Robustfying Shiller: Do Stock Prices Move *Enough* to be Consistent with Subsequent Changes in Dividends?

Kurt F. Lewis
Charles H. Whiteman

September 10, 2007
“Are you more comfortable in the time domain or the frequency domain?”

– Tom Sargent to Giorgio Primiceri
October 28, 2004
Outline

1. Shiller’s Variance Bound
2. Fixups
3. Least Squares Expectations
4. Robust Expectations
5. Robustness of the Robust Fixup
6. The Evil Agent Game
7. Evidence

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The Present Value Formula

- **Expected:**
  \[ p_t = E_t \sum_{j=0}^{\infty} \gamma^j d_{t+j} \]

- **Actual:**
  \[ p^*_t = \sum_{j=0}^{\infty} \gamma^j d_{t+j} \]
Orthogonal Projections

\[ P_t^* = E_t P_t^* + \varepsilon_t \quad \varepsilon_t \perp P_t \]
\[ = P_t + \varepsilon_t \]

- \( \text{var}(P_t^*) = \text{var}(P_t) + \text{var}(\varepsilon_t) \)
- Variance Bound: \( \text{var}(P_t^*) \geq \text{var}(P_t) \)
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Fixups

- Sampling Variability
  - Flavin, (*JPE*, 1983)

- Variability of Dividends
  - Marsh and Merton (*AER*, 1986)
    - If $\text{var}(d_t) = \infty$, $\geq \rightarrow \leq$

- Variability of the Discount Factor
  - Michener (*JPE*, 1982)
    “This reconciliation, while suggestive, is not as strong as one might hope for.”
The Problem

Suppose (abstracting from mean, trend) the predictor agent knows the DGP for dividends coincides with the Wold Representation:

\[ d_t = \sum_{j=0}^{\infty} q_j \varepsilon_{t-j} = q(L) \varepsilon_t \quad E(\varepsilon_t) = 0, \quad E(\varepsilon_t^2) = 1 \]

Now

\[ p_t^* = \sum_{j=0}^{\infty} \gamma^j d_{t+j} \]

Problem: Find \( p_t \) to min \( E(p_t^* - p_t)^2 \)
The Problem – Equivalent Formulation

- Write \( p_t = f(L)\varepsilon_t \)
- Find \( f_j \) in \( f(L) = f_0 + f_1L + f_2L^2 + \ldots = \arg\min E(f(L)\varepsilon_t - p^*_t)^2 \)

Equivalent Frequency Domain version

\[
\min_{f(z) \in H^2} \frac{1}{2\pi i} \oint \left| \frac{q(z)}{1 - \gamma z^{-1}} - f(z) \right|^2 \frac{dz}{z}
\]

- The Area Under the Spectrum (set \( z = e^{-i\omega} \), integrate from \( -\pi \) to \( \pi \))

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Key Fact

If:

\[ h(z) = \sum_{j=-\infty}^{\infty} h_j z^j \]

Then:

\[ \frac{1}{2\pi i} \oint z^{-j} h(z) \frac{dz}{z} = h_j \]

So:

\[ \frac{1}{2\pi i} \oint h(z) \frac{dz}{z} = h_{-1} \]

The residue of \( h(z) \) at \( z = 0 \).
Solution

The problem is written

$$\min_{f(z) \in H^2} \frac{1}{2\pi i} \oint \left| \frac{q(z)}{1 - \gamma z^{-1}} - f(z) \right|^2 \frac{dz}{z}$$

The FONC for \( \{f_j\} \ j = 0, 1, 2, \ldots \) is:

$$- \frac{2}{2\pi i} \oint z^{-j} \left[ \frac{q(z)}{1 - \gamma z^{-1}} - f(z) \right] \frac{dz}{z} = 0.$$
Solution – cont.

If

\[ H(z) = \frac{q(z)}{1 - \gamma z^{-1}} - f(z) \]

Then

\[ -\frac{2}{2\pi i} \oint z^{-j} H(z) \frac{dz}{z} = 0 \Rightarrow 2H_j = 0 \quad j = 0, 1, 2, \ldots \]

Multiply \( H_j \) by \( z^j \) and sum over all \( z \) to get

\[ H(z) = \sum_{-\infty}^{-1} \]
Thus we get the Wiener-Hopf equation

\[
\frac{q(z)}{1 - \gamma z^{-1}} - f(z) = \sum_{j=-\infty}^{-1} \quad \text{(1)}
\]

Apply the "plussing" (ignore negative powers of z) operator:

\[
\left[ \frac{q(z)}{1 - \gamma z^{-1}} \right]_+ - [f(z)]_+ = 0
\]

\[
f(z) = \left[ \frac{q(z)}{1 - \gamma z^{-1}} \right]_+ = \left[ \frac{zq(z)}{z - \gamma} \right]_+
\]
Solution – cont.

Plussing: remove principle part of Laurent expansion:

\[
\frac{z^q(z)}{z - \gamma} = \frac{b_{-1}}{z - \gamma} + b_0 + b_1(z - \gamma) + b_2(z - \gamma)^2 + \ldots
\]

Multiply by \((z - \gamma)\) and evaluate at \(z = \gamma\):

\[
b_{-1} = \gamma q(\gamma)
\]

\[
f(z) = \left[ \frac{z^q(z)}{z - \gamma} \right]_+ = \frac{z^q(z)}{z - \gamma} - \frac{\gamma q(\gamma)}{z - \gamma} = \frac{z^q(z) - \gamma q(\gamma)}{z - \gamma}.
\]
Solution Method

Solution – First Order Case

Detrended dividends assumed AR(1): $q(L) = (1 - \rho L)^{-1}$.

$$p_t = f(L)\epsilon_t = \frac{Lq(L) - \gamma q(\gamma)}{L - \gamma} \epsilon_t$$

$$= \frac{1}{(1 - \rho \gamma)(1 - \rho L)} \epsilon_t$$

$$= \left(\frac{1}{1 - \rho \gamma}\right) d_t.$$

With $\gamma = 0.943$, (SP data), the largest $\sigma(p)$ can be for $|\rho| < 1 \approx 22 \times \sigma(d)$. Estimating $\rho$, the ratio is about 11, far short of the factor of 34 needed to match the observed volatility.

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Results of *Least Squares* Pricing

LS optimal pricing results in serially correlated errors.

Note that:

\[
P_t = \frac{[Lq(L) - \gamma q(\gamma)]}{L - \gamma} \varepsilon_t
\]

\[
P_t^* = \frac{q(L)}{1 - \gamma L^{-1}} \varepsilon_t
\]
Least Squares Pricing Errors

\[ P_t - P_t^* = \left\{ \frac{Lq(L) - \gamma q(\gamma)}{L - \gamma} - \frac{q(L)}{1 - \gamma L^{-1}} \right\} \varepsilon_t \]

\[ = -\frac{\gamma q(\gamma)}{L - \gamma} \varepsilon_t = -\gamma q(\gamma) \frac{L^{-1}}{1 - \gamma L^{-1}} \varepsilon_t \]

\[ = -\gamma q(\gamma) \left\{ \varepsilon_{t+1} + \gamma \varepsilon_{t+2} + \gamma^2 \varepsilon_{t+3} + \ldots \right\} \]

Serially correlated, but not predictable.
Serially Correlated Errors

- Tolerance for serially correlated errors
  - When model is “known” to be correct.
  - In very few econometrics texts.
- What if you are unsure about the model?
Robust Prediction

Make the best of a bad situation.

- Replace

\[
\min_{\{f_j\}} \frac{1}{2\pi i} \oint \left| f(z) - \frac{q(z)}{1 - \gamma z^{-1}} \right|^2 \frac{dz}{z}
\]

- By

\[
\min_{\{f_j\}} \sup_{|z|=1} \left| f(z) - \frac{q(z)}{1 - \gamma z^{-1}} \right|^2
\]
Motivation for Robust Methods

The Ellsberg Paradox

- There are 300 balls in an urn, we know that 100 are Red, and the remaining 200 are Blue and Yellow.
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- Draw one ball at random, choose preferences over the following 4 gambles:

  ▶ A: $1000 if R, $0 if B or Y (Known Odds)
  ▶ B: $1000 if B, $0 if R or Y (Ambiguous Odds)
  ▶ C: $1000 if B or Y, $0 if R (Known Odds)
  ▶ D: $1000 if R or Y, $0 if B (Ambiguous Odds)

Empirically Observed Preferences: A ≻ B and C ≻ D

Anscombe-Auman & Savage: A ≻ B =⇒ D ≻ C

Min-max could rationalize

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Empirically Observed Preferences: $A \succ B$ and $C \succ D$

Anscombe-Auman & Savage: $A \succ B = \Rightarrow D \succ C$

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Empirically Observed Preferences:
- \( A \succeq B \) and \( C \succ D \)
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- Anscombe-Auman & Savage: \( A \succ B \implies D \succ C \)
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Robust Solution

Following Kasa (EL, 2001), find \( r(z) \) such that

\[
\min \sup_{\{r(z)\} \mid |z|=1} \left| r(z) - \frac{q(z)}{1 - \gamma z^{-1}} \right|^2
\]

Note that this is equivalent to:

\[
\min \sup_{\{r(z)\} \mid |z|=1} \left| r(z) - \frac{zq(z)}{z - \gamma} \right|^2
\]
For the “Blaschke factor” $B_\gamma(z) = (z - \gamma)/(1 - \gamma z)$,

$$\left| \frac{z - \gamma}{1 - \gamma z} \right|^2 = \frac{(z - \gamma)(z^{-1} - \gamma)}{(1 - \gamma z)(1 - \gamma z^{-1})} = \frac{(z - \gamma)z^{-1}(1 - \gamma z)}{(1 - \gamma z)z^{-1}(z - \gamma)} = 1$$

This allows us to write the problem as:

$$\min \sup \left\{ r(z) \right\} \left| \frac{z - \gamma}{1 - \gamma z} r(z) - \frac{zq(z)}{1 - \gamma z} \right|^2$$
Our problem can be written:

\[
\min_{\{r(z)\}} \sup_{|z|=1} \left| \frac{(z - \gamma)r(z) - zq(z)}{1 - \gamma z} \right|^2
\]

Define the term inside the conjugate square to be \(\phi(z)\).
Note that:

\[
\phi(\gamma) = \frac{-\gamma q(\gamma)}{1 - \gamma^2}
\]
Robust Solution – Cont.

This means that our problem can be written as:

\[
\min_{\phi(z)} \| \phi(z) \|_{\infty} 
\]

So, choose the analytic function \( \phi \) which minimizes the sup-norm subject to \( \phi(\gamma) = \frac{-\gamma q(\gamma)}{1-\gamma^2} \). The answer is given via the Maximum Modulus Theorem.
Maximum Modulus Theorem

For $f$ be an analytic function on $U$, the maximum value of $|f|$ on $\bar{U}$ occurs on the boundary $\partial U$.

Therefore, to minimize the worst case scenario, we level the whole function to be equal to a constant. But, we already know one value on the surface that must hold, so we know the value of the constant. Therefore:

$$\frac{(z - \gamma)r(z) - zq(z)}{1 - \gamma z} = \frac{-\gamma q(\gamma)}{1 - \gamma^2}$$
Robust Solution – Cont.

After some algebra:

\[ r(z) = \frac{zq(z) - \gamma q(\gamma)}{z - \gamma} + \frac{\gamma^2 q(\gamma)}{1 - \gamma^2} \]

where the first term on the RHS is the least squares, or \( H^2 \) solution to the original problem, and the second term is the extra piece resulting from the robust prediction.
Variance Comparison – $H^2$ vs. $H^\infty$

We denote the $H^2$ solution with a superscript $LS$ for “Least Squares”. We denote the $H^\infty$ solution with a superscript $R$, for robust.

$$P^R_t = P^LS_t + \frac{\gamma^2 q(\gamma)}{1 - \gamma^2} \varepsilon_t$$
Variance Comparison

Variance Calculation – AR(1)

In the first order AR case,

\[
\sigma^2(p_t) = \left(\frac{1}{1 - \rho \gamma}\right)^2 \sigma^2(d_t) + \frac{2\gamma^2 - \gamma^4}{(1 - \rho \gamma)^2(1 - \gamma^2)^2}.
\]

Using \(\rho\) and \(\gamma\) as in SP data

\[
\sigma(p^R) = 89.52.
\]

Actual

\[
\sigma(p) = 42.74.
\]

Robust prices are too volatile!
Robust Prediction Error

Recall that

$$r(z) = \frac{zq(z) - \gamma q(\gamma)}{z - \gamma} + \frac{\gamma^2 q(\gamma)}{1 - \gamma^2}$$

Which means that we know:

$$P_t^R = \left[ \frac{Lq(L) - \gamma q(\gamma)}{L - \gamma} + \frac{\gamma^2 q(\gamma)}{1 - \gamma^2} \right] \varepsilon_t$$

$$P_t^* = \frac{q(L)}{1 - \gamma L^{-1}} \varepsilon_t$$
Robust Prediction Error – Cont.

\[ P_t^R - P_t^* = \left\{ \frac{Lq(L) - \gamma q(\gamma)}{L - \gamma} + \frac{\gamma^2 q(\gamma)}{1 - \gamma^2} - \frac{q(L)}{1 - \gamma L^{-1}} \right\} \varepsilon_t \]

\[ = \left\{ \frac{\gamma^2 q(\gamma)}{1 - \gamma^2} - \frac{\gamma q(\gamma)}{L - \gamma} \right\} \varepsilon_t \]

\[ = \gamma q(\gamma) \left\{ \frac{\gamma(L - \gamma) - 1 + \gamma^2}{(1 - \gamma^2)(L - \gamma)} \right\} \varepsilon_t \]

\[ = \gamma q(\gamma) \left\{ \frac{\gamma L - 1}{(1 - \gamma^2)(L - \gamma)} \right\} \varepsilon_t \]
Robust Prediction Error – Cont.

Note that

$$\gamma q(\gamma) \left\{ \frac{\gamma L - 1}{(1 - \gamma^2)(L - \gamma)} \right\} \varepsilon_t = -\frac{\gamma q(\gamma)}{1 - \gamma^2} \left\{ \frac{1 - \gamma L}{L - \gamma} \right\} \varepsilon_t$$

The term in braces is an inverse Blaschke factor, and

$$\left( \frac{1 - \gamma z}{z - \gamma} \right) \left( \frac{1 - \gamma z^{-1}}{z^{-1} - \gamma} \right) = \left( \frac{1 - \gamma z}{z - \gamma} \right) \left( \frac{z^{-1}(z - \gamma)}{z^{-1}(1 - \gamma z)} \right) = 1$$

Therefore the prediction errors are not serially correlated.
Prediction Error Spectra

- **Standard $H^2$ Prediction:**
  \[ |f(z) - \frac{q(z)}{1 - \gamma z^{-1}}|^2 \]

- **Robust Prediction:**
  \[ |r(z) - \frac{q(z)}{1 - \gamma z^{-1}}|^2 \]
Prediction Error Spectra

For $z = e^{i\omega}$

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What are They Thinking?

- Robust agents act as if:

\[ d_t = [q(L) + C] \varepsilon_t \]

- Example:
  - If \( q(L) \varepsilon_t \sim AR(1) \)
  - Then \([q(L) + C] \varepsilon_t \sim ARMA(1, 1)\)

- How “detectable” is this?
Standard Orthogonality Conditions

In Shiller’s original formulation:

\[ P_t^* = E_t P_t^* + \epsilon_t \]
\[ = P_t + \epsilon_t \]

So, \( P_t^* - P_t \perp P_t \).

In Shiller’s data: \( \rho(P_t^* - P_t, P_t) = 0.99 \)
Robust Orthogonality Conditions

Robust formulation:

\[ P_t^* = E_t^R P_t^* + \nu_t \]
\[ = P_t + \nu_t \]

So, \( \nu_t \perp \nu_{t-1} : P_t^* - P_t \perp P_{t-1}^* - P_{t-1}. \)

In Shiller’s data: \( \rho(P_t^* - P_t, P_{t-1}^* - P_{t-1}) = 0.87 \)
Price Series

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“Intermediate” Case

- Note from previous figure:

\[ \text{var}(P_t^R) \gg \text{var}(P_t) \gg \text{var}(P_t^*) \]

- Only limited robustness is needed.
  - Some robust, some non-robust investors?
  - “Evil Agent” Game.
“Evil Agent” Game

- Deliver
  \[ d_t^n = d_t + n_t = [q(L) + m(L)] \varepsilon_t \]

- Make predictor as unhappy as possible subject to the constraint:
  \[ \text{var}(n_t) \leq \eta \]

- Discipline:
  - How detectable is \{n_t\}?
EA Game

The evil agent ("nature") delivers noisy dividends:

\[ d_t = [q(L) + m(L)] \epsilon_t \equiv C(L) \epsilon_t. \]

The investor agent’s problem becomes to choose an analytic function \( f(z) \) to

\[
\min_{f(z)} \frac{1}{2\pi i} \oint \left| \frac{C(z)}{1 - \gamma z^{-1}} - f(z) \right|^2 \frac{dz}{z}. 
\]

Evil Agent: Make investor miserable:

\[
\max_{C(z)} \frac{1}{2\pi i} \oint \left| \frac{C(z)}{1 - \gamma z^{-1}} - f(z) \right|^2 \frac{dz}{z} - \theta \left| \frac{C(z) - q(z)}{1 - \gamma z^{-1}} \right|^2 \frac{dz}{z}.
\]
EA Game – cont.

Nash equilibrium: solve the pair of Wiener-Hopf eqns

Investor:

\[ C(z) = (1 - \gamma z^{-1}) f(z) + \sum_{-\infty}^{-1} \]

Nature:

\[ \frac{(1 - \theta) C(z)}{1 - \gamma z} - \left[ \frac{(1 - \gamma z^{-1})}{1 - \gamma z} f(z) \right] + \frac{\theta q(z)}{1 - \gamma z} = \sum_{-\infty}^{-1}. \]
EA Game – cont.

Solutions:

\[ f(z) = \frac{zq(z) - \gamma q(\gamma)}{z - \gamma} + \frac{\gamma^2}{\theta - \gamma^2} q(\gamma) \]

\[ C(z) = q(z) + \frac{\gamma^2}{\theta - \gamma^2} q(\gamma) \]

Note

\[
\begin{cases} 
  \theta \to \infty \quad \to \text{LS Result} \\
  \theta \downarrow 1 \quad \to \text{Robust } H^\infty\text{-norm Result} 
\end{cases}
\]
How Much $\theta$ Does It Take?

- **Univariate Case:**
  - To match $\sigma(p)$: $\theta = 1.6$
  - Implied $\sigma(d) \approx 25 \times \text{actual } \sigma(d)$
θ: Bivariate Evidence

Bivariate MARs:

\[
\begin{pmatrix}
  d_t \\
  \rho_t
\end{pmatrix} =
\begin{pmatrix}
  A(L) & B(L) \\
  C(L) & D(L)
\end{pmatrix}
\begin{pmatrix}
  \varepsilon_{d_t} \\
  \varepsilon_{\rho_t}
\end{pmatrix}.
\]

The Modified ARMA(3,1) Game Model:

\[
A(L) = \frac{\rho_0}{(1 - \rho_3 L)(1 - \rho_4 L)}, \quad B(L) = \frac{\mu_0(1 - \mu_1 L)}{(1 - L)(1 - \mu_3 L)(1 - \mu_4 L)}.
\]

\[
C(L) = \frac{LA(L) - \gamma A(\gamma)}{L - \gamma} + \frac{\gamma^2}{\theta - \gamma^2} A(\gamma), \quad D(L) = \frac{LB(L) - \gamma B(\gamma)}{L - \gamma} + \frac{\gamma^2}{\theta - \gamma^2} B(\gamma).
\]
### ML Exercise

<table>
<thead>
<tr>
<th>Model</th>
<th>$\theta$</th>
<th># Parameters</th>
<th>Log-Likelihood</th>
</tr>
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<tr>
<td>Least-Squares</td>
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<tr>
<td>Game</td>
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<td>-569.32</td>
</tr>
</tbody>
</table>

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Conclusion

- LS Present Value Prediction Not Volatile Enough
- Robust Present Value Prediction Too Volatile
- Evil Agent Game Promising
  - Compromise on volatility between $d_t$ and $p_t$