and discussion furnished in [19].

1 The theory of distribution of set-valued functions has not received as extensive a treatment, but see [16, Section 3] and their references. A comprehensive theory had to await Sun’s results [32] modeled on Loeb spaces. For the theory of integration, see [33] and [34]. We may also note here that the so-called Debreu map studied in Claim 2 below is brought into explicit prominence in [8] which limits itself to set-valued functions taking finite values.
These counterexamples draw essentially on the following claims in which correspondence is used synonymously for a set-valued function (also referred to in the antecedent literature as a multi-function or a random set), \( I = [0, 1] \), \( \mathcal{L} \) the \( \sigma \)-algebra of Lebesgue measurable sets, \( \eta \) the Lebesgue measure defined on \( \mathcal{L} \), and hence, \( (I, \mathcal{L}, \eta) \), the Lebesgue unit interval.

**Claim 1.** There does not exist an \( \mathcal{L} \)-measurable selection \( \phi \) from the correspondence \( \Phi : I \to \{-1, 1\} \) with \( \Phi(i) = (-1, 1) \) for all \( i \in I \) such that for any \( t \in I \), \( \int_0^t \phi(i) \, d\eta(i) = 0 \).

**Claim 2.** There does not exist an \( \mathcal{L} \)-measurable selection \( \psi \) from the correspondence \( \Psi : I \to [-1, 1] \) with \( \Psi(i) = (-i, i) \) for all \( i \in I \) such that the induced distribution of \( \psi \) is the uniform distribution on the interval \([-1, 1] \), i.e., for any \( s \in [-1, 1] \), \( \eta(\{\psi < s\}) = (s + 1)/2 \).

For the validity of Claims 1 and 2, see [19] and their references.

Our next claim is taken from Lyapunov’s example on the range of vector measures (see p. 262 of [2]). We first review the set of Walsh functions \( W \). Our next claim is taken from Lyapunov’s example on the range of vector measures (see p. 262 of [2]). We first review the set of Walsh functions \( W \). We first review the set of Walsh functions \( W \).

Denote by \( \Phi(i) = \{\} \) the \( \mathcal{L} \)-valued measure defined as follows. For any \( i \in [0, 1] \), denote the binary representation by \( i = \sum_{k=0}^{a-1} \frac{i_k}{2^k} = \frac{n_0}{2} + \frac{i_1}{2^2} + \cdots + \frac{i_{a-1}}{2^a} + \cdots \), where each \( i_k \) is either 0 or 1,

\[
W_n(i) = (-1)^{n_0i_0 + n_1i_1 + \cdots + n_ai_a}.
\]

It is well known that the set of Walsh functions forms a complete orthogonal basis of the square-integrable functions on the Lebesgue unit interval.

Now consider a function \( f : [0, 1] \to \ell_2 \) where \( f(i) = (1 + W_n(i))_{n=0}^\infty \). Let \( \Pi(\cdot; \eta) \) be the \( \ell_2 \)-valued measure defined as follows. For any \( E \in \mathcal{L} \),

\[
\Pi(E; \eta) = \int_E f(i) \, d\eta(i) = \left( \int_E \frac{1 + W_n(i)}{2^{n+1}} \, d\eta(i) \right)_{n=0}^\infty.
\]

Denote by \( \Pi(\mathcal{L}; \eta) \) the range of the vector measure \( \Pi(\cdot; \eta) \) over \( \mathcal{L} \). Let \( \Pi([0, 1]; \eta) = e \), and it is clear that \( e = (1, \{2^{-n-1}\}_{n=1}^\infty) \).

**Claim 3.** There does not exist an \( \mathcal{L} \)-measurable selection \( \pi \) from the correspondence \( \Pi : I \to \ell_2 \) with \( \Pi(i) = \{0, f(i)\} \) for all \( i \in I \) such that the Bochner integral\(^2\) of \( \pi \) is \( e/2 \).

As a result of these claims, a robust theory of integration and distribution of set-valued functions has been constructed, but one that has been forced to jettison the Lebesgue interval. In the first instance, such a theory has been based on an atomless Loeb probability space, as in [22,32,33], and with the results finding significant application in the theory of large games, as surveyed in [18,19]. However, with the discovery of saturated probability spaces by Hoover and Keisler [11], and the realization that Loeb spaces are saturated, the results have been lifted to the general class of saturated probability spaces. Indeed, Keisler and Sun have recently shown these
spaces to be also necessary for the distributional theory constituted by the requisite properties listed above; see [13].

In this paper, we argue that in so far as the specific Claims 1 to 3 are concerned, one need not go all the way to the complexity of saturated probability spaces, and that the required selections (or measurable sets) can be found by a simple countably-generated extension of the Lebesgue interval. To put the matter another way, we propose a probability space with a countably-generated \( \sigma \)-algebra which suffices to negate the above claims, ensure the required properties for these correspondences, and thereby suffices for the substantive applications. Our results then point out the need for a compelling and a natural example for which our proposed probability space does not work, and for which a saturated probability space is essential. To be sure, given the Keisler–Sun necessity results, such an example exists!

This paper is organized as follows. In Section 2, we present definitions of a saturated probability space, and the proposed construction of the Lebesgue extension. The fact that this Lebesgue extension negates the above claims is presented in Section 3, that it allows the derivation of the general properties of the correspondences in Section 4, and that it connects to the necessity theory in Section 5. Section 6 draws the implications of these results for the theory of large games, and Section 7 allows us to make two remarks concerning questions opened up by our results.

2. Basic definitions and constructions

In this section we first present two equivalent definitions of saturated probability spaces, and then a countably-generated extension of the Lebesgue interval. Throughout this paper, probability space means complete countably additive probability space.

2.1. Saturated probability spaces

For a Polish (complete separable metric) space \( X \), denote its Borel \( \sigma \)-algebra by \( B_X \), and by \( M(X) \) the space of all Borel probability measures associated with the topology of weak convergence.

We present the definition of saturated probability spaces introduced by Hoover and Keisler [11] who were the first authors to provide a systematic study of such spaces.

**Definition 1.** A probability space \((I, \mathcal{I}, \lambda)\) is said to be saturated if for any two Polish spaces \( X \) and \( Y \), any Borel probability measure \( \tau \in M(X \times Y) \) with marginal probability measure \( \tau_X \) on \( X \), and any measurable mapping \( g \) from \((I, \mathcal{I}, \lambda)\) to \( X \) with distribution \( \tau_X \), there exists a measurable mapping \( h : (I, \mathcal{I}, \lambda) \rightarrow Y \) such that the measurable mapping \((g, h) : (I, \mathcal{I}, \lambda) \rightarrow X \times Y \) has distribution \( \tau \).

As shown in [11, Corollary 4.5], there is an equivalent definition of a saturated probability space that (simply but heuristically) requires that modulo sets of measure zero, the \( \sigma \)-algebra of the space, restricted to any set of positive measure, not be countably-generated. A rigorous development of this intuitive idea leads to Definition 3 below, and to proceed towards it, we first review some concepts related to the measure algebra for a probability space.\(^3\)

---

\(^3\) The reader not interested in the technical details can skip the next four paragraphs and proceed directly to Definition 3 below.
Let \((I, \mathcal{I}, \lambda)\) be a probability space. Consider a relation ‘\(\sim\)’ on \(\mathcal{I}\) as follows, for any \(E, F \in \mathcal{I}\), \(E \sim F\) if and only if \(\mu(E \Delta F) = 0\), where \(\Delta\) denotes the symmetric difference. It is clear that \(\sim\) is an equivalence relation on \(\mathcal{I}\). For any \(E \in \mathcal{I}\), let \(\hat{E} = \{F \in \mathcal{I}: F \sim E\}\) be the equivalence class of \(E\), and clearly \(E \in \hat{E}\); define the canonical epimorphism \(\pi_I : \mathcal{I} \to \hat{\mathcal{I}}\) by letting \(\pi_I(E) = \hat{E}\), for all \(E \in \mathcal{I}\). The pair \((\hat{\mathcal{I}}, \lambda)\) is said to be the measure algebra of \((I, \mathcal{I}, \lambda)\), here \(\hat{\mathcal{I}}\) is the quotient Boolean algebra for the equivalence relation, i.e., the set of equivalence classes in \(\mathcal{I}\) for \(\sim\), and \(\lambda : \hat{\mathcal{I}} \to [0, 1]\) is given by \(\hat{\lambda}(\hat{E}) = \lambda(E)\), for some \(E \in \hat{E}\). We can define the operations \(\cup, \cap, \setminus\) and \(\triangle\) on \(\hat{\mathcal{I}}\) in the following way: For any \(\hat{E}, \hat{F} \in \hat{\mathcal{I}}\) with \(E \in \hat{E}\) and \(F \in \hat{F}\), \(\hat{E} \subseteq \hat{F}\) if and only if \(\lambda(E \setminus F) = 0\), \(\hat{E} \cup \hat{F} = \hat{E} \cup F\), and analogously \(\cap, \setminus\) and \(\triangle\) are all well defined. It is clear that \(\hat{\mathcal{I}}\) is an algebra under \(\cap\) and \(\cup\).

Let \((\hat{\mathcal{I}}, \lambda)\) be the measure algebra associated to the probability space \((I, \mathcal{I}, \lambda)\). A subset of \(\hat{\mathcal{I}}\) is said to be a subalgebra of \(\hat{\mathcal{I}}\) if it contains \(\hat{I}\) (the equivalence class of \(I\)) and is closed under \(\cup\) and \(\cap\). A subalgebra \(\mathcal{E}\) is order-closed with respect to \(\triangle\) if for any non-empty upwards directed subset of \(\mathcal{E}\), its supremum exists and belongs to \(\hat{\mathcal{I}}\), then the supremum belongs to \(\mathcal{E}\) as well. A subset \(A\) of \(\hat{\mathcal{I}}\) is said to completely generate \(\hat{\mathcal{I}}\) if the smallest order-closed subalgebra in \(\hat{\mathcal{I}}\) containing \(A\) is \(\hat{\mathcal{I}}\) itself. Finally, the Maharam type of \((I, \mathcal{I}, \lambda)\) is the least cardinal number of any subset of \(\hat{\mathcal{I}}\) which completely generates \(\hat{\mathcal{I}}\).

Given a probability space \((I, \mathcal{I}, \lambda)\), for any subset \(S \in \mathcal{I}\) with \(\lambda(S) > 0\), denote by \((S, \mathcal{I}^S, \lambda^S)\) the probability space restricted to \(S\). Here \(\mathcal{I}^S := \{S \cap S' : S' \in \mathcal{I}\}\) and \(\lambda^S\) is the probability measure re-scaled from the restriction of \(\lambda\) to \(\mathcal{I}^S\).

**Definition 2.** A probability space \((I, \mathcal{I}, \lambda)\) is said to be countably-generated if the Maharam type of \((I, \mathcal{I}, \lambda)\) is countable. It is said to be saturated if, for any subset \(S \in \mathcal{I}\) with \(\lambda(S) > 0\), the Maharam type of the restricted probability space \((S, \mathcal{I}^S, \lambda^S)\) is uncountable.

**Remark 1.** This condition is originally called “\(\aleph_1\)-atomless” in [11], “nowhere separable” in [3], “rich” in an earlier version of [13], “super-atomless” in [29], and “nowhere countably-generated” in [25]; also see [4] for a comprehensive treatment, and the connection with the work of Maharam.

The Lebesgue unit interval \((I, \mathcal{L}, \eta)\) and the Lebesgue extension in Section 2.2 below are both countably-generated probability spaces, and therefore are not saturated probability spaces. In contrast, any atomless Loeb probability space is saturated (see [11]). By Maharam’s theorem [26], any two countably-generated atomless measure algebras are isomorphic, and moreover, a probability space is saturated if and only if its measure algebra is a countable convex combination of measure algebras of uncountable powers of the Borel \(\sigma\)-algebra on \([0, 1]\); see [4] for details. In the light of the discussion presented above, we shall use the following definition as the primitive notion of saturation, and see Definitions 1 and 2 as its fundamental characterizations, to be resorted to as required in applications.\(^4\)

**Definition 3.** A probability space \((I, \mathcal{I}, \lambda)\) is said to be saturated if it is nowhere countably generated, that is, for any subset \(S \in \mathcal{I}\) with \(\lambda(S) > 0\), \((S, \mathcal{I}^S, \lambda^S)\) is not countably generated.

\(^4\) One could argue that Definitions 2 and 3 constitute intrinsic as opposed to extrinsic characterizations of saturated probability spaces, though this intrinsic–extrinsic categorization deserves rigorous scrutiny. We are grateful to Conny Podczeck for conversation on this point.
2.2. Lebesgue extension: the construction

We now present an extension of the Lebesgue unit interval, and the extension is a countably generated probability space. This idea to construct such an extension can be traced back to [12,21]. To proceed, we first present Lemma 2 of [12]. It plays an important role in the construction. The proof of this lemma is based on transfinite induction.

Lemma 1. There is a disjoint family $C = \{C_k: k \in K\}$, $K = [0, 1]$, of subsets of $I = [0, 1]$ such that $\bigcup_{k \in K} C_k = I$, and for each $k \in K$, $\eta(C_k) = 0$ and $\eta^*(C_k) = 1$, where $\eta$ and $\eta^*$ are the respective inner and outer measures of the Lebesgue measure $\eta$.

Remark 2. The original version of this result, as presented by Kakutani, does not require $\bigcup_{k \in K} C_k = I$. However, if $\bigcup_{k \in K} C_k \neq I$, let $B = I \setminus \bigcup_{k \in K} C_k$. Since the cardinality of $B$ is at most the cardinality of $K$, the continuum, we can redistribute at most one point of $B$ into each $C_k$ in the family $C$ to obtain the required condition. Note also that we use both $K$ and $I$ to denote the unit interval.

We can now present the construction of the Lebesgue extension in five steps. Let $(K, \mathcal{K}, \kappa)$ be a copy of the Lebesgue interval, and therefore $K$ is a set with the cardinality of the continuum and $\mathcal{K}$ is countably-generated.

Step 1 (Existence of $C \subseteq I \times K$). Appeal to Lemma 1 to define a subset $C$ of $I \times K$ by letting $C = \{(i, k) \in I \times K: i \in C_k, k \in K\}$.

Step 2 (C has $(\eta \otimes \kappa)$-outer measure one). For any $\mathcal{L} \otimes \mathcal{K}$-measurable set $U$ that contains $C$, $C_k \subseteq U_k$ for each $k \in K$, where $U_k = \{i \in I: (i, k) \in U\}$ is the $k$-section of $U$. The Fubini property of $\eta \otimes \kappa$ implies that for $\kappa$-almost all $k \in K$, $U_k \in \mathcal{L}$-measurable, which means that $\eta(U_k) = 1$ (notice that $\eta^*(C_k) = 1$). Since $(\eta \otimes \kappa)(U) = \int_K \eta(U_k) \, d\kappa$, we have $(\eta \otimes \kappa)(U) = 1$. Therefore, the $(\eta \otimes \kappa)$-outer measure of $C$ is one.

Step 3 (Measure structure on $C$). Since the $(\eta \otimes \kappa)$-outer measure of $C$ is 1, one can extend $\eta \otimes \kappa$ to a measure $\gamma$ on the $\sigma$-algebra $\mathcal{U}$ generated by the set $C$ and the sets in $\mathcal{L} \otimes \mathcal{K}$ with $\gamma(C) = 1$. It is easy to see that $\mathcal{U} = \{(U^1 \cap C) \cup (U^2 \setminus C): U^1, U^2 \in \mathcal{L} \otimes \mathcal{K}\}$, and $\gamma((U^1 \cap C) \cup (U^2 \setminus C)) = (\eta \otimes \kappa)(U^1)$ for any measurable sets $U^1, U^2 \in \mathcal{L} \otimes \mathcal{K}$. Note that $(I \times K, \mathcal{U}, \gamma)$ is an extension of $(I \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$.

Let $\mathcal{T}$ be the $\sigma$-algebra $\{U \cap C: U \in \mathcal{L} \otimes \mathcal{K}\}$, which is the collection of all the measurable subsets of $C$ in $\mathcal{U}$. The restriction of $\gamma$ to $(C, \mathcal{T})$ is still denoted by $\gamma$. Then, $\gamma(U \cap C) = (\eta \otimes \kappa)(U)$, for every measurable set $U \in \mathcal{L} \otimes \mathcal{K}$.

Since the Lebesgue interval $(I, \mathcal{L}, \eta)$ (or $(K, \mathcal{K}, \kappa)$) is a countably-generated space, so is the product of two copies of Lebesgue intervals, $(I \times K, \mathcal{L} \otimes \mathcal{K}, \eta \otimes \kappa)$. And hence the probability space $(C, \mathcal{T}, \gamma)$ is also countably-generated.

Step 4 (New probability structure on $I$). Consider the projection mapping $p: I \times K \to I$ with $p(i, k) = i$. Let $\xi$ be the restriction of $p$ to $C$. Since the family $C$ is a partition of $I = [0, 1]$, $\xi$ is

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5 See also [6, Lemma 4191].
a bijection between $C$ and $I$. It is obvious that $p$ is a measure-preserving mapping from $(I \times K, \mathcal{L} \otimes K, \eta \otimes \kappa)$ to $(I, \mathcal{L}, \eta)$, i.e., for any $B \in \mathcal{L}$, $p^{-1}(B) \in \mathcal{L} \otimes K$ and $(\eta \otimes \kappa)[p^{-1}(B)] = \eta(B)$; and thus $p$ is a measure-preserving mapping from $(I \times K, \mathcal{U}, \gamma)$ to $(I, \mathcal{L}, \eta)$. Since $\gamma[\xi^{-1}(B)] = \gamma[C \cap p^{-1}(B)] = \eta \otimes \kappa[p^{-1}(B)] = \eta(B)$ for any $B \in \mathcal{L}$, $\xi$ is a measure-preserving mapping from $(C, T, \gamma)$ to $(I, \mathcal{L}, \eta)$.

Now let $\mathcal{I}$ be the $\sigma$-algebra $\{S \subseteq I : \xi^{-1}(S) \in \mathcal{T}\}$. Define a set function $\lambda$ on $\mathcal{I}$ by letting $\lambda(S) = \gamma[\xi^{-1}(S)]$ for each $S \in \mathcal{I}$. Since $\xi$ is a bijection, $\lambda$ is a well-defined probability measure on $(I, \mathcal{I})$. Hence $(I, \mathcal{I}, \lambda)$ is a probability space, and $\xi$ is also an isomorphism from $(C, T, \gamma)$ to $(I, \mathcal{I}, \lambda)$.

**Step 5 (Lebesgue extension).** Since $(I, \mathcal{I}, \lambda)$ is isomorphic to $(C, T, \gamma)$ as probability spaces, and $\xi$ is a measure-preserving mapping from $(C, T, \gamma)$ to $(I, \mathcal{L}, \eta)$, it is obvious that $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue unit interval $(I, \mathcal{L}, \eta)$.

The probability space $(I, \mathcal{I}, \lambda)$ is a countably-generated space since it is isomorphic to $(C, T, \gamma)$ via the isomorphism $\xi$.

This completes the construction, and we summarize it in Fig. 1.

**Remark 3.** The fact that the Lebesgue extension constructed here is a countably-generated space bears emphasis. The extensions of the Lebesgue interval provided in earlier work [29, Section 6], [36] and even [12, 21], generate spaces that are saturated, and therefore not countably-generated. In [29], $(K, K, \kappa)$ is the probability space $\{0, 1\}^\alpha$ with the cardinality $\alpha$ between the cardinality $c$ of the continuum and $2^c$; and in [36], it is the space obtained from a Loeb space via a bijection. Since both of these spaces are saturated, the Lebesgue extension $(I, \mathcal{I}, \lambda)$ is also a saturated probability space. These constructions are also based on an application of Lemma 1, but involve more complicated argumentation; see [29, Footnote 11] and the text it footnotes, and the reliance on a Loeb space in [36]. This construction of a saturated or non-separable extension of the Lebesgue unit interval can also be traced back to [12] and [21] who impose the additional requirement of invariance of the extension. Since we ask for much less, we need to work much less.

3. Claims 1 to 3 revisited

The importance of the construction reported in Section 2.2 lies not in itself, but in the fact that it is entirely successful in resolving the three negative claims presented in Section 1, which is to say that it allows the construction of $\mathcal{I}$-measurable mappings satisfying the required conditions. We turn to this.

Let $S_1 = \bigcup_{k \in [0, 1/2]} C_k$ and $S_2 = \bigcup_{k \in [1/2, 1]} C_k$, where $C_k$ is a subset of $I$ in the collection $\mathcal{C}$ in Lemma 1. Recall that $\mathcal{C}$ is a partition of $I$, then $S_2 \cap S_2 = \emptyset$, $S_1 \cup S_2 = I$. These subsets will be used in all of the three demonstrations in the sequel.
Lemma 2. For \( i = 1, 2 \), \( S_i \in \mathcal{I} \), but \( S_i \notin \mathcal{L} \). Moreover, \( \lambda(S_i \cap [0, t)) = t/2 \), for any \( t \in [0, 1] \), and in particular, \( \lambda(S_1) = 1/2 \).

Proof. Since \( S_2 \) is the complement of \( S_1 \) in \( I \), we only need to prove the result for \( S_1 \). By the construction of the subset \( C \subseteq I \times K \), \( \xi^{-1}(S_1) = \{(i, k): i \in C_k, k \in [0, 1/2)\} \); notice the latter is \( C \cap (I \times [0, 1/2)) \), then \( \xi^{-1}(S_1) \in \mathcal{I} \) by definition of \( \mathcal{I} \). It follows that \( S_1 \in \mathcal{I} \) because \( \xi \) is an isomorphism between \( (C, \mathcal{I}, \gamma) \) and \( (I, \mathcal{I}, \lambda) \).

For any \( t \in (0, 1] \), notice that \( (I, \mathcal{I}, \lambda) \) is an extension of the Lebesgue unit interval \( (I, \mathcal{L}, \eta) \), \( [0, t) \in \mathcal{I} \). Consequently, \( S_1 \cap [0, t) \in \mathcal{I} \) since \( S_1 \in \mathcal{I} \). Moreover, for any \( t \),

\[
\xi^{-1}(S_1 \cap [0, t)) = \{(i, k): i \in C_k; i \leq t, k \in [0, 1/2)\} = C \cap ([0, t) \times [0, 1/2)).
\]

Then we have

\[
\lambda[S_1 \cap [0, t)] = \gamma[\xi^{-1}(S_1 \cap [0, t))] = \gamma[C \cap ([0, t) \times [0, 1/2))]
\]

\[
= (\eta \otimes k)([0, t) \times [0, 1/2)) = t/2,
\]

where the first and third equalities follow from the definition of \( \lambda \) and \( \gamma \) respectively.

Next we show that \( S_1 \notin \mathcal{L} \). Suppose not. Since \( \lambda(S_i \cap [0, t)) = t/2 \) for all \( t \), it follows that for any open interval \( (t_1, t_2) \subseteq [0, 1] \), \( \lambda[S_1 \cap (t_1, t_2)] = (t_2 - t_1)/2 \). This contradicts [7, Theorem A]. \( \square \)

Given the Lebesgue extension \( (I, \mathcal{I}, \lambda) \) and \( S_1, S_2 \in \mathcal{I} \) as above, we are now ready to demonstrate the existence of the required selections from the correspondences in Claims 1 to 3.

Proposition 1. For the correspondence \( \Phi : I \to [-1, 1] \) with \( \Phi(i) = (-1, 1), \forall i \in I \), there exists an \( \mathcal{I} \)-measurable selection \( \phi \) such that for any \( t \in I \), \( \int_0^t \Phi(i) \, d\lambda(i) = 0 \).

Proof. Define \( \phi : I \to \{-1, -1\} \) by letting \( \phi(i) = 1 \) if \( i \in S_1 \), and \( \phi(i) = -1 \) if \( i \in S_2 \), where \( S_1 \) and \( S_2 \) are the disjoint subsets in Lemma 2. Since \( S_1 \) and \( S_2 \) are both \( \mathcal{I} \)-measurable subsets in \( I \), \( \phi \) is an \( \mathcal{I} \)-measurable function. Moreover, for any \( t \in [0, 1] \), on applying Lemma 2 again,

\[
\int_0^t \phi(i) \, d\lambda(i) = \lambda(S_1 \cap [0, t]) - \lambda(S_2 \cap [0, t)) = \frac{t}{2} - \frac{t}{2} = 0.
\]

Therefore, \( \phi \) is the required selection from the correspondence \( \Phi \). \( \square \)

Proposition 2. For the correspondence \( \Psi : I \to [-1, 1] \) with \( \Psi(i) = (-1, i) \) for all \( i \in I \), there exists an \( \mathcal{I} \)-measurable selection \( \psi \), such that the induced distribution of \( \psi \) is the uniform distribution on the interval \([-1, 1] \), i.e., for any \( s \in [-1, 1] \), \( \lambda(\{|\psi \leq s\}) = (s + 1)/2 \).

Proof. Define \( \psi : I \to [-1, 1] \) as follows:

\[
\psi(i) = \begin{cases} 
  i, & \text{if } i \in S_1; \\
  -i, & \text{if } i \in S_2.
\end{cases}
\]
It is well defined since $S_1 \cap S_2 = \emptyset$, and $S_1 \cup S_2 = I$. For any $s \in [-1, 1]$,

$$\{\psi \leq s\} = \begin{cases} S_2 \cup (S_1 \cap [0, s]), & \text{if } s \geq 0; \\ S_2 \cap [-s, 1], & \text{if } s < 0. \end{cases}$$

By Lemma 2, $S_1, S_2$ are $\mathcal{I}$-measurable subsets in $I$. Note that all the subsets of the form $[0, s]$ for any $s \geq 0$, and $[-s, 1]$ for any $s < 0$, are $\mathcal{L}$-measurable, and since $(I, \mathcal{I}, \lambda)$ is an extension of $(I, \mathcal{L}, \eta)$, they are all $\mathcal{I}$-measurable. Hence, $\psi$ is an $\mathcal{I}$-measurable function.

Moreover, for any $s \in [-1, 1]$, notice $S_1$ and $S_2$ are disjoint and applying Lemma 2,

$$\lambda\{\psi \leq s\} = \begin{cases} \lambda[S_2 \cup (S_1 \cap [0, s])] = \lambda(S_2) + \lambda(S_1 \cap [0, s]), & \text{if } s \geq 0 \\ \lambda[S_2 \cap [-s, 1)] = \lambda(S_2) - \lambda(S_2 \cap [0, -s]), & \text{if } s < 0 \end{cases} = \frac{1 + s}{2}.$$

Therefore, the $\mathcal{I}$-measurable function $\psi$ is the required selection from the correspondence $\Psi$.  

Finally we resolve a problem in Claim 3. Recall that the vector measure $\Pi$ is defined on the Lebesgue $\sigma$-algebra $\mathcal{L}$. Since $(I, \mathcal{I}, \lambda)$ is an extension of the Lebesgue interval, we next extend $\Pi$ to be a vector measure defined on $\mathcal{I}$, also denoted by $\Pi$. For each $S \in \mathcal{I}$, $\Pi(S; \lambda) = \int_S f(i) \, d\lambda(i) = (\int_S \frac{1 + W_n(i)}{2n+1} \, d\lambda(i))_{n=0}^{\infty}$. Similarly, let $\Pi(\mathcal{I}; \lambda)$ be the range of the extended vector measure.

For any nonnegative integer $n$, let

$$E_n = \{i \in [0, 1]: W_n(i) = 1\}. \quad (2)$$

By the definition of $W_n$, it is clear that $E_0 = [0, 1]$, $E_n$ for $n \geq 1$ is a disjoint union of several sub-intervals of $[0, 1]$ and $(1 + W_n)/2$ is the characteristic function of $E_n$. Moreover $\eta(E_0) = 1$ and $\eta(E_n) = 1/2$ for $n \geq 1$.

Next we calculate $\Pi(S_1; \lambda)$, the vector measure of $\Pi$ over $S_1$ in Lemma 2 with respect to $\lambda$. Note that $\int_{S_1} E_0 \, d\lambda = \lambda(S_1) = 1/2$. For $n \geq 1$, $\int_{S_1} \frac{1 + W_n}{2} \, d\lambda = \int_{S_1} E_n \, d\lambda = \lambda(S_1 \cap E_n) = 1/4$, where the last equation follows that $E_n$ is a finite union of disjoint sub-intervals with $\lambda(E_n) = \eta(E_n) = 1/2$ and $\lambda(S_1 \cap [0, t]) = t/2$ for any $t \in [0, 1]$ (see Lemma 2). Thus $\Pi(S_1; \lambda) = (1/2, (1/4 \cdot 1/2^m)_{m \geq 1}) = e/2$.

We thus proved that

**Proposition 3.** $e/2 \in \Pi(\mathcal{I}; \lambda)$.

4. General properties of the correspondences

The correspondence considered in Claim 2, originally due to Debreu (see [8]), is used in [32] to show that the distribution of a set-valued function on an abstract probability space is, in general, neither closed nor convex. The correspondence considered in Claim 3, originally due to Lyapunov (see [2, Chapter VII]), is used in [33] to show that the integral of a set-valued function on an abstract probability space is, in general, not convex. Sun used these facts to discredit the Lebesgue interval as a basis for the investigation of set-valued functions, arguing that the “Lebesgue interval fails to provide a suitable framework for a large class of problems,” and proposing instead the Loeb measure [22] based on hyperfinite models. This proposal has been
profoundly influential for applications in mathematical economics, as surveyed in [18,19]. However, given the existence of the required selections when these correspondences are based on the Lebesgue extension, one is led to ask whether the general results are themselves true for these specific correspondences on the proposed probability space. This is to ask whether the irregularities that these correspondences manifest in the context of the Lebesgue interval can be entirely subdued in the context of its simple extension that we propose here. We give an affirmative answer to this question, but one that turns out perhaps to be slightly more involved than one would anticipate.

In the first subsection we consider the correspondence involved in Claim 2, and in the second, that in Claim 3. Note that the irregularity associated with the correspondence considered in Claim 1 is of a totally different order: as far as the distribution and integral of this set-valued function is considered, it is entirely well-behaved as a consequence of the standard theory laid out in the text [10]. We shall return to this point in a subsequent section.

4.1. Distribution of the correspondence \( \Psi \)

First, we present a result which will be used in the sequel.

**Lemma 3.** Suppose a measure \( \mu \) on \([0, 1]\) satisfies the following: (i) \( \mu \) is absolutely continuous with respect to \( \eta \), and (ii) \( \mu([0, t]) \leq t \) for all \( t \in (0, 1] \). Then there exists an \( \mathcal{I} \)-measurable subset \( S_\mu \) in the countably-generated extension \((\mathcal{I}, \mathcal{I}, \lambda)\), a set not necessarily \( \mathcal{L} \)-measurable, such that for each \( t \), we have

\[
\mu([0, t]) = \lambda[S_\mu \cap [0, t)].
\]

**Proof.** According to (i), there exists an \( \mathcal{L} \)-measurable function \( f_\mu \) which is the Radon–Nikodym derivative of \( \mu \) with respect to \( \eta \). Because of (ii), without loss of generality, we can assume \( 0 \leq f_\mu \leq 1 \). It is clear that \( f_\mu \) is \( \eta \)-integrable and \( \mu([0, t]) = \int_0^t f_\mu \, d\eta \) for all \( t \in [0, 1] \).

Next, we construct \( S_\mu \). Let \( \Gamma(f_\mu) \subseteq \mathcal{I} \times \mathcal{K} \) be the hypograph of the function \( f_\mu : I \to [0, 1] \) (see Fig. 2 for the case that \( f_\mu \) is a continuous function), i.e.,

\[
\Gamma(f_\mu) = \{(i, k) \in \mathcal{I} \times \mathcal{K} : i \in I, k \leq f_\mu(i)\}.
\]

It is clear that \( \Gamma(f_\mu) \) is \((\mathcal{L} \otimes \mathcal{K})\)-measurable because \( f_\mu \) is Lebesgue measurable. Now define \( S_\mu = \xi[\Gamma(f_\mu) \cap C] \); see Section 2.2 for the construction of \( C \) and \( \xi \). By the construction
of \((C, \mathcal{T}, \gamma)\), \(\mathcal{F}(f_\mu) \cap C \in \mathcal{T}\). Moreover, the Lebesgue extension \((I, \mathcal{I}, \lambda)\) is isomorphic to \((C, \mathcal{T}, \gamma)\) via the 1–1 mapping \(\xi\). Consequently, \(S_\mu\) is an \(\mathcal{I}\)-measurable subset in \(I\).

Finally, we show that \(S_\mu \in \mathcal{I}\) is the required subset. Indeed, for any \(t \in [0, 1]\), we have

\[
\lambda[S_\mu \cap [0, t)] = \gamma[\xi^{-1}(S_\mu \cap [0, t])] = \gamma[C \cap \mathcal{F}(f_\mu) \cap ([0, t) \times [0, 1])]
\]

\[
= (\eta \otimes \kappa)[\mathcal{F}(f_\mu) \cap ([0, t) \times [0, 1])] = \int_0^t f_\mu \, d\eta = \mu([0, t]),
\]

where the first and third equations follow from the definition of \(\lambda\) and \(\gamma\) separately. \(\square\)

**Remark 4.** Note that Lemma 2 is a special case of Lemma 3 where \(\mu\) is the uniform distribution on \([-1, 1]\). We present a complete proof of Lemma 2 to orient the reader by presenting the argument for a special case.

Next, we turn to the correspondence \(\Psi : I \to [-1, 1]\) with \(\Psi(i) = (-i, i)\) for each \(i \in I\). Define

\[
\mathcal{D}_\Psi(\lambda) = \{\lambda \circ \psi^{-1} : \psi\text{ is an }\mathcal{I}\text{-measurable selection of }\Psi\},
\]

where \(\lambda \circ \psi^{-1}\) is the distribution induced by \(\psi\). For any Borel set \(B\) in \([-1, 1]\), let \(\Psi^{-1}(B) = \{i \in I : \Psi(i) \cap B \neq \emptyset\}\). Note that the correspondence \(\Psi\) is closed-valued and \(\Psi^{-1}(B) = \{[t] : t \in B\}\), which is also a Borel set in \([0, 1]\). We can now present the following result.

**Theorem 1.** \(\mathcal{D}_\Psi(\lambda)\) is closed and convex.

**Proof.** For the closedness part, Proposition 3.5 of Keisler and Sun [13] provides a useful characterization for Borel probability measures in the closure of \(\mathcal{D}_\Psi(\lambda)\). In particular, the Borel probability measure \(\mu\) on \([-1, 1]\) belongs to the closure of \(\mathcal{D}_\Psi(\lambda)\) if, and only if,

\[
\mu(O) \leq \lambda[\Psi^{-1}(O)] \quad \text{for any open set } O \subseteq [-1, 1].
\]

As a result, to prove the closedness of \(\mathcal{D}_\Psi(\lambda)\), we only need to show that any Borel probability measure satisfying (3) can be induced by some \(\mathcal{I}\)-measurable selection of \(\Psi\).

Let \(\mu\) be a Borel probability measure on \([-1, 1]\) satisfying (2). We next show that, as a Borel measure on \([0, 1]\), \(\mu\) satisfies the two conditions (i) and (ii) in Lemma 3. It is clear that (i) \(\mu\) is absolutely continuous with respect to \(\eta\). For condition (ii), notice that \(\mu([0, t)) \leq \mu((-t, t))\) for any \(t \in [0, 1]\), we next claim that

\[
\mu((-t, t)) = t, \quad \text{for any } t \in (0, 1].
\]

As a result, \(\mu([0, t)) \leq t\) for any \(t \in [0, 1]\), i.e., condition (ii) is also satisfied.

To prove the claim in (4), we first fix \(t \in (0, 1]\) and let \(B_1 = (-t, t)\) and \(B_2 = [-1, -t) \cup (t, 1]\). It is clear that \(\Psi^{-1}(B_1) = [0, t)\) with \(\lambda[\Psi^{-1}(B_1)] = t\), in addition, it follows from (2) that \(\mu(B_1) \leq t\). Similarly, we also have \(\Psi^{-1}(B_2) = (t, 1]\), \(\lambda[\Psi^{-1}(B_2)] = 1 - t\) and \(\mu(B_2) \leq 1 - t\). Notice that \(\mu(B_1 \cup B_2) = \mu(B_1) + \mu(B_2) = 1\), then \(\mu(B_1) = t\).
Due to Lemma 3, there exists a subset $S_\mu \in \mathcal{I}$ in the countably-generated extension $(I, \mathcal{I}, \lambda)$, such that for each $t \in [0, 1]$, $\mu([0, t)) = \lambda[S_\mu \cap [0, t))$. Then define $\psi : I \to [-1, 1]$ as follows,

$$\psi(i) = \begin{cases} i, & \text{if } i \in S_\mu; \\ -i, & \text{if } i \notin S_\mu. \end{cases}$$

It is clear that $\psi$ is an $\mathcal{I}$-measurable selection of $\Psi$.

We next show that the induced distribution of $\psi$ is $\mu$. It is clear that for any $0 \leq t_1 < t_2 \leq 1$, $\lambda[\psi^{-1}(t_1, t_2)] = \lambda[S_\mu \cap (t_1, t_2)] = \mu((t_1, t_2))$; moreover, $\lambda[\psi^{-1}(-t_2, -t_1)] = \lambda[(I \setminus S_\mu) \cap (t_1, t_2)] = t_2 - t_1 - \mu((t_1, t_2)) = \mu((-t_2, -t_1))$, where the last equation follows from $\mu(\mathcal{B}) = t$ for $t = t_1, t_2$. Notice that the Borel $\sigma$-algebra on $[-1, 1]$ is generated by these open intervals, thus, the induced distribution of the $\mathcal{I}$-measurable function $\psi$ is $\mu$. Therefore we proved that $\mathcal{D}_\psi(\lambda)$ is closed.

Finally, the convexity of $\mathcal{D}_\psi(\lambda)$ follows straightforward from the closedness of $\mathcal{D}_\psi(\lambda)$ and (2). In fact, let $\mu_1, \mu_2$ be two Borel probability measures in $\mathcal{D}_\psi(\lambda)$, then they both satisfy (2). For any $0 \leq \alpha \leq 1$, it is clear that the Borel probability measure $\alpha \mu_1 + (1 - \alpha)\mu_2$, denoted by $\mu'$, also satisfies (2). Due to [13, Proposition 3.5], $\mu'$ belongs to the closure of $\mathcal{D}_\psi(\lambda)$. Since $\mathcal{D}_\psi(\lambda)$ is closed, $\mu' = \alpha \mu_1 + (1 - \alpha)\mu_2$ belongs to $\mathcal{D}_\psi(\lambda)$ as well. \qed

**Remark 5.** Note that Proposition 2 is a special case of Theorem 1, and that more generally, it underscores the fact that all that is required to eliminate the irregularities of [32, Example 1] is that one work with the extended Lebesgue interval being proposed here.

### 4.2. Integral of the correspondence $\tilde{\Pi}$

We next turn to the integral of the correspondence $\tilde{\Pi}$ and the range of the vector measure $\Pi(\cdot; \lambda)$, as defined in Claim 3 and the text preceding it. By the definition of this vector measure, a point $x = (x_0, x_1, x_2, \ldots)$ in $\ell_2$ belongs to $\Pi(\mathcal{I}; \lambda)$ if and only if there exists an $S \in \mathcal{I}$ such that $x_0 = \lambda(S)$, and for $n \geq 1$,

$$x_n = \int_S \frac{1 + W_n(i)}{2n+1} d\lambda(i) = \frac{1}{2^n} \int_S 1_{E_n} d\lambda = \frac{1}{2^n} \lambda(E_n \cap S). \quad (5)$$

We now present the following result.

**Theorem 2.** $\Pi(\mathcal{I}; \lambda)$ is convex.

**Proof.** Given any $x = (x_n)_{n=0}^{\infty}$, $y = (y_n)_{n=0}^{\infty} \in \Pi(\mathcal{I}; \lambda)$, we need to show that $\alpha x + (1 - \alpha)y$ is also in $\int_I \Pi d\lambda$, for each $0 \leq \alpha \leq 1$. Let $S_x, S_y$ be the $\mathcal{I}$-measurable subsets such that $x = \Pi(S_x; \lambda)$, $y = \Pi(S_y; \lambda)$. Then we have that $x_0 = \lambda(S_x)$, $y_0 = \lambda(S_y)$, $x_n = 2^{-n} \lambda(E_n \cap S_x)$ and $y_n = 2^{-n} \lambda(E_n \cap S_y)$ for any natural number $n$ according to Eq. (5).

Fixing $\alpha$, we next define $\mu$ to be a Borel probability measure on $I = [0, 1]$ by letting $\mu(B) = \alpha \lambda(B \cap S_x) + (1 - \alpha)\lambda(B \cap S_y)$, for any Borel set $B$ in $I$. It is well defined since $(I, \mathcal{I}, \lambda)$ is a Lebesgue extension. We claim that $\mu$ satisfies the two conditions in Lemma 3. First, it is clear that $\mu$ is absolutely continuous with respect to $\eta$. Indeed, for any Lebesgue null set $N$ in $[0, 1]$, $\mu(N) \leq \lambda(N) = \eta(N) = 0$, where the last equality holds because $\lambda$ is an extension of $\eta$. Second,
for any $t \in [0, 1]$, $\mu([0, t)) \leq \lambda([0, t)) = t$. Then, by Lemma 3, there exists an $\mathcal{I}$-measurable subset $S$ such that, for each $t \in [0, 1]$, we have $\mu([0, t)) = \lambda([0, t) \cap S)$.

Finally, we show that $S$ is the $\mathcal{I}$-measurable subset such that, for each $t \in [0, 1]$, we have $\mu([0, t)) = \lambda([0, t) \cap S)$.

Remark 6. Note that Proposition 3 is a special case of Theorem 2, and that more generally, it underscores the fact that all that is required to eliminate the irregularities of [33, Example 1] is that one work with the extended Lebesgue interval being proposed here.

5. Necessity of saturation

In [13], Keisler and Sun make a persuasive and definitive case for the necessity of saturated probability spaces, arguing in [13, Introduction] as follows:

Atomless Loeb spaces have the desired properties for correspondences and large games. One realized that [such] spaces are very rich in the sense that they have many more measurable sets than the Lebesgue unit interval. [Once] it was shown that every atomless Loeb space is saturated, this gave a hint that these properties might hold for all saturated probability spaces. Here we also get converse results showing that the desired properties fail on every non-saturated probability space.

It is important to understand what is being claimed here. If the space is not saturated, as the space $(I, \mathcal{I}, \lambda)$ constructed in Section 2.2, there must exist a set-valued function defined on it whose distribution is not closed and/or convex, and whose integral is not convex. What is interesting, and somewhat of a surprise, is that this existential statement can be underscored by a constructive one; and furthermore, the set-valued function exhibiting “irregularities” on the extended space is a “simple” transformation of the originally given one. And the nature of the transformation is such that the extended space $(I, \mathcal{I}, \lambda)$ can be extended one more time to a countably-generated space $(I, \mathcal{I}_1, \lambda_1)$ to subdue the irregularities of the transformed set-valued function in precisely the same way that the original correspondence was subdued! But the fact that $(I, \mathcal{I}_1, \lambda_1)$ is also countably-generated, a second appeal to the Keisler–Sun results leads to a repetition of the process. And this repetition can be continued ad infinitum to obtain $(I, \mathcal{I}_n, \lambda_n)_{n \in \mathbb{N}}$, thereby giving insight into how rich a saturated probability space really is. It cannot be attained in a countably infinite number of extensions.

We formalize this verbal description by beginning with a result in Fremlin [5] that proves instrumental for the articulation of the process, and then turning successively to the two claims we have been entertaining so far.

5.1. A useful result

Definition 4. Let $(I, \mathcal{I}, \lambda)$ and $(I', \mathcal{I}', \lambda')$ be two measure spaces, and $(\mathcal{I}, \mathcal{I}_1, \lambda_1)$ and $(\mathcal{I}', \mathcal{I}_1', \lambda_1')$ be their measure algebras respectively. The measure preserving homomorphism $\rho : \mathcal{I} \rightarrow \mathcal{I}_1$, is said to be realized by a measure preserving map $h : I' \rightarrow I$ if for any $S \in \mathcal{I}$, $\pi_I[h^{-1}(S)] = \rho(\pi_I(S))$. 

where $\pi_I, \pi'_{I'}$ are the canonical epimorphisms. In other words, the following diagram commutes,

\[
\begin{array}{ccc}
\mathcal{I} & \xrightarrow{h^{-1}} & \mathcal{I}' \\
\downarrow{\pi_I} & & \downarrow{\pi'_{I'}} \\
\hat{\mathcal{I}} & \xrightarrow{\rho} & \hat{\mathcal{I}}'
\end{array}
\]

where the homomorphism $h^{-1}$ is naturally derived by $h$.

For the Lebesgue interval $(I, \mathcal{L}, \eta)$, denote by $(\hat{\mathcal{L}}, \hat{\eta})$ the measure-algebra. Let $(V, \mathcal{V}, \nu)$ be an atomless countably-generated probability space associated with the measure algebra $(\hat{\mathcal{V}}, \hat{\nu})$. By Maharam’s theorem (see [26]), there exists a measure-preserving isomorphism $\rho : \hat{\mathcal{L}} \to \hat{\mathcal{V}}$. The next result is Theorem 4.12 of Fremlin [5, p. 937]. As in [13, Theorem 2.7, p. 1589], we will use this result in the sequel to construct new counterexamples from old ones.

**Lemma 4.** Given an atomless countably generated probability space $(V, \mathcal{V}, \nu)$ and the measure algebra isomorphism $\rho$ as above, then $\rho$ can be realized by a measure-preserving mapping $h$ from $(V, \mathcal{V}, \nu)$ to the Lebesgue interval $(I, \mathcal{L}, \eta)$.

The next result is a corollary of the above, and is useful in the sequel.

**Corollary 1.** Let $(V, \mathcal{V}, \nu)$ be an atomless countably-generated probability space, $\rho$ and $h$ as in Lemma 4. Then for any $S \in \mathcal{V}$, there exists an $S' \in \mathcal{L}$ such that $\nu[S \Delta h^{-1}(S')] = 0$, where $\Delta$ is the symmetric difference operator.

**Proof.** By Lemma 4, the measure algebra isomorphism $\rho : \hat{\mathcal{L}} \to \hat{\mathcal{V}}$ is realized by $h : (V, \mathcal{V}, \nu) \to (I, \mathcal{L}, \eta)$. For any $S \in \mathcal{V}$, the corresponding equivalence class is $\pi_V(S)$ in the measure algebra $(\hat{\mathcal{V}}, \hat{\nu})$. Since $\rho$ is an isomorphism, consider $\rho^{-1}(\pi_V(S)) \in \hat{\mathcal{L}}$, and let $S' \in \mathcal{L}$ be a subset such that $\pi_I(S') = \rho^{-1}(\pi_V(S))$. By the communicate diagram in Definition 4, $\pi_V[h^{-1}(S')] = \rho[\pi_I(S')]$. By the definition of $S'$, $\rho[\pi_I(S')] = \rho[\rho^{-1}(\pi_V(S))] = \pi_V(S)$. Therefore, $\nu[S \Delta h^{-1}(S')] = 0$. □

### 5.2. Distribution of a new correspondence $\Psi_1$

In this subsection, we show that on $(I, \mathcal{I}, \lambda)$, the countably-generated extension of the Lebesgue interval $(I, \mathcal{L}, \eta)$, there exists a closed-valued correspondence $\Psi_1$ (to be defined below) whose distribution with respect to $\lambda$, $\mathcal{D}_{\Psi_1}(\lambda)$, is neither closed nor convex. Moreover, as in Section 2.2, we can then find a countably-generated extension of $(I, \mathcal{I}, \lambda)$, $(I, \mathcal{I}_1, \lambda_1)$, such that on this new extension, $\mathcal{D}_{\Psi_1}(\lambda_1)$ does not have to exhibit any such irregularities, which is to say that the distribution of $\Psi_1$ with respect to $\lambda_1$ is closed and convex.

The basic observation is simply this: Since both the Lebesgue unit interval $(I, \mathcal{L}, \eta)$ and its extension $(I, \mathcal{I}, \lambda)$ are countably-generated, by Maharam’s theorem, there is an isomorphism $\rho$ from $(\hat{\mathcal{L}}, \hat{\eta})$ to $(\hat{\mathcal{I}}, \hat{\lambda})$. By Lemma 4, there exists a measure-preserving mapping $h : (I, \mathcal{I}, \lambda) \to (I, \mathcal{L}, \eta)$ such that $\rho$ can be realized by $h$. Simply define $\Psi_1 = \Psi \circ h$, i.e., $\Psi_1(i) = \{h(i), -h(i)\}$ for all $i \in I$. 
5.2.1. $D_{\psi_1}(\lambda)$ is neither closed nor convex

We show that $D_{\psi_1}(\lambda) = D_{\psi}(\eta)$, and, as a result, $D_{\psi_1}(\lambda_1)$ is neither closed nor convex. First, for any $\mathcal{L}$-measurable selection $\psi$ of $\Psi$, it is clear that $\psi \circ h$ is an $\mathcal{I}$-measurable selection of $\Psi_1$ and $\lambda \circ (\psi \circ h)^{-1} = \lambda[h^{-1} \circ \psi^{-1}] = \eta \circ \psi^{-1}$. Hence $D_{\psi}(\eta) \subseteq D_{\psi_1}(\lambda)$.

We next prove the converse part, $D_{\psi_1}(\lambda) \subseteq D_{\psi}(\eta)$. Assume $\mu$ is a Borel probability measure in $D_{\psi_1}(\lambda)$ and it is induced by an $\mathcal{I}$-measurable selection $\psi_1$ of $\Psi_1$, i.e., $\lambda \circ \psi_1^{-1} = \mu$. Let $S = \{i \in I: \psi_1(i) = h(i) \geq 0\}$, then $S \in \mathcal{I}$. Since the measure algebra isomorphism $\rho$ is realized by the measure preserving map $h$, we can appeal to Corollary 1 to assert the existence of an $\lambda_1 \in \lambda$ such that $\lambda[(h^{-1}(S) \Delta S) = 0$. Now define $\psi : I \to [-1, 1]$ by letting $\psi(i) = i$, if $i \in S'$ and $-i$ if $i \notin S'$. It is clear that $\psi$ is an $\mathcal{L}$-measurable selection of $\Psi$. Moreover, the induced distribution by $\psi$ is $\mu$. Toward this end, we only need to show that for any Borel set $B \subseteq [0, 1]$, $\eta\psi^{-1}(B) = \mu(B)$. In fact,

$$
\mu(B) = \lambda\psi_1^{-1}(B) = \lambda[h^{-1}(B) \cap S] = \lambda[h^{-1}(B) \cap h^{-1}(S')] = \lambda[h^{-1}(B \cap S')] = \eta(B \cap S') = \eta\psi^{-1}(B).
$$

We have thus shown

**Proposition 4.** $D_{\psi_1}(\lambda)$ is neither closed nor convex.

5.2.2. Resolution of the irregularities of $D_{\psi_1}(\lambda)$

In Proposition 4, we know that there are some irregularities in $D_{\psi_1}(\lambda)$, i.e., it is neither convex nor closed. Actually, these irregularities can be resolved by extending the Lebesgue extension $(I, \mathcal{I}, \lambda)$, as in Section 2.2. In particular, we can find a countably generated extension of $(I, \mathcal{I}, \lambda)$, denoted by $(I, \mathcal{I}_1, \lambda_1)$, such that on this new extension, the distribution of $\psi_1$ with respect to $\lambda_1$, $D_{\psi_1}(\lambda_1)$, does not have to exhibit any such irregularities, which is to say that it is closed and convex.

Following the construction of $(I, \mathcal{I}, \lambda)$ in Section 2.2, we provide an outline on how to construct the new countably-generated extension of this probability space. First, we reproduce an analogue of Lemma 1. Recall that $(K, \mathcal{K}, \kappa)$ is a copy of the Lebesgue interval.

**Lemma 5.** On the probability space $(I = [0, 1], \mathcal{I}, \lambda)$, there is a disjoint family $\{D_k: k \in [0, 1]\}$ of subsets of $I$ such that $\bigcup_{k \in K} D_k = I$, and for each $k \in [0, 1]$ $\lambda_s(D_k) = 0$ and $\lambda^*(D_k) = 1$, where $\lambda_s$ and $\lambda^*$ are the respective inner and outer measures of the probability measure $\lambda$.

**Proof.** The construction is based on $(C, \mathcal{T}, \gamma)$ in Step 3 of Section 2.2, where $C = \bigcup_{k \in [0, 1]} \{i, k\}: i \in C_k$ and $\{C_k: k \in [0, 1]\}$ is a partition of the Lebesgue unit interval $(I, \mathcal{L}, \eta)$ with $\eta_s(C_k) = 0$, $\eta^*(C_k) = 1$; see Lemma 1. Now for each $k \in [0, 1]$, define a subset $C'_k \subseteq C$ where $C'_k = \bigcup_{k' \in C_k} \{i, k': i \in C_{k'}\}$. Notice that $\{C_k: k \in [0, 1]\}$ is a partition of $(I, \mathcal{L}, \eta)$, it is clear that $\{C'_k: k \in [0, 1]\}$ is a partition of $C$ as well. Moreover, for all $k$, as a subset in the product probability space of two copies of the Lebesgue unit interval, $C'_k$ has $(\eta \otimes \kappa)$-inner measure 0, outer measure 1. By the construction of $(C, \mathcal{T}, \gamma)$ (see Step 3 thereof), every $C'_k$ also has $\gamma$-inner measure 0, outer measure 1. Finally, let $D_k = \xi(C'_k) = \bigcup_{k' \in C_{k'}} C_{k'}$ for each $k \in [0, 1]$. It is clear that $\{D_k: k \in [0, 1]\}$ is the required partition of $(I, \mathcal{I}, \lambda)$ because that $(C, \mathcal{T}, \gamma)$ and $(I, \mathcal{I}, \lambda)$ are isomorphic through the 1–1 map $\xi$. $\square$
With Lemma 5 in place, we can construct a countably-generated extension of \((I, \mathcal{I}, \lambda)\) by applying Steps 1–5 in Section 2.2 as follows: With \(\{D_k: k \in [0, 1]\}\), we can define \(C' = \{(i, k): i \in D_k, k \in K\}\). It is a subset of the product probability space \((I \times K, \mathcal{I} \otimes \mathcal{K}, \lambda \otimes \kappa)\). Also, \(C'\) has \((\lambda \otimes \kappa)\)-outer measure 1, a probability structure on \(C'\) can be constructed, and new probability structure on \(I, (I, \mathcal{I}_1, \lambda_1)\), can be derived from that on \(C'\) through the 1–1 mapping \(\xi': C' \rightarrow I\), where \(\xi'\) is the projection map between \(C'\) and \(I\). Finally, it is worthwhile to note that \((I, \mathcal{I}_1, \lambda_1)\) is a countably generated extension of \((I, \mathcal{I}, \lambda)\).

Now we are ready to introduce the following result.

**Theorem 1’**. \(D_{\Psi_1}(\lambda_1)\) is closed and convex.

**Proof.** The argument is analogous to that used in the proof of Theorem 1. Here we only prove the closedness part. According to Proposition 3.5 of [13], a Borel probability measure \(\mu\) on \([-1, 1]\) is in the closure of \(D_{\Psi_1}(\lambda_1)\) if, and only if,

\[
\mu(O) \leq \lambda_1[\Psi_1^{-1}(O)], \quad \text{for all Borel open set } O \subseteq [-1, 1]. \tag{6}
\]

As a result, for any Borel probability measure \(\mu\) satisfying (3), we only need to show that there exists an \(\mathcal{I}_1\)-measurable selection \(\psi_1\) of \(\Psi_1\) with \(\lambda_1 \circ \psi_1^{-1} = \mu\).

Next, we claim that as a Borel probability measure on \([0, 1]\), \(\mu\) satisfies the two conditions in Lemma 3. In fact, for any \(t \in [0, 1]\),

\[
\mu([0, t)) \leq \lambda_1[\Psi_1^{-1}([0, t)) = \lambda_1[\psi^{-1}([0, t)))] = \lambda_1[h^{-1}((0, t))]] = \lambda_1[h^{-1}((0, t))]] \leq \lambda_1(h^{-1}((0, t))) = \lambda_1(0, t) = t.
\]

Moreover, it is clear that \(\mu\) is absolutely continuous w.r.t. \(\eta\). As in the proof of Lemma 3, let \(f_\mu : (I, \mathcal{L}, \eta) \rightarrow [0, 1]\) be the Radon–Nikodym derivative of \(\mu\) with respect to the Lebesgue measure \(\eta\). Now define \(g_\mu = f_\mu \circ h\), which is an \(\mathcal{I}\)-measurable function on \((I, \mathcal{I}, \lambda)\). Let \(\Gamma(g_\mu)\) be the hypograph of \(g_\mu\) in the product probability space of \((I, \mathcal{I}, \lambda)\) and the Lebesgue unit interval. Let \(S'_\mu = \xi'[\Gamma(g_\mu) \cap C']\), where \(\xi'\) is the projection of \(C'\) to \(I\) and it is a \(\sigma\)-algebra isomorphism. Define the function \(\psi_1\) as follows,

\[
\psi_1(i) = \begin{cases} h(i), & \text{if } i \in S'_\mu; \\ -h(i), & \text{if } i \notin S'_\mu. \end{cases}
\]

It is clear that \(\psi_1\) is a selection of the correspondence of \(\Psi_1 : (I, \mathcal{I}_1, \lambda_1) \rightarrow [-1, 1]\).

Finally, we complete the proof by showing that the induced distribution of \(\psi_1\) is \(\mu\). As in the proof of Theorem 1, it suffices to prove that \(\lambda_1[\psi_1^{-1}([0, t))] = \mu([0, t])\) for any \(t \in [0, 1]\). In fact,

\[
\lambda_1[\psi_1^{-1}([0, t))] = \lambda_1[S'_\mu \cap h^{-1}((0, t))] = \int_{h^{-1}([0, t])} f_\mu \circ h \, d\lambda = \int_0^t f_\mu \, d\eta = \mu([0, t]), \tag{7}
\]
where the second equation follows from the definition of $\lambda_1$, the third equation from substitution of variables, and the fourth from the definition of Radon–Nikodym derivative function $f_{\mu}$. □

5.2.3. Necessity of saturation

As mentioned in the beginning, the procedure in Sections 5.2.1 and 5.2.2 can be repeated again and again. For example, since $(I, I_1, \lambda_1)$ is also a countably generated probability space, according to Theorem 3.7 of [13], one can construct a correspondence $\Psi_2$ from $\Psi$ such that $D\Psi_2(\lambda_1)$ is neither closed nor convex, however, one can construct a countably generated extension of $(I, I_1, \lambda_1)$, denoted the new extension by $(I, I_2, \lambda_2)$, such that the irregularities on $D\Psi_2(\lambda_1)$ can be subdued, that is, $D\Psi_2(\lambda_2)$ is both closed and convex. This repetition can be continued ad infinitum to obtain $(I, I_n, \lambda_n)_{n \in \mathbb{N}}$.

We now explain briefly about the construction of $(I, I_n, \lambda_n)$ for $n \geq 2$, which is a countably generated probability space. As in Section 5.2.2 above, the probability space $(I, I_n, \lambda_n)$ can be constructed inductively from the probability product space between the Lebesgue interval and $(I, I_{n-1}, \lambda_{n-1})$. The key idea is to establish an analogue of Lemma 1, which is to construct a continuum of subsets in $(I, I_{n-1}, \lambda_{n-1})$ such that each subset has $\lambda_{n-1}$ inner measure 0, outer measure 1, and all subsets consist of a partition of $[0, 1]$.

Now we explain how to attain similar results as in Sections 5.2.1 and 5.2.2 on $(I, I_n, \lambda_n)$ for $n \geq 2$. Since $(I, I_n, \lambda_n)$ is a countably generated probability space, for each nonnegative integer $n$, due to Maharam’s theorem again, there exists a measure algebra isomorphism between the measure algebra of the Lebesgue interval $(I, L, \eta)$ and the measure algebra of $(I, I_n, \lambda_n)$. According to Lemma 4, there exists a measure-preserving mapping $h_n : (I, I_n, \lambda_n) \to (I, L, \eta)$ such that this measure algebra isomorphism can be realized by $h_n$. Moreover, define a set-valued function $\Psi_n : (I, I_{n+1}, \lambda_{n+1}) \to [-1, 1]$ as $\Psi_n = \Psi \circ h_{n-1}$. As in the argument in Proposition 4, for each $n \geq 2$, $D\Psi_n(\lambda_{n-1}) = D\Psi(\eta)$, hence $D\Psi_n(\lambda_{n-1})$ is neither closed nor convex; however, $D\Psi_n(\lambda_n)$ is both closed and convex. This can be achieved by using the similar argument in the proof of Theorem 1’. The procedure can be illustrated in Fig. 3 in which $D\Psi_n(\lambda_{n-1})$ is neither closed nor convex, but $D\Psi_n(\lambda_n)$ is both closed and convex.

Such irregularities on the distribution of closed-valued correspondences can be remedied by working with saturated probability spaces. It is shown in [13, Theorem 3.6] that on a saturated probability space, the distribution for any closed-valued correspondence is both closed and convex. Moreover, Theorem 3.7 therein assures us that the saturation property is also necessary for the validity of such regular properties of any closed-valued correspondence on a probability space. From this viewpoint, the contribution of this note can also be seen as an emphasis on an approximate approach to demonstrate the necessity of the saturated probability space in the context of keeping the regular properties of the distributions for closed-valued correspondences.
5.3. Integral of a new correspondence $\tilde{\Pi}$

In this section, we rework the ideas of Section 5.2 in the context of Lyapunov’s example on the range of vector measures. Recall that $h : (I, \mathcal{I}, \lambda) \to (I, \mathcal{L}, \eta)$ is the measure-preserving mapping which induces the measure algebra isomorphism $\rho$, and $(I, \mathcal{I}_1, \lambda_1)$ is a countably-generated extension of $(I, \mathcal{I}, \lambda)$ obtained in Section 5.2.2. Define $\Pi_1$ to be a vector measure defined on $\mathcal{I}$. For any $S \in \mathcal{I},$

$$\Pi_1(S; \lambda) = \int_S f \circ h \, d\lambda = \left( \int_S \frac{1 + W_n \circ h(i)}{2^{n+1}} \, d\lambda(i) \right)_{n=0}^{\infty}.$$  

For simplicity, we also denote $\Pi_1$ by $\Pi \circ h$. Let $\Pi_1(\mathcal{I}; \lambda)$ be the range of this new vector measure, and as before, we emphasize the associated correspondence by $\tilde{\Pi}_1 : I \to \ell_2$ with $\tilde{\Pi}(i) = \{0, f \circ h(i)\}$ for all $i \in I$.

5.3.1. $\Pi_1(\mathcal{I}; \lambda)$ is not convex

For any $x \in \Pi_1(\mathcal{I}; \lambda)$, assume that $x = \Pi_1(S_x; \lambda)$ for some $S_x \in \mathcal{I}$. By Corollary 1, there exists an $S'_x \in \mathcal{L}$ with $\lambda[S_x \Delta h^{-1}(S'_x)] = 0$. Now

$$x = \int_{S_x} f \circ h \, d\lambda = \int_{h^{-1}(S'_x)} f \circ h \, d\lambda = \int_{S'_x} f \, d\eta \in \Pi(\mathcal{L}; \eta),$$

where the last equation follows from the substitution of variables and $\eta = \lambda h^{-1}$. As a result, $\Pi_1(\mathcal{I}; \lambda) \subseteq \Pi(\mathcal{L}; \eta)$. The converse is clear. Therefore, we have $\Pi_1(\mathcal{I}; \lambda) = \Pi(\mathcal{L}; \eta)$. Notice that $\Pi(\mathcal{L}; \eta)$ is not convex by Claim 3, and therefore neither is $\Pi_1(\mathcal{I}; \lambda)$. We have thus shown

**Proposition 5.** $\Pi_1(\mathcal{I}; \lambda)$ is not convex.

5.3.2. Resolution of the irregularity of $\Pi_1(\mathcal{I}; \lambda)$

Though $\Pi_1(\mathcal{I}; \lambda)$ is not convex, however, this irregularity about the vector measure $\Pi_1$ can be resolved by working with the countably-generated extension of $(I, \mathcal{I}, \lambda)$, i.e., $(I, \mathcal{I}_1, \lambda_1)$.

**Theorem 2’.** $\Pi_1(\mathcal{I}_1; \lambda_1)$ is convex.

**Proof.** The argument is almost the same as Theorem 2. Here is a sketch. Given any $x = (x_n)_{n=0}^{\infty}, y = (y_n)_{n=0}^{\infty} \in \Pi_1(\mathcal{I}_1; \lambda_1), 0 \leq \alpha \leq 1$. We next show that $\alpha x + (1 - \alpha)y$ is also in $\Pi_1(\mathcal{I}_1; \lambda_1)$. Similar to Eq. (5), there are two $\mathcal{I}_1$-measurable subsets $S_x, S_y$ such that, $x_n = 2^{-n}\lambda_1[h^{-1}(E_n) \cap S_x]$ and $y_n = 2^{-n}\lambda_1[h^{-1}(E_n) \cap S_y]$ for all nonnegative number $n$.

Now define a Borel probability measure $\mu$ on $I = [0, 1]$ by letting $\mu(B) = \alpha \lambda_1[h^{-1}(B) \cap S_x] + (1 - \alpha)\lambda_1[h^{-1}(B) \cap S_y]$, for all Borel set $B$ in $I$. We show that $\mu$ satisfies the conditions in Lemma 3. First, it is clear that it is absolutely continuous with respect to $\eta$. Second, for any $t \in [0, 1], \mu([0, t]) \leq \lambda_1[h^{-1}([0, t])] = \lambda(h^{-1}([0, t])] = \eta([0, t]) = t$, where the first equation and the second hold because $\lambda_1$ extends $\lambda$ and $\lambda$ extends $\eta$ with $\lambda h^{-1} = \eta$ respectively. Then, as in Lemma 3, let $f_\mu$ be the Radon–Nikodym derivative of $\mu$ with respect to $\eta$. Let $g_\mu = f_\mu \circ h$
and $S'_\mu \in \mathcal{I}_1$ where $S'_\mu = \xi'[\Gamma(g_\mu) \cap C']$ as in the proof of Theorem 1'. Thus, for each $t \in [0, 1]$, we have $\mu([0, t)) = \lambda_1[h^{-1}([0, t)) \cap S'_\mu]$, see Eq. (7).

Notice that for each nonnegative integer $n$, $E_n$ is a finite union of disjoint sub-intervals in $[0, 1]$, then we have $\mu(E_n) = \lambda_1[h^{-1}(E_n) \cap S'_\mu]$. Recall that, $\mu(E_n) = \alpha \lambda_1[h^{-1}(E_n) \cap S_x^\prime] + (1 - \alpha)\lambda_1[h^{-1}(E_n) \cap S_y]$ for any nonnegative integer $n$. Hence, $\alpha x_n + (1 - \alpha)y_n = 2^{-n}\lambda_1[h^{-1}(E_n) \cap S'_\mu]$ for any nonnegative integer $n$. Therefore, $\alpha x + (1 - \alpha)y = \Pi_1(S'_\mu; \lambda_1)$. Notice $\alpha$ is arbitrary, we have thus proved the convexity of $\Pi_1(I_1; \lambda_1)$. □

5.3.3. Necessity of saturation

As in Section 5.2.3, we can argue for the necessity of the saturation property in terms of integral of the specific Banach-valued correspondence $\Pi_n, \forall n \in \mathbb{N}$. Take as given the sequence of countably-generated extensions $(I, \mathcal{I}_n, \lambda_n)$ and the measure preserving map $h_n : (I, \mathcal{I}_n, \lambda_n) \to (I, \mathcal{L}, \eta)$, for any nonnegative integer $n$. For $n \geq 2$, we define $\Pi_n = \Pi \circ h_{n-1}$. The necessity of saturation in this situation can be illustrated in Fig. 4, in which for each $n \geq 2$, $\Pi_n(I_n; \lambda_n)$ is convex.

6. Large games on Lebesgue extensions: an application

In this section we further develop our underlying themes in the context of games. One main concern here is how to model the set of players. If the set of players is the Lebesgue interval, counterexamples in [14,31] show that the Nash equilibrium for certain games with an uncountably infinite set of actions does not exist. Nevertheless, if the set of players is an atomless Loeb probability space, or a general saturated probability space, an elegant property can be achieved that every game with a large number of players, without any cardinality restrictions on the set of actions, has a Nash equilibrium; see [17] and [13] respectively. Moreover, it is also shown in [13] that saturated probability spaces are also necessary for such an elegant property. The objective of this section is to provide an understanding, as in Section 5.2, of the necessity of the saturation property by focusing on certain games as in [14,31].

We shall first give a formal definition of a game based on a probability space of players $(\Omega, \mathcal{F}, P)$. Let $A$ be a compact metric space, and let $\mathcal{U}_A$ be the space of real-valued continuous functions on $A \times \mathcal{M}(A)$ endowed with the sup-norm topology, where $\mathcal{M}(A)$ is the space of Borel probability measures on $A$ associated with the weak convergence topology. By a game $G$ with player space $\Omega$ and action space $A$ we will mean a random element of $\mathcal{U}_A$ on $(\Omega, \mathcal{F}, P)$. Thus, a game simply associates each player $\omega \in \Omega$ with a payoff function $G(\omega)(a, \nu)$ that depends on the player’s own action $a$ and the distribution of actions by all the players, $\mu$. To improve readability, we also use $G_\omega$ to denote $G(\omega)$.
Definition 5. A Nash equilibrium of a game \( G \) is a random element \( g : \Omega \rightarrow A \) such that for \( P \)-almost all \( \omega \in \Omega \),

\[
G_\omega \left[ g(\omega), P \circ g^{-1} \right] \geq G_\omega \left[ a, P \circ g^{-1} \right] \quad \text{for all } a \in A.
\]

Example. This game is taken from Section 2 of [14]. Consider a game \( G \) in which the set of players \( (\Omega, \mathcal{F}, P) \) is the Lebesgue unit interval \( (I = [0, 1], \mathcal{L}, \eta) \), \( A \) is the interval \([-1, 1] \), and the payoff function of any player \( i \in I \) is given by

\[
G_i(a, \mu) = q \left[ a, \beta d(\mu^*, \mu) \right] - |i - |a||, \quad \forall a \in [-1, 1], \forall \mu \in \mathcal{M}([-1, 1]),
\]

where \( \beta \in (0, 1) \) is a constant, \( \mu^* \) the uniform distribution on \([-1, 1] \), \( d(\mu^*, v) \) the Prohorov distance between \( \mu^* \) and \( \mu \) based on the natural metric on \([-1, 1] \), and \( q : [-1, 1] \times [0, 1] \rightarrow \mathbb{R}_+ \) defined as follows. For any \( a \in [-1, 1] \), \( q(a, 0) = 0 \), and for any \( \ell \in (0, 1] \), \( q(\cdot, \ell) \) is a periodic function with least period \( 2\ell \), defined on one period \([0, 2\ell]\) by

\[
q(a, \ell) = \begin{cases} 
  a/2, & \text{for } 0 \leq a \leq \ell/2; \\
  (\ell - a)/2, & \text{for } \ell/2 \leq a \leq \ell; \\
  -q(a - \ell, \ell), & \text{for } \ell \leq a \leq 2\ell.
\end{cases}
\]

Notice that \( q(a, \ell) = -q(-a, \ell) \) for \( a < 0 \).

For such a game, as proved in [14], there does not exist a Nash equilibrium. Suppose not, let \( g \) be a Nash equilibrium in \( G \) associated with the distribution \( \mu \). On the one hand, \( \mu \) cannot be the uniform distribution \( \mu^* \). Indeed, if so, for any player \( i \in I \), the set of her best response is \( \{i, -i\} \), this is nothing but the set-valued function \( \Psi \) above. That is to say that \( g \) is an \( \mathcal{L} \)-measurable selection of \( \Psi \) which induces the uniform distribution \( \mu^* \) on \([-1, 1] \). This contradicts Claim 2. On the other hand, if \( \mu \neq \mu^* \), the choice of the function \( q \) guarantees that the distribution induced by the best-response correspondence cannot be the one given by \( \mu \), and therefore precludes the existence of a Nash equilibrium.

However, if we do not use the Lebesgue extension to model the set of players, and use instead \((I, \mathcal{T}, \lambda)\), the countably-generated Lebesgue extension constructed in Section 2.2, we can get the following positive result.

Proposition 6. There exists a Nash equilibrium in game \( G \) if the player space is replaced by \((I, \mathcal{T}, \lambda)\).

Proof. By Proposition 2, there exists an \( \mathcal{T} \)-measurable selection \( \psi \) of the correspondence \( \Psi \), where \( \Psi(i) = \{i, -i\} \) for any \( i \in I \), such that the induced distribution is the uniform distribution \( \mu^* \). We next claim that \( \psi \) is a Nash equilibrium of game \( G \) if the player space is replaced by \((I, \mathcal{T}, \lambda)\). Indeed, for any player \( i \in I \), if all the other players take actions as described by \( \psi \), the distribution of all the other players is the uniform distribution \( \mu^* \). In this case, player \( i \)'s action \( \psi(i) \), either \( i \) or \(-i\), is the best response by the definition of the payoff function. Hence \( \psi \) is a Nash equilibrium in game \( G \) if the player space is replaced by \((I, \mathcal{T}, \lambda)\). \( \square \)
Notice that \((I, \mathcal{I}, \lambda)\) is a countably-generated probability space, take \(h : (I, \mathcal{I}, \lambda) \to (I, \mathcal{L}, \eta)\) to be the measure preserving map where a measure algebra isomorphism between these two measure algebras can be realized by \(h\). Now consider the following game \(G^1\) on \((I, \mathcal{I}, \lambda)\) with the same action set \(A = [-1, 1]\), for any player \(i\), the payoff function is,

\[
G^1_i(a, \mu) := G_h(i)(a, \mu) = q[a, \beta \delta(J^\mu, \mu)] - |h(i) - |a||
\]

for all \(a \in [-1, 1]\), \(\mu \in \mathcal{M}([-1, 1])\), where \(\beta \in (0, 1)\) is a constant. Note that, in the new game \(G^1\), player \(i\) receives the identical payoff of player \(h(i)\) in the old game \(G\), for all \(i \in I\). Put differently, \(G^1\) can be played in two steps, first, there is roughly a fixed permutation among players where the new order of player \(i\) is \(h(i)\), then the old game \(G\) is played in which player \(i\) plays the same role as player \(h(i)\). In the sequel, we also denote \(G^1\) by \(G \circ h\).

**Theorem 3.** There does not exist Nash equilibrium in the game \(G^1\).

**Proof.** We prove this by contradiction. Suppose \(g^1 : (I, \mathcal{I}, \lambda) \to [-1, 1]\) is a Nash equilibrium of \(G^1\). Let \(\mu\) be the induced distribution of \(g^1\), i.e., \(\lambda \circ (g^1)^{-1} = \mu\).

We first show that \(\mu\) cannot be the uniform distribution \(\mu^*\) on \([-1, 1]\). In fact, if \(\mu = \mu^*\), then for any player \(i\), her best response is either \(h(i)\) or \(-h(i)\). Recall that \(\Psi_1(i) = (\Psi \circ h)(i) = \{h(i), -h(i)\}\), that is to say, \(g^1\) is an \(\mathcal{I}\)-measurable selection of \(\Psi_1\), and \(g^1\) induces the uniform distribution \(\mu^*\) on \([-1, 1]\). However, this contradicts Proposition 4 in which \(\mathcal{D}_{\Psi_1}(\lambda) = \mathcal{D}_{\Psi}(\eta)\) and \(\mu^*\) is not in \(\mathcal{D}_{\Psi}(\eta)\).

Now \(\mu \neq \mu^*\), let \(\ell = \beta \delta(J^\mu, \mu)\). Note that \(\ell > 0\). It is straightforward (as in [31]) to show that the best response for player \(i \in I\) is the following,

\[
B_i(\mu) := \begin{cases} 
\{h(i)\}, & \text{for } h(i) \in \bigcup_{k \in \mathbb{N}}(2k\ell, (2k + 1)\ell); \\
\{-h(i)\}, & \text{for } h(i) \in \bigcup_{k \in \mathbb{N}}((2k + 1)\ell, (2k + 2)\ell); \\
\{h(i), -h(i)\}, & \text{for } h(i) = k\ell, \text{for some } k \in \mathbb{N}.
\end{cases}
\]

By definition of Nash equilibrium, without loss of generality, we assume that \(g^1(i) \in B_i(\mu)\) for all \(i \in I\). Define \(g : (I, \mathcal{L}, \eta) \to [-1, 1]\) as follows,

\[
g(i) = \begin{cases} 
i, & \text{for } i \in \bigcup_{k \in \mathbb{N}}[2k\ell, (2k + 1)\ell]; \\
-i, & \text{for } i \in \bigcup_{k \in \mathbb{N}}[(2k + 1)\ell, (2k + 2)\ell].
\end{cases}
\]

It is clear that \(g^1(i) = g \circ h(i)\) for \(i \neq k\ell\) with \(k \in \mathbb{N}\). Take into account that all the points in \(I\) of the form \(k\ell, k \in \mathbb{N}\) form an \(\eta\)-null set, then we have \(\mu = \lambda \circ (g^1)^{-1} = \lambda(h^{-1} \circ g^{-1}) = \eta \circ g^{-1}\), where the last equation holds since that \(h\) is measure preserving.

Finally, we show that \(g\) is a Nash equilibrium of the game \(G\) where the player space is modeled by the Lebesgue unit interval \((I, \mathcal{L}, \eta)\), which contradicts the non-existence result of [14], and we thus complete the proof. Towards this end, first, notice that \(g^1\) and \(g\) induce the same distribution \(\mu\) on \([-1, 1]\). Next, we claim that for any player \(t \in h(I)\) with \(t \neq k\ell\) for any \(k \in \mathbb{N}\), \(g(t)\) is the best response for her against \(\mu\) in game \(G\). Let \(i \in I\) be a player such that \(t = h(i)\). By the definition of \(B_i(\mu)\) above, in game \(G^1\), for player \(i\), \(g^1(i) = g[h(i)] = g(t)\) is her unique best response. As a result, in game \(G\), for such a player \(t\), \(g(t)\) is the unique best response for her against \(\mu\). Finally, by Definition 5, \(g\) is a Nash equilibrium of \(G\) if we can show that all players in the set \([t \in h(I) : t \neq k\ell \text{ for any } k \in \mathbb{N}]\) form an \(\eta\)-full set. Indeed, since a measure-algebra
isomorphism is induced by $h$, without loss of generality, $h(I) \in \mathcal{L}$ is of full $\eta$-measure. As a result, $[t \in h(I): t \neq k\ell$ for any $k \in \mathbb{N}]$ is also an $\eta$-full set. We thus complete the proof. \hfill $\square$

The negativity of Theorem 3 can be resolved in a similar way as the resolution exhibited by Proposition 6. To be precise, even though $G^1$ has no Nash equilibrium based on the countably-generated Lebesgue extension $(I, \mathcal{I}, \lambda)$, we can construct a countably-generated extension of the extension to assure that the game $G^1$ has a Nash equilibrium based on the new extension. However, a new game can be constructed from $G$ such that it does not have Nash equilibria based on the new extension. Moreover, this procedure can be repeated \emph{ad infinitum}, just like the repetition we discussed in Sections 5.2.3 and 5.3.3. In such a way, we can explain the necessity of saturation in the context of specific game with a continuum of number of players by the following Fig. 5, which is similar to Figs. 3 and 4.

In Fig. 5, $(I, \mathcal{I}_n, \lambda_n)$ and the measure preserving map $h_n : (I, \mathcal{I}_n, \lambda_n) \to (I, \mathcal{L}, \eta)$, for any nonnegative integer $n$. For $n \geq 2$ are the same as in Figs. 3 and 4. For $n \geq 2$, the game $G^n$ where the player space is modeled by $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$ is defined as $G^n = G \circ h_{n-1}$. For $n \geq 2$, there does not exist a Nash equilibrium in the game $G^n$, however, there does exist a Nash equilibrium if the player space is modeled by $(I, \mathcal{I}_n, \lambda_n)$, which is a countably-generated extension of $(I, \mathcal{I}_{n-1}, \lambda_{n-1})$.

\textbf{Remark 7.} Note that we have focussed attention on the first counterexample considered in [14]. An additional example based on a weakly compact action set in an infinite-dimensional separable Banach space is also considered in [14], our resolution of Claim 3 can be carried through to deal with that example, see [35].

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The authors of [14] (Khan, Rath and Sun) are indebted to Xiang Sun for pointing out that the functions $W_n(t)$ used in [14] are only part of the set of Walsh functions. Here is a simple correction in the context of [14]. For $n \geq 1$, let $n_0 + n_1 2^1 + \cdots + n_q 2^q$ be the binary expansion of $n-1$, where $n_q = 0$ or 1 for $0 \leq \ell \leq q$. For $t \in [0, 1]$, let $t = \sum_{\ell=0}^{\infty} t_\ell 2^{-\ell-1}$ with $t_\ell = 0$ or 1, $W_n(t) = \sum_{\ell=0}^q n_\ell t_\ell$, and $\psi_n(t) = 2^{-q} \left(1 - W_n(t)\right)$. Denote $\sum_{\ell=0}^{q-1} n_\ell t_\ell$ by $\tilde{n}$, and $\sum_{\ell=0}^{q-1} t_\ell 2^{-\ell-1}$ by $\tilde{t}$. Next, fix $n \geq 2$ and assume that $n_q = 1$. We need to estimate $H_n = |\sum_{t_\ell=0}^q (-1)^{\tilde{n}_\ell t_\ell} \psi_n(t_{\tilde{n}, t})\, d\lambda(t_{\tilde{n}, t})|$. It is easy to see that

$$H_n = |\sum_{t_\ell=0}^q (-1)^{\tilde{n}_\ell} \psi_n(t_{\tilde{n}, t})\, d\lambda(t_{\tilde{n}, t})| = \frac{1}{2} |\sum_{t_\ell=0}^q (-1)^{\tilde{n}_\ell} \psi_n(t_{\tilde{n}, t})\, d\lambda(t_{\tilde{n}, t})|.
$$

By change of variables, we can obtain that

$$H_n \leq \frac{1}{2} \left|\int_0^1 (-1)^{u_\ell+\alpha} (1 - W_{n+1}(t))\, d\lambda(t)\right| + \sum_{t_\ell=0}^q (-1)^{\tilde{n}_\ell} \psi_n(t_{\tilde{n}, t})\, d\lambda(t_{\tilde{n}, t}).$$

It follows from Lemma 5 that $H_n \leq \frac{1}{2} 2^q+1 \alpha 2^q = 2^{2q+1} \alpha \leq 2(n-1)\alpha \leq 2^{n-1} \alpha$. Hence, $|\int_0^1 (-1)^{u_\ell+\alpha} (1 - W_{n+1}(t))\, d\lambda(t)| \leq \min(2, 2^n \alpha)$ for $n \geq 1$. Therefore Lemma 6 and hence Lemma 4 still hold when $W_n(t) = (-1)^{2^n-1} t$ in [14] is substituted by $W_n(t) = (-1)^{2^n-1} t$ here. All the other proofs remain the same.
7. Concluding remarks

The results presented in this paper both underscore and question the “necessity theory” presented by Keisler and Sun [13]. On the one hand, in the context of specific set-valued functions, they show that no finite number of extensions of the underlying measure space is enough to bypass the necessity of saturated probability spaces: as Figs. 3–5 illustrate, irrespective of the number of extensions, there always exists a correspondence which exhibits irregularities. On the other hand, if a particular set-valued function that arises in applied work is at issue, it may not at all be necessary to go to a saturated probability space. Depending on the complexity of the function, even a single extension may suffice, as is the case with Claim 2 above. This latter point attains especial salience in the context of the theory of large games. To be sure, in keeping with the necessity results in [13], there exist games without Nash equilibria on a space extended a finite number of times, but such games may be entirely without interest in terms of the particular economic or game-theoretic phenomena being modelled. To repeat the point, as far as the specific model is at issue, a simple countably-generated extension of the space of players may fully suffice.

This being said, we should like to conclude this paper with two open questions. The first concerns the correspondence in Claim 1 about which we have been silent in the context of the necessity theory. This was inevitable in light of the fact that the two operations of distribution and integration over the entire domain space, as considered in this paper, are not really at issue in Claim 1. What is important there is the existence of a selection for which the integral is null for each interval \([0, t], t \in [0, 1]\). Whereas such a selection can surely be found for a Loeb counting space, it is not at all clear as to the shape of this requirement for a general saturated space. In particular, there is no necessity theory here to appeal to and to circumvent. One suspects that the issue hinges on the homogeneity property, one that is discussed in [4, Section 2D], and in the context of large non-anonymous games, in [17], but a clear formulation and possible resolution would have to await future work. Such work would presumably have to build on the reformulated theory of large games presented in [15] that is constructed around the precise correspondence considered in Claim 1.

The second question concerns the necessity of saturation in the context of purification issues, as considered in [23, 25, 30, 37], and in the context of finite games with private information. One can inquire into the shape of the Keisler–Sun necessity theory for such games, and more specifically, whether the countably-generated extension of the Lebesgue interval presented in this paper answers the counterexample in [23] concerning the impossibility of a purification from a set-valued function based on the Debreu map used in Claim 2 in this paper. This question too we leave for future work.

References