Fall 2011 Macro Group Brown Bag

Fei TAN

Department of Economics
Indiana University Bloomington

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Solving Linear Rational Expectations Models: A $z$-Transform Approach

by

Fei Tan & Todd B. Walker

Department of Economics, Indiana University
A Road Map...

1. Introduction
2. Minimal Tools
3. Solution Technique
   - Univariate Case: A Toy Model
   - A Summary of Solution Tenets
   - Multivariate Case: A Monster Model
   - Connection to Gensys
4. Illustrative Examples
   - Example 1: Leeper (1991)
   - Example 2: Simple RBC Model
   - Example 3: Decoupled System
5. Concluding Remarks
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Motivation: Call For New Solution Device

- Fact: multivariate linear rational expectations models are extensively employed, solved, and estimated for empirical research on cause and effect in macroeconomy. [Sims & Sargent]

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Technical Difficulties And Problems

- **Infinite regress in expectations**: when forming expectations, agents forecast forecasts of others under incomplete information, making state vector explode over time. [Townsend (1983)]

- **Numerical rather than analytical solutions**: explicit characterization of equilibrium will make cross-equation restrictions transparent and facilitate econometric implementation. [This work]

- **Stringent driving processes**: inclusion of oversimplified stochastic processes for exogenous shocks poses serious question: “how general is our macro model?” [Todd in E550 class]
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**Observational equivalence**: two distinct policy regimes may generate identical equilibrium time series and empirical test based on simple correlations in data may lead to spurious results. Such identification challenge may extend to more general setups. [Leeper and Walker (2011)]

**Decoupling issue**: existing methods, e.g. root-counting approach and winding number criterion, give incorrect existence condition to decoupled systems. [Sims (2007)]

**Not user-friendly perhaps**: conventional device, e.g. Gensys, requires cleverness in models containing multiple leads and lags, withholding equations, general driving processes, etc. [Sims (2002)]
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Main Results

A summary of key points in this work:

- We generalized linear rational expectations solution method of White-man (1983) to multivariate case, allowing use of generic exogenous driving process which must only satisfy covariance stationarity;
- We derived multivariate cross-equation restrictions linking Wold representation of exogenous process to endogenous variables. This mapping is multivariate version of Hansen-Sargent formula;
- We connected our solution methodology to other popular approaches, e.g. Sims (2002);
- We introduced several motivating examples that highlight usefulness of our approach.
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Minimal Tools

- A stochastic process \( \{ y_t \} \) is said to be **covariance stationary** if \( E y_t = \mu \) for all \( t \in T \), and \( \text{cov}(y_t, y_s) = \sigma_{t,s} \) depends only on \( t - s \).

- An overview of basic tools in this work:
  - **Wold Representation Theorem**: any covariance stationary process \( \{ y_t \} \) can be represented as \( y_t = \mu_t + \sum_{j=0}^{\infty} \theta_j \varepsilon_{t-j} \) with \( \sum_{j=0}^{\infty} \theta^2_j < \infty \). Here \( \{ \varepsilon_t \} \sim WN(0, \sigma^2) \) (our basic building blocks);
  - **Corollary of Riesz-Fischer Theorem**: any square summable sequence \( \{ c_n \}_{-\infty}^{\infty} \) has \( z \)-transform \( g(z) = \sum_{j=-\infty}^{\infty} c_j z^j \) where \( c_k \) can be recovered from \( g(z) \) via inversion formula;
  - **Wiener-Kolmogorov Prediction Formula**: expectations are formed optimally via \( E_t y_{t+k} = \left[ \frac{C(L)}{L^k} \right]_+ \varepsilon_t \);
  - **Smith Normal Form**: useful decomposition technique for polynomial matrices.
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Univariate Case: A Toy Model

- Consider a simple asset valuation model

\[ P_t = \beta E_t(P_{t+1}) + d_t, \quad 0 < \beta < 1 \quad (1) \]

- Dividend process \( d_t \) has Wold representation

\[ d_t = \sum_{j=0}^{\infty} \rho^j \varepsilon_{t-j} = \frac{1}{1 - \rho L} \varepsilon_t = A(L)\varepsilon_t, \quad 0 < \rho < 1 \quad (2) \]

- “Guess” a solution form for \( P_t \) with square-summable coefficients

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Univariate Case: A Toy Model (Cont’d)

- **Wiener-Komolgorov optimal prediction formula**

\[ E_t P_{t+1} = L^{-1} [C(L) - C_0] \varepsilon_t = \left[ \frac{C(L)}{L} \right] + \varepsilon_t \quad (4) \]

- Plugging in expressions for \( P_t, E_t P_{t+1}, \) and \( d_t \) gives

\[ C(L) \varepsilon_t = \beta L^{-1} [C(L) - C_0] \varepsilon_t + A(L) \varepsilon_t \quad (5) \]

- Above equation holds for all realizations of \( \{ \varepsilon_t \} \), implying \( z \)-transforms of LHS and RHS are identical as analytic functions on open unit disk, i.e.

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- Solving for $C(z)$ gives $z$-transform of $P_t$

\[ C(z) = \frac{zA(z) - \beta C_0}{z - \beta} \]  

(7)

- How to pin down $C_0$? Square summability in time domain is tantamount to $z$-transform being analytic in frequency domain

\[ 0 < \beta < 1 \Rightarrow C_0 = A(\beta) = \frac{1}{1 - \rho \beta} \]  

(8)

- Unique solution coincides with solution by iterating forward

\[ P_t = \frac{1}{(1 - \rho \beta)(1 - \rho L)} \varepsilon_t = \sum_{k=0}^{\infty} (\rho \beta)^k d_t = E_t \sum_{k=0}^{\infty} \beta^k d_{t+k} \]  

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More insights about selection of $C_0$...

- Consistency of solution: any candidate for $C_0$ satisfying existence condition leads to a $C(z)$, whose “$C_0$” must be identical to that candidate so that solution is consistent with rational expectations restrictions imposed by model.

- A rational expectations equilibrium is a fixed point of mapping between perceived law of motion and law of motion generated by those beliefs.

- Each candidate for $C_0$ designates a perceived law of motion and a solution generated by such $C_0$ forms a new law of motion with equilibrium conditions imposed.

- Consistency of solution requires that two laws of motion be identical and thus $z$-transform approach reduces to solving fixed point problem.
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A comparison of determinacy conditions: $z$-transform v.s. “usual” approach

- By conventional approach, e.g. Sims (2002), one forecast error requires one unstable eigenvalue ($1/\beta > 1 \Rightarrow \beta < 1$) so that exogenous shocks pin down all error terms influenced by endogenous shocks. This is tantamount to pinning down $C_0$ by making $C(z)$ analytic at pole inside unit circle, $z = \beta < 1$.

- What if $\beta \geq 1$? There is no unstable eigenvalue ($0 < 1/\beta \leq 1$) imposing restrictions between exogenous and endogenous shocks and we have infinite solutions, each of which is indexed by a sunspot. This is tantamount to pole of $C(z)$ being outside unit circle and $C_0$ can be set in arbitrary manner, each of which indexes an equilibrium.
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A comparison of determinacy conditions: z-transform v.s. “usual” approach

- By conventional approach, e.g. Sims (2002), one forecast error requires one unstable eigenvalue \((1/\beta > 1 \Rightarrow \beta < 1)\) so that exogenous shocks pin down all error terms influenced by endogenous shocks. This is tantamount to pinning down \(C_0\) by making \(C(z)\) analytic at pole inside unit circle, \(z = \beta < 1\).

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Solution Principles

- Four tenets embedded in previous toy model [Whiteman (1983)]:
  - Driving process is taken to be zero-mean linearly regular covariance stationary process with known Wold representation;
  - Expectations are formed rationally and are computed using Wiener-Kolmogorov formula;
  - Solutions are sought in space spanned by time-independent square-summable linear combinations of process fundamental for driving process;
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Multivariate Case: A Monster Model

- Let’s be ambitious and solve a monster model (user-friendly)

\[
\left[ \sum_{k=1}^{n} F_k L^{-k} + \sum_{k=0}^{m} G_k L^k \right] y_t = \left[ \sum_{k=1}^{n} \Phi_k L^{-k} + \sum_{k=0}^{l} \Psi_k L^k \right] x_t + \sum_{k=1}^{n} \left[ F_k \Pi_k^y L^{-k} \eta_t - \Phi_k \Pi_k^x L^{-k} \nu_t \right]
\]  
(10)

- Wold representations for vector exogenous and endogenous processes

\[ x_t = A(L) \varepsilon_t \quad \text{and} \quad y_t = C(L) \varepsilon_t \]  
(11)

Further assumptions:
- \( F_n \) is of full rank (can be relaxed though...);
- Roots of \( \text{det}[z^n (F(z^{-1}) + G(z))] = \sum_{k=0}^{h} f_k z^k \) are distinct, with \( rp \) of them lie inside unit circle while \( h - rp \) of them lie outside unit circle.
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Wiener-Komolgorov formula for endogenous & exogenous forecast errors

\[
\eta_{t+k} = y_{t+k} - E_t y_{t+k} = L^{-k} \left( \sum_{i=0}^{k-1} C_i L^i \right) \varepsilon_t
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\[
\nu_{t+k} = x_{t+k} - E_t x_{t+k} = L^{-k} \left( \sum_{i=0}^{k-1} A_i L^i \right) \varepsilon_t
\]

z-transform of equilibrium conditions become

\[
(F(z^{-1}) + G(z)) C(z) = (\Phi(z^{-1}) + \Psi(z)) A(z) + \sum_{t=1}^{n} \sum_{s=t}^{n} (F_s \Pi_s^y C_{t-1} - \Phi_s \Pi_s^x A_{t-1}) z^{-s+t-1}
\]

Applying Smith canonical form gives

\[
z^n (F(z^{-1}) + G(z)) = \underbrace{U(z)^{-1} P_1(z)}_{S(z)} \underbrace{P_2(z) V(z)^{-1}}_{T(z)}
\]
Multivariate Case: A Monster Model (Cont’d)

- Wiener-Komolgorov formula for endogenous & exogenous forecast errors

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(14)

- Applying Smith canonical form gives

\[ z^n (F(z^{-1}) + G(z)) = \frac{U(z)^{-1} P_1(z)}{S(z)} \frac{P_2(z) V(z)^{-1}}{T(z)} \]  

(15)
Wiener-Komolgorov formula for endogenous & exogenous forecast errors

\[ \eta_{t+k} = y_{t+k} - E_t y_{t+k} = L^{-k} \left( \sum_{i=0}^{k-1} C_i L^i \right) \varepsilon_t \]  
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Multivariate Case: A Monster Model (Cont’d)

- Inverting $S(z)$ gives $z$-transform identity

$$T(z)C(z) = \frac{z^n \tilde{U}(z)}{\prod_{j=1}^{k}(z - z_j)} \left\{ (\Phi(z^{-1}) + \Psi(z))A(z) + \sum_{t=1}^{n} \sum_{s=t}^{n} (F_s \Pi^y_s C_{t-1} - \Phi_s \Pi^x_s A_{t-1})z^{-s+t-1} \right\}$$

(16)

- How to pin down $C_0, C_1, \ldots, C_{n-1}$? Square summability in time domain is tantamount to $z$-transform being analytic in frequency domain

$$(z - z_j)T(z)C(z)|_{z=z_j} = 0, \quad j = 1, \ldots, k$$

(17)

- Stacking restrictions yield a compact expression

$$A_{<} = - R_{<} C_{\left[ k \times q \right], \left[ k \times np \right]} \left[ np \times q \right]$$

(18)
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Existence and Uniqueness Conditions

- Existence problems arise if endogenous shocks $\eta$ cannot adjust to offset exogenous shocks $x$. Solution exists iff

$$\text{span}(A_<) \subseteq \text{span}(R_<)$$  \hspace{1cm} (19)

- Analytical solution for $y_t$ is given by

$$y_t = y_t^P + y_t^R$$

Perfect foresight solution \hspace{2cm} Remainder term $y_t - y_t^P$  \hspace{1cm} (20)

- In order for solution to be unique, one must be able to determine remainder term from knowledge of $R<C$. Solution is unique iff

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THEOREM Consider a special case of monster model

$$[F_0L^0 + F_1L^{-1}]y_t = \Phi_1L^{-1}x_t + F_1\Pi_1L^{-1}\eta_t$$ (22)

where $F_1$ is of full rank. Assume both eigenvalues of $-F_1^{-1}F_0$ and roots of $\det[zF_0 + F_1] = 0$ are nonzero and distinct. Then...

1. **Factorization equivalence:** eigenvalues of $-F_1^{-1}F_0$ are exactly inverse of corresponding roots of $\det[zF_0 + F_1] = 0$, or, those roots of determinant of Smith Normal Form for $zF_0 + F_1$;

2. **Existence equivalence:** restrictions imposed by unstable eigenvalues in Gensys are exactly those imposed by roots inside unit circle in $z$-transform approach;

3. **Uniqueness equivalence:** Gensys and $z$-transform approach yield identical uniqueness condition.
Connection to Gensys

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Example 1: Leeper (1991)

- The following bivariate system in \((\pi_t, b_t)\) delivers a cashless version of Leeper (1991) model:

\[
E_t\pi_{t+1} = \alpha\pi_t + \theta_t \tag{23}
\]

\[
b_t + \beta^{-1}\pi_t = [\beta^{-1} - \gamma(\beta^{-1} - 1)]b_{t-1} + \alpha\beta^{-1}\pi_{t-1} - (\beta^{-1} - 1)\psi_t \tag{24}
\]

- Decoupling problem: merely counting \# of unstable eigenvalues does not necessarily deliver existence conditions!
- Casting above system into monster model gives

\[
\begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
F_1 L^{-1} + \begin{pmatrix}
-\alpha \\
1/eta \\
0
\end{pmatrix}
G_0 L^0 + \begin{pmatrix}
0 \\
0 \\
-\alpha /eta
\end{pmatrix}
G_1 L + \begin{pmatrix}
1 \\
0 \\
1/eta
\end{pmatrix}
\begin{pmatrix}
\theta_t \\
\psi_t \\
\eta_t^\pi
\end{pmatrix}
\]

\[
= \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\psi_0 + \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
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\theta_t \\
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L^{-1} \begin{pmatrix}
1 \\
0 \\
0
\end{pmatrix}
\eta_t^b
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\[
\begin{pmatrix}
L^{-1} + \begin{pmatrix}
-\alpha \\
\frac{1}{\beta} \\
0 \\
0
\end{pmatrix} & \begin{pmatrix}
0 \\
0 \\
0 \\
1
\end{pmatrix} \\
F_1 & G_0 & G_1
\end{pmatrix}
\begin{pmatrix}
\pi_t \\
b_t
\end{pmatrix} = \begin{pmatrix}
L^{-1} \\
\psi_0 \\
x_t \\
F_1 \\
\Pi_1 \\
\eta_t
\end{pmatrix}
\begin{pmatrix}
\theta_t \\
\psi_t \\
0 \\
0 \\
0 \\
0
\end{pmatrix} + \begin{pmatrix}
1 \\
0 \\
0 \\
1 \\
0 \\
0
\end{pmatrix} \begin{pmatrix}
\eta_t^{\pi} \\
\eta_t^b
\end{pmatrix}
\]
Applying Smith Form decomposition gives

\[ z(F(z^{-1}) + G(z)) = U(z)^{-1} \begin{pmatrix} 1 & 0 \\ 0 & z(z - \frac{1}{\alpha}) \end{pmatrix} \begin{pmatrix} 0 & \frac{1}{\beta - \gamma(\frac{1}{\beta}-1)} \\ z(z - \frac{1}{\alpha}) & \frac{1}{\beta - \gamma(\frac{1}{\beta}-1)} \end{pmatrix} V(z)^{-1} \]  

(25)

\[ \text{det}[z^n(F(z^{-1}) + G(z))] \] has three distinct roots: \( z_1 = 0, z_2 = \frac{1}{\alpha}, \) and \( z_3 = \frac{1}{\beta - \gamma(\frac{1}{\beta}-1)}. \)

Existence and unique conditions depend on whether these roots are inside or outside unit circle.
Roots Inside & Outside Unit Circle

- Applying Smith Form decomposition gives

\[ z(F(z^{-1}) + G(z)) = U(z)^{-1} \left( \begin{array}{cc}
1 & 0 \\
0 & z \left( z - \frac{1}{\alpha} \right) \left( z - \frac{1}{\beta - \gamma(\frac{1}{\beta} - 1)} \right) \end{array} \right) V(z)^{-1} \] (25)

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\[ P(z) \]

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Existence and unique conditions depend on whether these roots are inside or outside unit circle.
Case 1: Passive Monetary ($\alpha < 1$) & Passive Fiscal ($\gamma > 1$)

- One root inside unit circle, $z = 0$, and two roots outside unit circle, $z = \frac{1}{\alpha} > 1$ and $z = \frac{1}{\beta - \gamma(\frac{1}{\beta} - 1)} > 1$.

- Existence requires $\text{span}(A_<) \subseteq \text{span}(R_<)$ (satisfied). However, all restrictions on $C_0$ are redundant and $C_0$ can be set in arbitrary way.

- Uniqueness requires $\text{span}(R'_>) \subseteq \text{span}(R'_<)$ (not satisfied). Thus, there are infinite solutions.

- Characterization of set of sunspot equilibria. [Lubik and Schorfheide (2003)]
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- Uniqueness requires $\text{span}(R'_{\succ}) \subseteq \text{span}(R'_\prec)$ (not satisfied). Thus, there are infinite solutions.

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- Existence requires $\text{span}(A_-) \subseteq \text{span}(R_-)$ (satisfied). However, all restrictions on $C_0$ are redundant and $C_0$ can be set in arbitrary way.

- Uniqueness requires $\text{span}(R'_+) \subseteq \text{span}(R'_-)$ (not satisfied). Thus, there are infinite solutions.

- Characterization of set of sunspot equilibria. [Lubik and Schorfheide (2003)]
Case 2: Active Monetary ($\alpha > 1$) & Passive Fiscal ($\gamma > 1$)

- Two roots inside unit circle, $z = 0$ and $z = \frac{1}{\alpha} < 1$, and one root outside unit circle, $z = \frac{1}{\beta - \gamma(\frac{1}{\beta} - 1)} > 1$.

- Existence requires $\text{span}(A_<) \subseteq \text{span}(R_<)$ (satisfied). Restrictions imply $C_{01}^{11} = -\frac{1}{\alpha}$ and $C_{01}^{12} = 0$. However, $C_{01}^{21}$ and $C_{01}^{22}$ can be set in arbitrary way.

- Uniqueness requires $\text{span}(R'_>) \subseteq \text{span}(R'_<)$ (satisfied). Thus, solution is unique and is given by

$$
\begin{pmatrix}
\pi_t \\
\theta_t \\
\psi_t
\end{pmatrix} = \begin{pmatrix}
\frac{-1}{\alpha} & 0 & 0 \\
\frac{1}{\alpha \beta} & 1 - \beta & 0 \\
1 - \gamma + \gamma \beta & 1 - \gamma + \gamma \beta & 0
\end{pmatrix}
\begin{pmatrix}
\theta_t \\
\psi_t \\
C_0
\end{pmatrix}
+ \sum_{k=1}^{\infty}
\begin{pmatrix}
0 \\
\rho^k - \frac{\rho^{k-1}}{\beta} & 0 \\
\frac{1 - \beta}{1 - \gamma + \gamma \beta} \rho^k & C_k
\end{pmatrix}
\begin{pmatrix}
\theta_{t-k} \\
\psi_{t-k}
\end{pmatrix}
$$

(26)
Case 2: Active Monetary ($\alpha > 1$) & Passive Fiscal ($\gamma > 1$)

- Two roots inside unit circle, $z = 0$ and $z = \frac{1}{\alpha} < 1$, and one root outside unit circle, $z = \frac{1}{\beta - \gamma(\frac{1}{\beta} - 1)} > 1$.

- Existence requires $\text{span}(A_{<}) \subseteq \text{span}(R_{<})$ (satisfied). Restrictions imply $C_{01}^{11} = -\frac{1}{\alpha}$ and $C_{01}^{12} = 0$. However, $C_{01}^{21}$ and $C_{01}^{22}$ can be set in arbitrary way.

- Uniqueness requires $\text{span}(R_{>}) \subseteq \text{span}(R_{<})$ (satisfied). Thus, solution is unique and is given by

$$
\begin{pmatrix}
\pi_t \\
\theta_t
\end{pmatrix}
= 
\begin{pmatrix}
\frac{1}{\alpha \beta} & 0 \\
\frac{1 - \beta}{1 - \gamma + \gamma \beta}
\end{pmatrix}
\begin{pmatrix}
\theta_t \\
\psi_t
\end{pmatrix}
+ 
\sum_{k=1}^{\infty}
\begin{pmatrix}
0 \\
\rho^k - \frac{\rho^{k-1}}{\beta}
\end{pmatrix}
\begin{pmatrix}
0 \\
\frac{1 - \beta}{1 - \gamma + \gamma \beta} \rho^k
\end{pmatrix}
\begin{pmatrix}
\theta_{t-k} \\
\psi_{t-k}
\end{pmatrix}
$$

(26)
Case 2: Active Monetary ($\alpha > 1$) & Passive Fiscal ($\gamma > 1$)

- Two roots inside unit circle, $z = 0$ and $z = \frac{1}{\alpha} < 1$, and one root outside unit circle, $z = \frac{1}{\beta - \gamma(\frac{1}{\beta} - 1)} > 1$.

- Existence requires $\text{span}(A_{<}) \subseteq \text{span}(R_{<})$ (satisfied). Restrictions imply $C_{011} = -\frac{1}{\alpha}$ and $C_{012} = 0$. However, $C_{021}$ and $C_{022}$ can be set in arbitrary way.

- Uniqueness requires $\text{span}(R'_{>}) \subseteq \text{span}(R'_{<})$ (satisfied). Thus, solution is unique and is given by

$$
\begin{pmatrix}
\pi_t \\
b_t
\end{pmatrix} = 
\begin{pmatrix}
\frac{-1}{\alpha} & 0 \\
\frac{1}{\alpha \beta} & \frac{1-\beta}{1-\gamma+\gamma \beta}
\end{pmatrix}
\begin{pmatrix}
\theta_t \\
\psi_t
\end{pmatrix} + \sum_{k=1}^{\infty}
\begin{pmatrix}
0 & 0 \\
\frac{\rho^k - \rho^{k-1}}{\alpha \beta} & \frac{1-\beta}{1-\gamma+\gamma \beta} \rho^k
\end{pmatrix}
\begin{pmatrix}
\theta_{t-k} \\
\psi_{t-k}
\end{pmatrix}
$$

(26)
A Comparison Of Impulse Responses (Case 2)
Case 3: Passive Monetary ($\alpha < 1$) & Active Fiscal ($\gamma < 1$)

- Two roots inside unit circle, $z = 0$ and $z = \frac{1}{\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1)} > 1$, and one root outside unit circle, $z = \frac{1}{\alpha} < 1$.

- Existence requires span($A_<$) $\subseteq$ span($R_<$) (satisfied). Restrictions imply $C_{01}^{11} = -\frac{1}{\frac{1}{\beta} - \gamma(\frac{1}{\beta} - 1)}$ and $C_{01}^{12} = \beta - 1$. However, $C_{0}^{21}$ and $C_{0}^{22}$ can be set in arbitrary way.

- Uniqueness requires span($R'_>$) $\subseteq$ span($R'_<$) (satisfied). Thus, solution is unique and is given by

$$
\begin{pmatrix}
\pi_t \\
b_t
\end{pmatrix} = \begin{pmatrix}
-\rho & \beta - 1 \\
\frac{1}{1 - \gamma + \gamma \beta} & 0
\end{pmatrix} \begin{pmatrix}
\theta_t \\
\psi_t
\end{pmatrix} + \sum_{k=1}^{\infty} \begin{pmatrix}
\alpha^{k-1} - \alpha^k \rho & (\beta - 1) \alpha^k \\
0 & 0
\end{pmatrix} \begin{pmatrix}
\theta_{t-k} \\
\psi_{t-k}
\end{pmatrix} 
$$

(27)
Case 3: Passive Monetary ($\alpha < 1$) & Active Fiscal ($\gamma < 1$)

- Two roots inside unit circle, $z = 0$ and $z = \frac{1}{\beta - \gamma (\frac{1}{\beta} - 1)} > 1$, and one root outside unit circle, $z = \frac{1}{\alpha} < 1$.

- Existence requires $\text{span}(A_<) \subseteq \text{span}(R_<)$ (satisfied). Restrictions imply $C_{01}^{11} = -\frac{1}{\beta - \gamma (\frac{1}{\beta} - 1)}$ and $C_{02}^{12} = \beta - 1$. However, $C_{01}^{21}$ and $C_{02}^{22}$ can be set in arbitrary way.

- Uniqueness requires $\text{span}(R'_>) \subseteq \text{span}(R'_<)$ (satisfied). Thus, solution is unique and is given by

$$
\begin{pmatrix}
\pi_t \\
\theta_t
\end{pmatrix}
= \begin{pmatrix}
-\rho & \beta - 1 \\
\frac{1}{1-\gamma+\gamma \beta} & 0
\end{pmatrix}
\begin{pmatrix}
\theta_t \\
\psi_t
\end{pmatrix}
+ \sum_{k=1}^{\infty}
\begin{pmatrix}
\alpha^{k-1} - \alpha^k \rho & (\beta - 1) \alpha^k \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\theta_{t-k} \\
\psi_{t-k}
\end{pmatrix}
$$

(27)
Case 3: Passive Monetary ($\alpha < 1$) & Active Fiscal ($\gamma < 1$)

- Two roots inside unit circle, $z = 0$ and $z = \frac{1}{\beta - \gamma(\frac{1}{\beta} - 1)} > 1$, and one root outside unit circle, $z = \frac{1}{\alpha} < 1$.
- Existence requires $\text{span}(A_{<}) \subseteq \text{span}(R_{<})$ (satisfied). Restrictions imply $C_{0}^{11} = -\frac{1}{\beta - \gamma(\frac{1}{\beta} - 1)}$ and $C_{0}^{12} = \beta - 1$. However, $C_{0}^{21}$ and $C_{0}^{22}$ can be set in arbitrary way.
- Uniqueness requires $\text{span}(R_{>}) \subseteq \text{span}(R_{<}')$ (satisfied). Thus, solution is unique and is given by

$$
\begin{pmatrix}
\pi_t \\
\theta_t \\
\psi_t
\end{pmatrix} = 
\begin{pmatrix}
\frac{-\rho}{1-\gamma+\gamma\beta} & \beta - 1 & 0 \\
1 & 0 & 0
\end{pmatrix}
\begin{pmatrix}
\theta_t \\
\psi_t
\end{pmatrix} + 
\sum_{k=1}^{\infty}
\begin{pmatrix}
\alpha^{k-1} - \alpha^k \rho & (\beta - 1)\alpha^k \\
0 & 0
\end{pmatrix}
\begin{pmatrix}
\theta_{t-k} \\
\psi_{t-k}
\end{pmatrix}
$$

(27)
A Comparison Of Impulse Responses (Case 3)

Solving Linear RE Models

Pres. by Fei Tan

Department of Economics, Indiana Univ.
Case 4: Active Monetary ($\alpha > 1$) & Active Fiscal ($\gamma < 1$)

- All roots are inside unit circle, $z = 0$, $z = \frac{1}{\alpha} < 1$, $z = \frac{1}{\frac{1}{\beta} - \gamma (\frac{1}{\beta} - 1)} < 1$.
- Existence requires span$(A_\prec) \subseteq$ span$(R_\prec)$ (not satisfied). Thus, model is too restrictive and has no stationary solutions in space we are interested in. However, unbounded solutions or bounded solutions in some other spaces may exist!
Case 4: Active Monetary ($\alpha > 1$) & Active Fiscal ($\gamma < 1$)

- All roots are inside unit circle, $z = 0$, $z = \frac{1}{\alpha} < 1$, $z = \frac{1}{\beta - \gamma \left( \frac{1}{\beta} - 1 \right)} < 1$.
- Existence requires $\text{span}(A_<) \subseteq \text{span}(R_<)$ (not satisfied). Thus, model is too restrictive and has no stationary solutions in space we are interested in. However, unbounded solutions or bounded solutions in some other spaces may exist!
Example 2: Simple RBC Model

- The following bivariate system in \((c_t, k_t)\) delivers a simple RBC model:

\[
E_t c_{t+1} = c_t + (\alpha - 1)k_t + E_t a_{t+1} \\
\left(\frac{1 - \alpha \beta}{\alpha \beta}\right) c_t + k_t = \frac{1}{\alpha \beta} a_t + \frac{1}{\beta} k_{t-1}
\]

- Assume \(a_t = A(L)\varepsilon_t\) satisfies covariance stationarity. Conventional approaches are not readily available since \(A(L)\) is unknown here.
- Hold on! \(z\)-transform method gives solution

\[
\begin{pmatrix} c_t \\ k_t \end{pmatrix} = C(L) \begin{pmatrix} \varepsilon_t \\ b_t \end{pmatrix} = \begin{pmatrix} \frac{A(L)}{1 - \alpha L} \varepsilon_t \\ \frac{A(L)}{1 - \alpha L} \varepsilon_t \end{pmatrix}
\]
Example 2: Simple RBC Model

- The following bivariate system in \((c_t, k_t)\) delivers a simple RBC model:

\[
E_t c_{t+1} = c_t + (\alpha - 1) k_t + E_t a_{t+1} \tag{28}
\]

\[
\frac{1 - \alpha \beta}{\alpha \beta} c_t + k_t = \frac{1}{\alpha \beta} a_t + \frac{1}{\beta} k_{t-1} \tag{29}
\]

- Assume \(a_t = A(L) \varepsilon_t\) satisfies covariance stationarity. Conventional approaches are not readily available since \(A(L)\) is unknown here.

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\[
\begin{pmatrix} c_t \\ k_t \end{pmatrix} = C(L) \begin{pmatrix} \varepsilon_t \\ b_t \end{pmatrix} = \begin{pmatrix} \frac{A(L)}{1 - \alpha L} \varepsilon_t \\ \frac{A(L)}{1 - \alpha L} \varepsilon_t \end{pmatrix}
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Example 2: Simple RBC Model

- The following bivariate system in \((c_t, k_t)\) delivers a simple RBC model:

\[
E_t c_{t+1} = c_t + (\alpha - 1)k_t + E_t a_{t+1} \\
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\]

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\[
\begin{pmatrix} c_t \\ k_t \end{pmatrix} = C(L) \begin{pmatrix} \varepsilon_t \\ b_t \end{pmatrix} = \begin{pmatrix} A(L) \\ \frac{1 - \alpha L}{1 - \alpha L} \varepsilon_t \end{pmatrix}
\]
Example 3: Decoupled System

Consider a decoupled linear rational expectations model in \((x_t, y_t)\) that has no stable solution:

\[
\begin{align*}
x_t &= \alpha x_{t-1} + \varepsilon_t \\
E_t y_{t+1} &= \beta y_t + v_t
\end{align*}
\]

\(\alpha > 1\) and \(\beta < 1\) \(\Rightarrow\) root-counting approach and winding number criterion fail to give correct existence condition.

Unstable root occurs in a part of system that is decoupled from expectational equation. [Sims (2007)]

Hold on! \(z\)-transform method says no solution since existence condition is violated:

\[
\begin{pmatrix}
\frac{\beta - \alpha}{\alpha^2 \beta} & 0 \\
\frac{1}{C_0^{21}} & C_0^{22} + \frac{1}{\alpha}
\end{pmatrix}
= \begin{pmatrix} 0 & 0 \end{pmatrix} \Rightarrow \alpha = \beta
Example 3: Decoupled System

- Consider a decoupled linear rational expectations model in \((x_t, y_t)\) that has no stable solution:

\[
x_t = \alpha x_{t-1} + \varepsilon_t
\]  
\[
E_t y_{t+1} = \beta y_t + \nu_t
\]  

- \(|\alpha| > 1\) and \(|\beta| < 1\) \(\Rightarrow\) root-counting approach and winding number criterion fail to give correct existence condition.
- Unstable root occurs in a part of system that is decoupled from expectational equation. [Sims (2007)]
- Hold on! \(z\)-transform method says no solution since existence condition is violated

\[
\begin{pmatrix}
\frac{\beta - \alpha}{\alpha^2 \beta} & 0 \\
\frac{1}{\alpha} C_{0}^{21} & C_{0}^{22} + \frac{1}{\alpha} \\
\end{pmatrix}
= 
\begin{pmatrix}
0 & 0
\end{pmatrix}
\Rightarrow
\alpha = \beta
Example 3: Decoupled System

- Consider a decoupled linear rational expectations model in \((x_t, y_t)\) that has no stable solution:

\[
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\]

\[
E_t y_{t+1} = \beta y_t + v_t
\]

- \(|\alpha| > 1\) and \(|\beta| < 1\) ⇒ root-counting approach and winding number criterion fail to give correct existence condition.

- Unstable root occurs in a part of system that is decoupled from expectational equation. [Sims (2007)]

- Hold on! \(z\)-transform method says no solution since existence condition is violated

\[
\begin{pmatrix}
\frac{\beta - \alpha}{\alpha^2 \beta} & 0 \\
\frac{1}{C_{01}^{21}} & C_{02}^{22} + \frac{1}{\alpha}
\end{pmatrix}
\begin{pmatrix}
0 \\
0
\end{pmatrix}
= \begin{pmatrix}
0 \\
0
\end{pmatrix}
\Rightarrow\ \alpha = \beta
\]
Example 3: Decoupled System

- Consider a decoupled linear rational expectations model in \((x_t, y_t)\) that has no stable solution:

\[
x_t = \alpha x_{t-1} + \varepsilon_t \quad (30)
\]
\[
E_t y_{t+1} = \beta y_t + v_t \quad (31)
\]

- \(|\alpha| > 1\) and \(|\beta| < 1\) \implies\ root-counting approach and winding number criterion fail to give correct existence condition.

- Unstable root occurs in a part of system that is decoupled from expectational equation. [Sims (2007)]

- Hold on! \(z\)-transform method says no solution since existence condition is violated

\[
\begin{pmatrix}
\frac{\beta - \alpha}{\alpha^2 \beta} & 0 \\
\frac{1}{\alpha} C_{0}^{21} & C_{0}^{22} + \frac{1}{\alpha}
\end{pmatrix} = \begin{pmatrix}
0 \\
0
\end{pmatrix} \quad \Rightarrow \quad \alpha = \beta
\]
A summary of what we did in this work:

- We generalized linear rational expectations solution method of White- man (1983) to multivariate case. This permits use of generic driving process which must only satisfy covariance stationarity;

- We derived multivariate cross-equation restrictions linking Wold representation of driving process to endogenous variables;

- We connected our solution methodology to Gensys and showed an equivalence relationship between two techniques;

- We introduced several motivating examples that highlight usefulness of our approach.
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Wrapping up

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Research Agenda

Two applications of this work:

- **Incomplete information**: one may solve and estimate incomplete information models where agents have heterogeneous beliefs.

- **Observational equivalence**: two distinct parameter regions may generate empirically indistinguishable equilibrium time series. Identification problem becomes less severe as model complexity grows. Investigation of this point requires a **tractable** tool.
  
  - Monster model completely separates endogenous variables (LHS) and exogenous variables (RHS) (decomposition of parameter space).
  - Factorization of polynomial matrix depends on private parameters, independent of selection of driving processes and model complexity.
  - “Semi-analytic” approach: (1) fix numerical private parameters; (2) pick numerical policy parameters for distinct regimes; (3) employ symbolic representations for general driving processes.
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