Recursive Utility and Thompson Aggregators I: Constructive Existence Theory for the Koopmans Equation\textsuperscript{1}

Macro Brown Bag (IUB) Presentation Slides

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Exponential Utility & the TAS Utility Function

- The Exponential Discounting Model (infinite horizon, discrete time) aka the familiar **TAS Utility** is:

\[
\sum_{t=1}^{\infty} \delta^{t-1} u (c_t),
\]

where \(0 < \delta < 1\) and \(u\) is a bounded, continuous, and concave (differentiable) function on \(\mathbb{R}_+ = [0, \infty)\). Usual Inada conditions apply (e.g. \(0 \leq u (c) = \arctan (\sqrt{c}) \leq (\pi/2))\).

- Discounted Optimal Growth Model w/One Sector: Euler Equation:

\[
u'(c_t) = \delta f'(k_t) u'(c_{t+1})\]

each \(t\).

- Steady state condition: \(\delta f'(k^*) = 1\) — independent of the form of \(u\).

- **Dynamic Nonsubstitution Theorem:** Steady state depends on technology and pure rate of time preference (\(\delta = 1 / (1 + \rho)\) with \(\rho > 0\)).
TAS and the Impatience Problem

- **Impatience Problem** – one example where the exponential discounting model produces an extreme result.

- RB’s papers on “Ramsey Equilibria.” Only most patient household holds capital: **IF 2 households and:**

  \[
  1 > \delta_1 > \delta_2 > 0,
  \]

  then

  \[
  \delta_2 f'(k^*) < \delta_1 f'(k^*) = 1.
  \]

  Impossible for both to hold capital in a steady state.

- **Recursive Utility functions create alternative specifications for an infinitely-lived household’s “lifetime” utility function where the stationary equilibrium discount factor depends on the underlying consumption sequence. Are more robust results available than w/TAS?**

- Need ways to describe recursive utility functions and some of their economic properties. **Focus today: The description of recursive utility functions via aggregators.**
Recursive Property of the TAS Utility

\[ U(C) = \sum_{t=1}^{T} \delta^{t-1} u(c_t) + \delta^T \sum_{t=1}^{\infty} \delta^{t-1} u(c_{t+T}) . \]

- Let \( S^T C = (c_{T+1}, c_{T+2}, \ldots) \). The decision maker’s behavior over the infinite horizon is guided by the behavior over the tail horizon \( t = T + 1, T + 2, \ldots \) for each \( T \) that is hidden inside the original horizon. **Recursivity is a self-referential property.**

- **Recursive Utility functions abstract this self-referential property** in order to relax the fixed discount factor assumption and maintain the preference structure to derive **time consistent decision rules** in stationary infinite horizon optimization problems.

**Non-Recursive Example:** **Quasi-Geometric Utility (behavioral theory):**

\[ 0 < \beta < 1, \beta \neq \delta, \]

\[ U(C) = u(c_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(c_t) \]
Recursive Utility Background

- Recursive Utility Generalizes TAS Time Stationarity and Time Consistency Properties
- Koopmans (1960s) axiomatic approach: preference structures induce utility representations of the form:

\[ U(C) = W(c_1, U(SC)), \]  

(1)

where \( C = \{ c_t \}_{t=1}^{\infty} \in \ell^+_\infty \) — the positive cone of \( \ell_\infty \) with its sup norm topology is the commodity space and \( S \) is the shift operator:

\[ SC = \{ c_2, c_3, \ldots \}. \]

- The function \( W \) is the aggregator and has two arguments: \( x \) — present consumption; \( y \) — future utility. Write \( W(x, y) \) for the aggregator. \( W \) has a 2-period Fisherian Interpretation.

- A recursive utility function satisfies (1) — the aggregator is derived from an axiomatization of the preference ordering.
**Example**

Time Additive Utility (TAS): $u \geq 0$ & bounded, has the aggregator

$$W(x, y) = u(x) + \delta y$$ with $0 < \delta < 1$ and utility function,

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t) := U(C).$$

**Example**

The Koopmans, Diamond and Williamson (KDW) aggregator is

$$W(x, y) = \frac{\delta}{d} \ln \left(1 + ax^b + dy\right); \ a, b, d, \delta > 0. \tag{2}$$

Assume $b < 1$ and $\delta < 1$. Then $W$ is concave and a Lipschitz condition obtains: $0 \leq \sup_y W_2(x, y) < 1.$
Lucas and Stokey’s (1984) idea is to let aggregators be the primitive concept: given $W$ find $U$ satisfying (1). **Fisherian “two-period interpretation” motivation.**

Consider the TAS aggregator: Successive approximations initiated from $\theta(C) = 0$ (input “no information” about $U$) yields the sequence of **partial sums**:

\[
U_1(C) = u(c_1) = W(c_1, \theta(SC)) = W(c_1, 0)
\]
\[
U_2(C) = u(c_1) + \delta u(c_2) = W(c_1, W(c_2, 0))
\]
\[
\vdots
\]
\[
U_N(C) = \sum_{t=1}^{N} \delta^{t-1}(c_t) = W(c_1, W(c_2, W(c_3, \ldots, W(c_N, 0))).
\]

Clearly $U_N(C) \uparrow \sum_{t=1}^{\infty} \delta^{t-1}(c_t)$. Each partial sum $U_N(C)$ approximates $\sum_{t=1}^{\infty} \delta^{t-1}(c_t)$ from below.

Abstract **partial summation method** in today’s **LFP construction.**
Introduce the **Koopmans operator**, denoted $T_W$, and defined by:

$$T_W U (C) = W (c_1, U (SC)).$$

A recursive utility function is a fixed point of this operator:

$$T_W U = U.$$

- The **RECOVERY PROBLEM** is to show that this operator equation has at least one solution.
- The **UNIQUENESS PROBLEM** is to show there is at most one solution.
Contraction Mappings

- Banach’s Contraction Mapping Theorem, when applicable, resolves the Recovery (or, Existence) Problem and the Uniqueness Problem at once (and contains a successive approximations construction of the solution).

- $X \neq \emptyset$. Define $B(X) = \{ f \text{ such that } f : X \to \mathbb{R} \text{ and } \| f \|_\infty = \sup_{x \in X} |f(x)| < \infty \}$. This is a complete metric space when $B(X)$ is assigned its norm topology.

- **Pointwise order:** $f \geq g$ if and only if $f(x) \geq g(x)$ for each $x \in X$.

**Definition**

An operator $T : B(X) \to B(X)$ is a **contraction mapping with modulus** $\delta \in (0, 1)$ if for each $f, g \in B(X)$:

$$\| Tf - Tg \|_\infty \leq \delta \| f - g \|_\infty.$$
**Theorem**

(Banach) If $T : B(X) \rightarrow B(X)$ is a contraction mapping with modulus $\delta \in (0, 1)$, then there is a unique $f^* \in B(X)$ such that $Tf^* = f^*$.

**Theorem**

(Blackwell (1965) Let $T : B(X) \rightarrow B(X)$ be an operator satisfying:

- **(M)** $f \succeq g \implies Tf \succeq Tg$ — **Monotonicity**;
- **(D)** There exists some $\delta \in (0, 1)$ such that

  \[
  [T(f + a)](x) \leq (Tf)(x) + \delta a
  \]

  for each nonnegative scalar $a$, each $f \in B(X)$, and each $x \in X$ — **Discounting**.

  Then, $T$ is a contraction with modulus $\delta$.

Here: $(f + a)(x) = f(x) + a$ for each $x \in X$. 

RAB (IUB) & JPRZ (U. Carlos III de Madrid) (IUB & Carlos III de Madrid) Slides - Thompson Aggregators

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We look today at a new class of aggregators known as **Thompson Aggregators**. The Blackwell aggregator Lipschitz/discounting condition fails in this scenario.

**Marinacci and Montrucchio (MM)** (JET 2010) introduced this family and gave sufficient conditions for Recovery and Uniqueness Theorems. Blackwell’s Sufficient Condition for Contraction Maps fails, but **Monotonicity** holds.

We emphasize **constructive methods for Recovery Theory** (successive approximations — partial summation methods) and focus on Least Fixed Point (LFP) construction by Monotone Operator methods only. Defer a discussion of uniqueness theory for our second paper (see references).

Need this foundation to infer qualitative properties of extremal fixed points (e.g. concavity and continuity).

The LFP corresponds to **Kantorovich’s (1939) Principal Fixed Point (Principal Solution)**. We discuss why this might be a reasonable interpretation of LFP.
## Thompson Aggregators: Two Examples

### Example

KDW is Thompson whenever $\delta \geq 1$. Is jointly concave in $(x, y)$ and unbounded (above). It is Lipschitz in $y$, but $T_W$ is NOT a contraction mapping.

### Example

Constant Elasticity of Substitution (CES) aggregators have the form:

$$W(x, y) = (1 - \beta)x^\rho + \beta y^\rho$$

(3)

where $0 < \beta < 1$ and $0 < \rho < 1$ implying the elasticity of substitution, $1/(1 - \rho) > 1$. This $W$ is Thompson & jointly concave in $(x, y)$. It fails the Blackwell Lipschitz condition and is unbounded (above).

- A Thompson aggregator jointly concave in $(x, y)$ is a **concave Thompson aggregator**.
Graph of $W(x, y) = (1 - \beta) x^\rho + \beta y^\rho$ for $\beta = \rho = 1/2$ as $y$ varies for a fixed value of $x = 1$: $\forall x \geq 0 \exists! y_x \geq 0$ such that $W(x, y_x) = y_x$. 
The Weight Function

- Define a **weight function on the commodity space** $\ell_\infty^+$, $\varphi_\gamma$, by the formula
  \[
  \varphi_\gamma (C) = (1 + \| C \|_\infty)^{1/\gamma}.
  \]  
  (4)

- This function is uniformly continuous on $\ell_\infty^+$ with respect to the sup norm topology.

- Clearly $\varphi_\gamma (C) \geq 1$ for each $C$.

- $\gamma > 0$ is a **subhomogeniety** parameter. It comes from the formal Thompson aggregator assumptions.

- Thompson KDW has $\gamma = b^{-1}$;

- Thompson CES aggregators has $\gamma = 1$. 

Setup for the Utility Function Space

- $U : \ell^+_\infty \to \mathbb{R}$ is $\phi_\gamma$-bounded provided
  \[
  \|U\|_\gamma := \sup_{C \in \ell^+_\infty} \frac{|U(C)|}{(1 + \|C\|_\infty)^{1/\gamma}} < +\infty.
  \] (5)

- $B = \left\{ U : \ell^+_\infty \to \mathbb{R} : U$ is $\phi_\gamma$-bounded $\right\}.$

- Provide $B$ with the pointwise ordering: the positive cone is defined by $U \in B^+$ if and only if $U \geq \theta$ pointwise; i.e., $U(C) \geq \theta(C) = 0$ for each $C \in \ell^+_\infty.$

- Define the standard pointwise lattice operations, $\vee$ (sup) and $\wedge$ (inf).

- $B$ is a Dedekind complete Riesz space & Banach lattice with the norm, $\|U\|_\gamma$, and order unit, $\phi_\gamma$.

- The positive cone is $B^+$ is a norm-closed, convex set and has a nonempty norm interior.
If $W$ is a Thompson aggregator, then $T_W$ is a **monotone** self-map on $B^+$, i.e. $U \geq V \in B^+$ implies $T_W U \geq T_W V$. In particular, $T_W \theta \geq \theta$.

Define $U^T \in B^+$ (pointwise) by the formula

$$U^T (C) = W (1, y^*) \phi_\gamma (C)$$

where

$y^* > 0$ is the unique solution to $W (1, y) = y$.

Define the order interval $\langle \theta, U^T \rangle \subset B^+$.

$\langle \theta, U^T \rangle$ is a complete lattice (in the induced partial order) contained in $B^+$.

**MM prove** $T_W : \langle \theta, U^T \rangle \rightarrow \langle \theta, U^T \rangle$. 
MM’s Recovery Theorem

- \( \text{fix}(T_W) \) is the set of fixed points of the Koopmans operator restricted to the domain \( \langle \theta, U^T \rangle \).

**Theorem**

*(MM 2010)* Suppose \( W \) is a Thompson aggregator. Then there are functions \( U_\infty \leq U^\infty \) such that each is a fixed point of the Koopmans operator. Moreover,

1. \( \text{fix}(T_W) \subseteq \langle U_\infty, U^\infty \rangle \subseteq \langle \theta, U^T \rangle \);
2. \( \text{fix}(T_W) \) is a complete lattice in the induced order;

- \( U_\infty \) & \( U^\infty \) are the **extremal fixed points** of \( T_W \).
- \( U_\infty \) is the **Least Fixed Point (LFP)** & \( U^\infty \) the **Greatest Fixed Point (GFP)**.
- MM’s proof rests on the “non-constructive” Tarski FPT.
Tarski-Kantorovich FPT.

- Kantorovich (1939) proved a “constructive” FPT applicable to Dedekind complete Riesz spaces. The so-called Tarski-Kantorovich (TK) FPT (Granas and Dugundji (2003)) extends this result.
- TK FPT weakens the operator’s domain and adds a “continuity” property to monotonicity for the operator in comparison to Tarski’s FPT.
- The interpretation and implementation of “continuity” underlies today’s main LFP theme. There turn out to be 2 interesting interpretations for LFP theory! Only focus on 1 case today.
- We follow the TK FPT developed by Balbus, Dziewulski, Reffett, and Woźny, *Int. J. Game Th.* (2015).

Show blackboard illustrations for $F : [0, 1] \rightarrow [0, 1]$. 
Theorem (Tarski-Kantorovich). Suppose that $X$ is a countably chain complete partially ordered set with the least element, $x$ and the greatest element, $\bar{x}$. Let $F$ be a monotone self-map on $X$.

1. If $F$ is monotonically sup-preserving; then $\bigvee F^N(x)$ is the least fixed point of $F$, denoted $x_\infty$.
2. If $F$ is monotonically inf-preserving; then $\bigwedge F^N(\bar{x})$ is the greatest fixed point of $F$, denoted $x^\infty$;
3. $\text{fix}(F)$ is a nonempty countably chain complete poset in $X$.

- $F$ is monotonically sup-inf preserving if it is both sup and inf-preserving.
- Primary focus on Least Fixed Point Theory and monotonic sup preservation.
We prove that $T_W$ is **monotonically sup-inf preserving**. This continuity property rests only on order theoretic structures available in the commodity and utility spaces.

**Constructive** means that the fixed points are found by iteration of $T_W$ with an initial seed $U$, and denoted by $T_W^N U = T_W (T_W^{N-1} U)$ for $T_W^0 U = U$. The 2 interesting initial seeds are: $\theta$ and $U^T$.

We show $T_W^N \theta \uparrow U_\infty$ is the **least fixed point** of $T_W$.

Likewise, $T_W^N U^T \downarrow U_\infty$ is the **greatest fixed point** of $T_W$.

$\text{fix}(T_W)$ is a countably chain complete poset in $\langle \theta, U^T \rangle$. 
Approximations and Properties of the Extremal Fixed Points

Why Focus on the Least Fixed Point? Answer: Desirable Economic & Math Properties

- The LFP partial sum method (successive approximations) approximates \( U_{\infty} \) from below (starting with no information . . .):
  \[
  T^N_W \theta(C) = W(c_1, W(c_2, \ldots, W(c_N, 0)) \uparrow U_{\infty}.
  \]

- \( U_{\infty} \) is norm LSC on \( \ell_\infty^+ \). It is also monotone and concave on that domain. Concavity requires a concave Thompson aggregator (as is the case with our examples).

- Concavity implies \( U_{\infty} \) is norm continuous on the interior of its effective domain, which is the interior of \( \ell_\infty^+ \), denoted by \( \ell_\infty^{++} \).

- Concavity also implies \( U_{\infty} \) is weakly (and product) continuous (for the dual pair \((\ell_\infty, ba)\)) on each closed convex subset of \( \ell_\infty^{++} \).

- \( U_{\infty} \) is norm USC on \( \ell_\infty^+ \) and monotone. Our iterative methods alone do not imply \( U_{\infty} \) is a concave function as the input function \( U^T \) is a convex function! And, INPUTS ALL component of \( C \) vs a finitely many in for LFP approximations.
Monotonic Sup-Preservation, Order Continuity and the Least FP

- \( \{ T^N_W \theta \} \) is a **monotone** (isotone or nondecreasing) sequence. First, define

\[
\liminf_N T^N_W \theta \equiv \sup_N \left( \inf_{K \geq N} T^K_W \theta \right) = \bigvee_N T^N_W \theta, \tag{7}
\]

and note the supremum and infima exist in \( \langle \theta, U^T \rangle \) as it is an order bounded subset of the Dedekind complete lattice defining the utility function space.

- **Monotonic sup-preservation for sequences** means:

\[
T_W \left( \liminf_N T^N_W \theta \right) = \liminf_N T^N_W \theta, \quad \text{or} \quad T_W \left( \bigvee_N T^N_W \theta \right) = \bigvee_N T^N_W \theta.
\]

This implies \( T_W (U_\infty) = U_\infty \) **when** \( \bigvee_N T^N_W \theta \equiv U_\infty \).
Monotonic sup-preservation for sequences is the order continuity condition needed to construct the least fixed point in the TK FPT.

It is an order theoretic concept based on the Dedekind complete order structure of the Riesz space of possible utility functions. It is a NOT A TOPOLOGICAL CONTINUITY notion.

This condition, suitably abstracted in the Scott topology, underlies the LFP construction combining topological and order theoretic methods covered in the paper. It is EXTREMELY TECHNICAL.

Austrian Capital Theory Idea: $T_W$ is Scott continuous provides an alternative, more roundabout (capital intensive), constructive method than the TK FPT for proving a LFP exists. Scott continuity for $T_W$ supports the proposition that $U_\infty$ is the principal solution of the Koopmans equation. This topological method does NOT construct the GFP!

See our paper for details of this second interpretation of order continuity.
Recall this means: for the monotonic (isotonic) sequence \( \{ T_N^W \theta \}_{N=1}^{\infty} \), it follows that:

\[
T_W \left( \bigvee_N T_N^W \theta \right) = \bigvee_N T_N^W \theta. \tag{8}
\]

This equality can be broken up into two inequalities:

\[
T_W \left( \bigvee_N T_N^W \theta \right) \leq \bigvee_N T_N^W \theta; \tag{9}
\]

and

\[
T_W \left( \bigvee_N T_N^W \theta \right) \geq \bigvee_N T_N^W \theta. \tag{10}
\]

An analogous condition to (8) holds for monotone (antitone) sequences and inf-preservation.
The Koopmans operator is a monotonic sup/inf preserving self map on the order interval $\langle \theta, U^T \rangle$.

- Provide a heuristic interpretation of why we expect (8) to obtain.
- NOTE: $T_W$ monotone and $\{T_N^W \theta\}_{N=1}^\infty$ monotonic (nondecreasing pointwise) imply (10).

$$\bigvee_N T_N^W \theta \geq T_K^W \theta \text{ for each } K \in \mathbb{N};$$

$$T_W \left( \bigvee_N T_N^W \theta \right) \geq T_W \left( T_K^W \theta \right) = T_K^{W+1} \theta.$$ 

Taking the sup on the RHS (above); Re-indexing with $N$:

$$T_W \left( \bigvee_N T_N^W \theta \right) \geq \bigvee_N T_N^W \theta.$$
Goal: Show the monotonic sup-preservation property is a reasonable requirement!

\[ T^N_W \theta (C) \leq T^{N+1}_W \theta (C) \] for each \( C \) if and only if

\[ W (c_1, W(c_2, \ldots, W(c_N, 0) \ldots)) \leq W (c_1, W(c_2, \ldots, W(c_{N+1}, 0) \ldots) \]

That is, more information about the utility value \( U_\infty (C) \) is given by \( T^{N+1}_W \theta (C) \) than \( T^N_W \theta (C) \). If we interpret

\[ \forall_N T^N_W \theta (C) = \lim \inf_N T^N_W \theta (C) = U_\infty (C) \]

as a notion of maximal information about \( U_\infty (C) \), then it stands to reason we cannot deduce additional information by applying \( T_W \) to \( \lim \inf_N T^N_W \theta (C) \) again! That is, pointwise,

\[ T_W \left( \lim \inf_N T^N_W \theta \right) \leq \lim \inf_N T^N_W \theta \]

should hold for the monotonic sequence \( \{ T^N_W \theta \} \). But this is just (9).
Continuation: Heuristics for (9).

- Suppose \( T_w \left( \liminf_N T^N_w \theta \right) \) contained more information about \( U_\infty \) than \( \liminf_N T^N_w \theta \), i.e.

\[
\liminf_N T^N_w \theta < T_w \left( \liminf_N T^N_w \theta \right).
\]

- Information contained in RHS becomes known to us at Vickers’ ominous *Crack of Doom* — the time when ALL infinite computations are completed! And, that time comes TOO LATE!!

- Hence \( T_w \left( \liminf_N T^N_w \theta \right) \leq \liminf_N T^N_w \theta \) holds. Inequality (9) is the critical property of \( \{ T^N_w \theta \} \) for LFP Theory – extends to the Scott Topology continuity case.

- Remark: This is NOT the Theorem’s FORMAL PROOF — see the paper!

- Caveat: This interpretation does NOT apply to monotonic inf-preservation and GFP construction.
Theorem

(Least Fixed Point Existence and Construction Theorem) The monotonic sup-preserving Koopmans operator has a least fixed point, \( U_\infty \). Moreover, \( U_\infty = \bigvee_N T_N^W \theta \) and it is constructed by successive approximations indexed on the natural numbers.

Proof.

The existence and construction of \( U_\infty \) follows from the Tarski-Kantorovich Theorem since \( T_W \) preserves the supremum of the monotonic sequence \( \{ T_N^W \theta \} \). Hence, \( U_\infty = \bigvee_N T_N^W \theta = T_W U_\infty \) and \( U_\infty \in \text{fix}(T_W) \).

Suppose that \( U \in \text{fix}(T_W) \). Then \( \theta \leq U \) and \( T_W \) monotone implies \( T_W \theta \leq T_W U = U \). Iterate this to yield the inequality \( T_N^W \theta \leq U \). Hence, passing to the limit we find \( U_\infty \leq U \) and \( U_\infty \) is the least fixed point of the Koopmans operator acting on \( \langle \theta, U^T \rangle \). \( \square \)
Absent a uniqueness theorem the Koopmans operator may have many fixed points.

We present reasons why the least fixed point should be singled out as the principal fixed point (Kantorovich).

We give economic properties, mathematical features, and theoretical computational advantages belonging to the Least Fixed Point in support of our favoring $U_\infty$ over $U^\infty$.

Of course, an adequate uniqueness theory would make this distinction among the possibly multiple fixed points an “academic exercise.”

However, we know from counterexamples that uniqueness cannot hold for all consumption sequences in the domain $\ell^+_\infty$, hence the prospect of spurious solutions to the Koopmans equation may be partially ameliorated by concentration on the Least Fixed Point, $U_\infty$. Put differently, treat the LFP as a selection criteria to choose one solution from the possible ones in $\text{fix}(T_W)$. 
Both the Recovery and Uniqueness Problems require a domain of functions for the Koopmans operator to act on as a self-map.

The class of aggregators for which this approach applies (under suitable restrictions) forms the *Blackwell aggregator class.*

The TAS and KDW cases shown above are Blackwell aggregators.

Becker and Boyd’s (1997) book covers the theory up to their publication date and focus exclusively on the Blackwell cases.

Recent work by Rincón-Zapatero and Rodríguez-Palmero, Martins-Da-Rocha and Vailakis, extends this work to a number of previously uncovered Blackwell aggregators using a range of *local contraction arguments.* Le Van and Vailakis examine partial sum methods for aggregators that are unbounded. Their examples include many Blackwell aggregators.
### Annex 2: Thompson Aggregators: Defining Assumptions

**Definition**

$W : \mathbb{R}^2_+ \rightarrow \mathbb{R}$ is said to be a **Thompson aggregator** if it satisfies properties (T1) – (T4):

1. **(T1)** $W \geq 0$, continuous, and monotone: $(x, y) \leq (x', y')$ implies $W(x, y) \leq W(x', y')$;

2. **(T2)** $W(x, y) = y$ has at least one nonnegative solution for each $x \geq 0$;

3. **(T3)** $W(x, \bullet)$ is concave at 0 for each $x \geq 0$, that is

\[ W(x, \mu y) \geq \mu W(x, y) + (1 - \mu) W(x, 0) \]

for each $\mu \in [0, 1]$ and each $(x, y) \in \mathbb{R}^2_+$;

4. **(T4)** $W(x, 0) > 0$ for each $x > 0$. 

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**Slides - Thompson Aggregators**

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Additional MM Aggregator Assumptions

MM assume Thompson aggregators satisfy (T5) & (T6) below.

(T5) \( W \) is \( \gamma \)– subhomogeneous — there is some \( \gamma > 0 \) such that:

\[
W(\mu^\gamma x, \mu y) \geq \mu W(x, y)
\]

for each \( \mu \in (0, 1] \) and each \( (x, y) \in \mathbb{R}_+^2 \).

- Thompson KDW satisfies (T5) with \( \gamma = b^{-1} \);
- Thompson CES aggregators satisfy (T5) with \( \gamma = 1 \).
MM also assume:

\[(T6)\] \(W\) satisfies the \textit{MM-Limit Condition}: for each \(\alpha \geq 1\) and \(\gamma > 0\),

\[
\lim_{t \to \infty} \frac{W(1, t)}{t} < \alpha^{-1/\gamma},
\]

with \(t > 0\).

- The parameter \(\alpha\) in \((T6)\) is the economy's maximum long-run possible consumption growth factor.
- We set \(\alpha = 1\) (no long-run growth) today and note \((T6)\) holds whenever the LHS of 11 is less than 1.
- Parameter \(\gamma\) is taken from \((T5)\).
- KDW, CES and Quasi-Linear aggregators satisfy \((T6)\).
Annex 3: The Commodity Space Setup

- Commodity Space: $\ell_\infty^+$ with sup norm with $\|C\|_\infty = \sup_t |c_t|$ whenever $C \in \ell_\infty$. The zero sequence is 0.

- Commodity spaces admitting exponential growth are admissible in the paper’s more general setup. Focus on this special case of bounded growth in today’s talk.

- Imagine “bounded” production possibilities – e.g. diminishing returns to capital & one-sector model.

- The commodity space $\ell_\infty$ is a Banach lattice with the usual pointwise partial order. The positive cone is $\ell_\infty^+$.

- It is also an AM - space with unit given by $e = (1, 1, \ldots)$. This fact implies: the sup norm interior of $\ell_\infty^+$ is non-empty. Denote the positive cone’s interior by $\ell_\infty^{++}$; thus, $\ell_\infty^+$ is a solid cone. Clearly $e \in \ell_\infty^{++}$. 
Annex 4: Industrial Strength Version of Blackwell’s Theorem

The basic Blackwell aggregator theory in Becker and Boyd (1997) rests on the following mathematical theorem:

**Monotone Contraction Mapping Theorem for Ideals:** Let \( A_\omega \) be a principal Riesz ideal of the Riesz space \( E \) that is complete in the associated lattice norm. Suppose that \( T : A_\omega \to E \) obeys:

1. \( Tx \leq Ty \) whenever \( x \leq y \);
2. \( T\theta \in A_\omega \);
3. \( T(x + \lambda \omega) \leq Tx + \lambda \delta \omega \) with \( 0 \leq \delta < 1 \) and \( \lambda > 0 \).

Then \( T \) is a strict contraction and has a unique fixed point.

**FOCUS TODAY:** \( T \) is a monotone operator.
Fix the vector $\omega \in E$ — an **order unit** in the norm interior of $E$. $A_\omega$ is the subset of $E$ defined by:

$$x \in A_\omega \text{ iff } |x| \leq \lambda \omega \text{ for some scalar } \lambda \geq 0.$$ 

Here: $|x| = \sup (x, -x)$.

**Lattice Norm:** $|x| \leq y \Rightarrow \|x\| \leq \|y\|$ where

$$\|x\| = \inf \{\lambda > 0 : |x| \leq \lambda \omega\} \text{ for } x \in E \text{ & } \omega > 0 \text{ an order unit.}$$

**Example:** Let $E$ be the space of real sequences & $\omega = (1, 1, 1, \ldots)$. Then $A_\omega = \ell_\infty$.

**Example:** Suppose for $\alpha \geq 1$, $\omega = (\alpha, \alpha^2, \alpha^3, \ldots)$. $A_\omega$ contains sequences that grow exponentially.

**Example:** The space $B$ introduced below is a Principal Ideal: The order unit is the weight function $\varphi_\gamma$. 

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