Recursive Utility and Thompson Aggregators I: Constructive Existence Theory for the Koopmans Equation[1]

Macro Brown Bag (IUB) Presentation Slides

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Exponential Utility & the TAS Utility Function

- The Exponential Discounting Model (infinite horizon, discrete time) aka the familiar **TAS Utility** is:

$$\sum_{t=1}^{\infty} \delta^{t-1} u\left(c_t\right),$$

  where $0 < \delta < 1$ and u is a bounded, continuous, and concave (differentiable) function on $\mathbb{R}_+ = [0, \infty)$. Usual Inada conditions apply (e.g. $0 \leq u\left(c\right) = \arctan\left(\sqrt{c}\right) \leq \left(\pi/2\right)$).
- Discounted Optimal Growth Model w/One Sector: Euler Equation:

$$u'\left(c_t\right) = \delta f'\left(k_t\right) u'\left(c_{t+1}\right) \text{ each } t.$$

- Steady state condition: $\delta f'\left(k^*\right) = 1$ — independent of the form of u.
- **Dynamic Nonsubstitution Theorem:** Steady state depends on technology and pure rate of time preference ($\delta = 1/\left(1+\rho\right)$ with $\rho > 0$).

TAS and the Impatience Problem

- **Impatience Problem** – one example where the exponential discounting model produces an extreme result.
- RB's papers on "Ramsey Equilibria." Only most patient household holds capital: **IF 2 households and**:

  $$1 > \delta_1 > \delta_2 > 0,$$

  then

  $$\delta_2 f'\left(k^*\right) < \delta_1 f'\left(k^*\right) = 1.$$

  Impossible for both to hold capital in a steady state.
- **Recursive Utility functions create alternative specifications for an infinitely-lived household's "lifetime" utility function where the stationary equilibrium discount factor depends on the underlying consumption sequence. Are more robust results available than w/TAS?**
- Need ways to describe recursive utility functions and some of their economic properties. **Focus today: The description of recursive utility functions via aggregators.**

Recursive Property of the TAS Utility

$$U(C) = \sum_{t=1}^{T} \delta^{t-1} u(c_t) + \delta^T \sum_{t=1}^{\infty} \delta^{t-1} u(c_{t+T}).$$

- Let $S^T C = (c_{T+1}, c_{T+2}, \ldots)$. The decision maker's behavior over the infinite horizon is guided by the behavior over the tail horizon $t = T+1, T+2, \ldots$ for each T that is hidden inside the original horizon. *Recursivity is a self-referential property.*
- **Recursive Utility functions abstract this self-referential property** in order to relax the fixed discount factor assumption and maintain the preference structure to derive **time consistent decision rules** in stationary infinite horizon optimization problems.

Non-Recursive Example: *Quasi-Geometric Utility (behavioral theory):* $0 < \beta < 1, \beta \neq \delta,$

$$U(C) = u(c_1) + \beta \sum_{t=2}^{\infty} \delta^{t-1} u(c_t)$$

Recursive Utility Background

- Recursive Utility Generalizes TAS Time Stationarity and Time Consistency Properties
- Koopmans (1960s) axiomatic approach: preference structures induce utility representations of the form:

$$U(C) = W(c_1, U(SC)), \qquad (1)$$

where $C = \{c_t\}_{t=1}^{\infty} \in \ell_{\infty}^{+}$ — the positive cone of ℓ_{∞} with its sup norm topology is the **commodity space** and S is the **shift operator**:

$$SC = \{c_2, c_3, \ldots\}.$$

- The function W is the **aggregator** and has two arguments: x — present consumption; y — future utility. Write $W(x, y)$ for the aggregator. W has a 2-period *Fisherian Interpretation*.
- **A recursive utility function satisfies (1)** — the aggregator is derived from an axiomatization of the preference ordering.

Classic Aggregators

Example

Time Additive Utility (TAS): $u \geq 0$ & bounded, has the aggregator

$$W(x,y) = u(x) + \delta y \text{ with } 0 < \delta < 1 \text{ and utility function,}$$

$$\sum_{t=1}^{\infty} \delta^{t-1} u(c_t) := U(C).$$

Example

The Koopmans, Diamond and Williamson (KDW) aggregator is

$$W(x,y) = \frac{\delta}{d} \ln\left(1 + ax^b + dy\right); \; a, b, d, \delta > 0. \tag{2}$$

Assume $b < 1$ and $\delta < 1$. Then W is concave and a Lipschitz condition obtains: $0 \leq \sup_y W_2(x,y) < 1$.

Blackwell Aggregators and Partial Summation

- Lucas and Stokey's (1984) idea is to let aggregators be the primitive concept: given W find U satisfying (1). **Fisherian "two-period interpretation" motivation.**
- Consider the TAS aggregator: Successive approximations initiated from $\theta(C) = 0$ (input "no information" about U) yields the sequence of **partial sums** :

$$
\begin{aligned}
U_1(C) &= u(c_1) = W(c_1, \theta(SC)) = W(c_1, 0) \\
U_2(C) &= u(c_1) + \delta u(c_2) = W(c_1, W(c_2, 0)) \\
&\vdots \\
U_N(C) &= \sum_{t=1}^{N} \delta^{t-1}(c_t) = W(c_1, W(c_2, W(c_{3,\dots} \dots, W(c_N, 0)).
\end{aligned}
$$

- Clearly $U_N(C) \nearrow \sum_{t=1}^{\infty} \delta^{t-1}(c_t)$. Each partial sum $U_N(C)$ **approximates** $\sum_{t=1}^{\infty} \delta^{t-1}(c_t)$ **from below.**
- Abstract **partial summation method** *in today's LFP construction*.

- Introduce the **Koopmans operator**, denoted T_W, and defined by:

$$T_W U(C) = W(c_1, U(SC)).$$

A recursive utility function is a fixed point of this operator:

$$T_W U = U.$$

- The **RECOVERY PROBLEM** is to show that this operator equation has at least one solution.
- The **UNIQUENESS PROBLEM** is to show there is at most one solution.

Contraction Mappings

- Banach's Contraction Mapping Theorem, when applicable, resolves the Recovery (or, Existence) Problem and the Uniqueness Problem at once (and contains a successive approximations construction of the solution).
- $X \neq \varnothing$. Define
  $B(X) = \{f \text{ such that } f : X \to \mathbb{R} \text{ and } \|f\|_\infty = \sup_{x \in X} |f(x)| < \infty\}$.
- This is a complete metric space when $B(X)$ is assigned its norm topology.
- **Pointwise order:** $f \geq g$ if and only if $f(x) \geq g(x)$ for each $x \in X$.

Definition

An operator $T : B(X) \to B(X)$ is a **contraction mapping with modulus** $\delta \in (0,1)$ if for each $f, g \in B(X)$:

$$\|Tf - Tg\|_\infty \leq \delta \|f - g\|_\infty .$$

Blackwell's Sufficient Condition for A Contraction Mapping

Theorem

(Banach) If $T : B(X) \to B(X)$ is a contraction mapping with modulus $\delta \in (0,1)$, then there is a unique $f^* \in B(X)$ such that $Tf^* = f^*$.

Theorem

(Blackwell (1965) Let $T : B(X) \to B(X)$ be an operator satisfying:

(M) $f \geq g \Longrightarrow Tf \geq Tg$ — **Monotonicity;**

(D) *There exists some $\delta \in (0,1)$ such that*

$$[T(f+a)](x) \leq (Tf)(x) + \delta a$$

for each nonnegative scalar a, each $f \in B(X)$, and each $x \in X$ — **Discounting**.

Then, T is a contraction with modulus δ.

- Here: $(f+a)(x) = f(x) + a$ for each $x \in X$.

- We look today at a new class of aggregators known as **Thompson Aggregators**. The Blackwell aggregator Lipschitz/discounting condition fails in this scenario.
- **Marinacci and Montrucchio (MM)** (JET 2010) introduced this family and gave sufficient conditions for Recovery and Uniqueness Theorems. Blackwell's Sufficient Condition for Contraction Maps fails, but **Monotonicity** holds.
- We emphasize **constructive methods for Recovery Theory (successive approximations — partial summation methods) and focus on Least Fixed Point (LFP) construction by Monotone Operator methods only.** Defer a discussion of uniqueness theory for our second paper (see references).
- Need this foundation to infer qualitative properties of extremal fixed points (e.g. concavity and continuity).
- The LFP corresponds to **Kantorovich's (1939) Principal Fixed Point (Principal Solution). We discuss why this might be a reasonable interpretation of LFP.**

Thompson Aggregators: Two Examples

Example

KDW is Thompson whenever $\delta \geq 1$. Is jointly concave in (x, y) and unbounded (above). It is Lipschitz in y, but T_W is NOT a contraction mapping.

Example

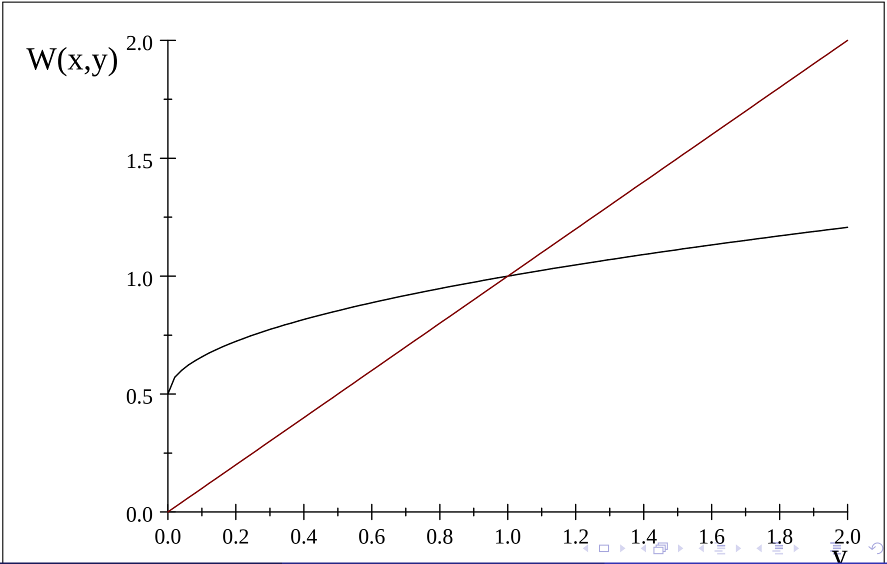
Constant Elasticity of Substitution (CES) aggregators have the form:

$$W(x, y) = (1-\beta) x^{\rho} + \beta y^{\rho} \qquad (3)$$

where $0 < \beta < 1$ and $0 < \rho < 1$ implying the elasticity of substitution, $1/(1-\rho) > 1$. This W is Thompson & jointly concave in (x, y). It fails the Blackwell Lipschitz condition and is unbounded (above).

- A Thompson aggregator jointly concave in (x, y) is a **concave Thompson aggregator**.

- Graph of $W(x,y) = (1-\beta)x^{\rho} + \beta y^{\rho}$ for $\beta = \rho = 1/2$ as y varies for a fixed value of $x = 1$: $\forall x \geq 0\ \exists!\ y_x \geq 0$ such that $W(x, y_x) = y_x$.



The Weight Function

- Define a **weight function on the commodity space** ℓ_∞^+, φ_γ, by the formula
$$\varphi_\gamma(C) = (1 + \|C\|_\infty)^{1/\gamma}. \tag{4}$$
- This function is uniformly continuous on ℓ_∞^+ with respect to the sup norm topology.
- Clearly $\varphi_\gamma(C) \geq 1$ for each C.
- $\gamma > 0$ is a **subhomogeniety** parameter. It comes from the formal Thompson aggregator assumptions.
- Thompson KDW has $\gamma = b^{-1}$;
- Thompson CES aggregators has $\gamma = 1$.

Setup for the Utility Function Space

- $U : \ell_\infty^+ \to \mathbb{R}$ is $\varphi_\gamma -$ **bounded** provided

$$\|U\|_\gamma := \sup_{C \in \ell_\infty^+} \frac{|U(C)|}{(1 + \|C\|_\infty)^{1/\gamma}} < +\infty. \quad (5)$$

- $B = \left\{ U : \ell_\infty^+ \to \mathbb{R} : U \text{ is } \varphi_\gamma - \text{bounded} \right\}$.
- Provide B with the pointwise ordering: the positive cone is defined by $U \in B^+$ if and only if $U \geq \theta$ pointwise; i.e., $U(C) \geq \theta(C) = 0$ for each $C \in \ell_\infty^+$.
- Define the standard pointwise lattice operations, $\vee$ (sup) and $\wedge$ (inf).
- B is a **Dedekind complete Riesz space** & Banach lattice with the norm, $\|U\|_\gamma$, and *order unit*, φ_γ.
- The positive cone is B^+ is a norm-closed, convex set and has a nonempty norm interior.

Monotone Koopmans Operator

- If W is a Thompson aggregator, then T_W is a **monotone** self-map on B^+, i.e. $U \geq V \in B^+$ implies $T_W U \geq T_W V$. In particular, $T_W \theta \geq \theta$.
- Define $U^T \in B^+$ (pointwise) by the formula

$$U^T(C) = W(1, y^*)\ \varphi_\gamma(C) \tag{6}$$

where

$$y^* > 0 \text{ is the unique solution to } W(1, y) = y.$$

- Define the order interval $\langle \theta, U^T \rangle \subset B^+$.
- $\langle \theta, U^T \rangle$ is a complete lattice (in the induced partial order) contained in B^+.
- **MM prove** $T_W : \langle \theta, U^T \rangle \to \langle \theta, U^T \rangle$.

MM's Recovery Theorem

- $\text{fix}(T_W)$ is the set of fixed points of the Koopmans operator restricted to the domain $\langle \theta, U^T \rangle$.

Theorem

(MM 2010) Suppose W is a Thompson aggregator. Then there are functions $U_\infty \leq U^\infty$ such that each is a fixed point of the Koopmans operator. Moreover,

1. $\text{fix}\,(T_W) \subseteq \langle U_\infty, U^\infty \rangle \subset \langle \theta, U^T \rangle$;
2. $\text{fix}\,(T_W)$ *is a complete lattice in the induced order;*

- U_∞ & U^∞ are the **extremal fixed points** of T_W.
- U_∞ is the **Least Fixed Point (LFP) &** U^∞ the **Greatest Fixed Point (GFP).**
- MM's proof rests on the "non-constructive" Tarski FPT.

Tarski-Kantorovich FPT.

- Kantorovich (1939) proved a "constructive" FPT applicable to Dedekind complete Riesz spaces. The so-called Tarski-Kantorovich (TK) FPT (Granas and Dugundji (2003)) extends this result.
- TK FPT weakens the the operator's domain and adds a "continuity" property to monotonicity for the operator in comparison to Tarski's FPT.
- **The interpretation and implementation of "continuity" underlies today's main LFP theme. There turn out to be 2 interesting interpretations for LFP theory! Only focus on 1 case today.**
- We follow the TK FPT developed by Balbus, Dziewulski, Reffett, and Woźny, *Int. J. Game Th.* (2015).

Show blackboard illustrations for $F : [0, 1] \to [0, 1]$.

Theorem

(**Tarski-Kantorovich**). *Suppose that X is a countably chain complete partially ordered set with the least element, $\underline{x}$ and the greatest element, $\bar{x}$. Let F be a monotone self-map on X.*

1. *if F is monotonically sup-preserving; then $\bigvee F^N(\underline{x})$ is the least fixed point of F, denoted x_∞.*
2. *If F is monotonically inf-preserving; then $\bigwedge F^N(\bar{x})$ is the greatest fixed point of F, denoted x^∞;*
3. $\mathrm{fix}(F)$ *is a nonempty countably chain complete poset in X.*

- F **is monotonically sup-inf preserving** if it is both sup and inf-preserving.
- Primary focus on Least Fixed Point Theory and monotonic sup preservation.

MM's Theorem and our Order Theoretic Construction

- We prove that T_W is **monotonically sup-inf preserving**. This continuity property rests only on order theoretic structures available in the commodity and utility spaces.
- **Constructive** means that the fixed points are found by iteration of T_W with an initial seed U, and denoted by $T_W^N U = T_W \left(T_W^{N-1} U \right)$ for $T_W^0 U = U$. The 2 interesting initial seeds are: θ and U^T.
- We show $T_W^N \theta \nearrow U_\infty$ is the **least fixed point** of T_W.
- Likewise, $T_W^N U^T \searrow U^\infty$ is the **greatest fixed point** of T_W.
- $\mathrm{fix}(T_W)$ is a countably chain complete poset in $\langle \theta, U^T \rangle$.

Approximations and Properties of the Extremal Fixed Points

Why Focus on the Least Fixed Point? Answer: Desirable Economic & Math Properties

- The LFP partial sum method (successive approximations) **approximates U_∞ from below** (starting with no information...):

$$T_W^N \theta(C) = W(c_1, W(c_2, W(c_{3,...} \ldots, W(c_N, 0)) \nearrow U_\infty.$$

- U_∞ is **norm LSC** on ℓ_∞^+. It is also **monotone** and **concave** on that domain. **Concavity requires a concave Thompson aggregator (as is the case with our examples).**
- Concavity implies U_∞ is **norm continuous on the interior of its effective domain**, which is the interior of ℓ_∞^+, denoted by ℓ_∞^{++}.
- Concavity also implies U_∞ is **weakly (and product) continuous** (for the dual pair (ℓ_∞, ba)) on each closed convex subset of ℓ_∞^{++}.
- U^∞ is **norm USC** on ℓ_∞^+ and **monotone**. Our iterative methods alone do not imply U^∞ is a concave function as the **input function U^T is a convex function**! And, INPUTS ALL component of C vs a finitely many in for LFP approximations.

Monotonic Sup-Preservation, Order Continuity and the Least FP

- $\left\{ T_W^N \theta \right\}$ is a **monotone** (isotone or nondecreasing) sequence. First, define

$$\liminf_N T_W^N \theta \equiv \sup_N \left(\inf_{K \geq N} T_W^K \theta \right) = \bigvee_N T_W^N \theta, \tag{7}$$

and note the supremum and infima exist in $\left\langle \theta, U^T \right\rangle$ as it is an order bounded subset of the Dedekind complete lattice defining the utility function space.

- **Monotonic sup-preservation for sequences** means:

$$\begin{aligned} T_W \left(\liminf_N T_W^N \theta \right) &= \liminf_N T_W^N \theta, \text{ or} \\ T_W \left(\bigvee_N T_W^N \theta \right) &= \bigvee_N T_W^N \theta. \end{aligned}$$

This implies $T_W \left(U_\infty \right) = U_\infty$ **when** $\vee_N T_W^N \theta \equiv U_\infty$.

- **Monotonic sup-preservation for sequences is the order continuity condition needed to construct the least fixed point in the TK FPT.**
- It is an order theoretic concept based on the Dedekind complete order structure of the Riesz space of possible utility functions. It is a NOT A TOPOLOGICAL CONTINUITY notion.
- *This condition, suitably abstracted in the Scott topology, underlies the LFP construction combining* **topological and order theoretic methods covered in the paper. It is EXTREMELY TECHNICAL.**
- *Austrian Capital Theory Idea:* T_W is Scott continuous provides an alternative, **more roundabout (capital intensive),** constructive method than the TK FPT for proving a LFP exists. Scott continuity for T_W supports the *proposition that* U_∞ *is the principal solution of the Koopmans equation.* This topological method does NOT construct the GFP!
- See our paper for details of this second interpretation of order continuity.

Monotonic Sup-Preservation

- Recall this means: for the monotonic (isotonic) sequence $\left\{ T_W^N \theta \right\}_{N=1}^{\infty}$, it follows that:

$$T_W \left(\bigvee_N T_W^N \theta \right) = \bigvee_N T_W^N \theta. \tag{8}$$

This equality can be broken up into two inequalities:

$$T_W \left(\bigvee_N T_W^N \theta \right) \leq \bigvee_N T_W^N \theta; \tag{9}$$

and

$$T_W \left(\bigvee_N T_W^N \theta \right) \geq \bigvee_N T_W^N \theta. \tag{10}$$

- An analogous condition to (8) holds for monotone (antitone) sequences and inf-preservation.

Monotonic Sup/Inf Preservation Theorem

Theorem

The Koopmans operator is a monotonic sup/inf preserving self map on the order interval $\langle \theta, U^T \rangle$.

- Provide a heuristic interpretation of why we expect (8) to obtain.
- NOTE: T_W monotone and $\left\{ T_W^N \theta \right\}_{N=1}^{\infty}$ monotonic (nondecreasing pointwise) imply (10).

$$\bigvee_N T_W^N \theta \quad \geq \quad T_W^K \theta \text{ for each } K \in \mathbb{N};$$

$$T_W \left(\bigvee_N T_W^N \theta \right) \quad \geq \quad T_W \left(T_W^K \theta \right) = T_W^{K+1} \theta.$$

Taking the sup on the RHS (above); Re-indexing with N:

$T_W \left(\bigvee_N T_W^N \theta \right) \geq \bigvee_N T_W^N \theta.$

Heuristic Argument for Inequality(9) After Vickers (1989).

- **Goal**: Show the monotonic sup-preservation property is a reasonable requirement!
- $T_W^N \theta(C) \leq T_W^{N+1}\theta(C)$ for each C if and only if

$$W(c_1, W(c_2, \ldots, W(c_N, 0)\cdots)) \leq W(c_1, W(c_2, \ldots, W(c_{N+1}, 0)\cdots$$

That is, **more information about the utility value** $U_\infty(C)$ **is given by** $T_W^{N+1}\theta(C)$ **than** $T_W^N\theta(C)$. If we interpret $\vee_N T_W^N \theta(C) = \liminf_N T_W^N \theta(C) = U_\infty(C)$ as a notion of **maximal information** about $U_\infty(C)$, then it stands to reason we cannot deduce additional information by applying T_W to $\liminf_N T_W^N \theta(C)$ again! That is, pointwise,

$$T_W\left(\liminf_N T_W^N \theta\right) \leq \liminf_N T_W^N \theta$$

should hold for the monotonic sequence $\{T_W^N \theta\}$. But this is just (9).

Continuation: Heuristics for (9).

- Suppose $T_W\left(\liminf_N T_W^N\theta\right)$ contained **more information about** U_∞ **than** $\liminf_N T_W^N\theta$, i.e.

$$\liminf_N T_W^N\theta < T_W\left(\liminf_N T_W^N\theta\right).$$

- Information contained in RHS becomes known to us at Vickers' ominous *Crack of Doom* — the time when ALL infinite computations are completed! And, that time comes TOO LATE!!
- Hence $T_W\left(\liminf_N T_W^N\theta\right) \leq \liminf_N T_W^N\theta$ holds. Inequality (9) is the critical property of $\left\{T_W^N\theta\right\}$ for LFP Theory – extends to the Scott Topology continuity case.
- Remark: This is NOT the Theorem's FORMAL PROOF — see the paper!
- **Caveat: This interpretation does NOT apply to monotonic inf-preservation and GFP construction.**

A Constructive Least Fixed Point Theorem

Theorem

(Least Fixed Point Existence and Construction Theorem) The monotonic sup-preserving Koopmans operator has a least fixed point, U_∞. Moreover, $U_\infty = \vee_N T_W^N \theta$ and it is constructed by successive approximations indexed on the natural numbers.

Proof.

The existence and construction of U_∞ follows from the Tarski-Kantorovich Theorem since T_W preserves the supremum of the monotonic sequence $\{T_W^N \theta\}$. Hence, $U_\infty = \vee_N T_W^N \theta = T_W U_\infty$ and $U_\infty \in \text{fix}(T_W)$. Suppose that $U \in \text{fix}(T_W)$. Then $\theta \leq U$ and T_W monotone implies $T_W \theta \leq T_W U = U$. Iterate this to yield the inequality $T_W^N \theta \leq U$. Hence, passing to the limit we find $U_\infty \leq U$ and U_∞ is the least fixed point of the Koopmans operator acting on $\langle \theta, U^T \rangle$. □

Concluding Comments

- Absent a uniqueness theorem the Koopmans operator may have many fixed points.
- We present reasons why the least fixed point should be singled out as the **principal fixed point** (Kantorovich).
- We give *economic* properties, *mathematical* features, and *theoretical* computational advantages belonging to the Least Fixed Point in support of our favoring U_∞ over U^∞.
- Of course, an adequate uniqueness theory would make this distinction among the possibly multiple fixed points an "academic exercise."
- However, we know from counterexamples that uniqueness cannot hold for all consumption sequences in the domain ℓ_∞^+, hence the prospect of spurious solutions to the Koopmans equation may be partially ameliorated by concentration on the Least Fixed Point, U_∞. Put differently, treat the LFP as a selection criteria to choose one solution from the possible ones in $\text{fix}(T_W)$.

Annex 1: Blackwell Literature Comments

- Both the Recovery and Uniqueness Problems require a domain of functions for the Koopmans operator to act on as a self-map.
- The class of aggregators for which this approach applies (under suitable restrictions) forms the *Blackwell aggregator class.*
- The TAS and KDW cases shown above are Blackwell aggregators.
- Becker and Boyd's (1997) book covers the theory up to their publication date and focus exclusively on the Blackwell cases.
- Recent work by Rincón-Zapatero and Rodriquez-Palmero, Martins-Da-Rocha and Vailakis, extends this work to a number of previously uncovered Blackwell aggregators using a range of *local contraction arguments.* Le Van and Vailakis examine partial sum methods for aggregators that are unbounded. Their examples include many Blackwell aggregators.

Annex 2: Thompson Aggregators: Defining Assumptions

Definition

$W : \mathbb{R}^2_+ \to \mathbb{R}$ is said to be a **Thompson aggregator** if it satisfies properties (T1) – (T4):

(T1) $W \geq 0$, continuous, and monotone: $(x, y) \leq (x', y')$ implies $W(x, y) \leq W(x', y')$;

(T2) $W(x, y) = y$ has at least one nonnegative solution for each $x \geq 0$;

(T3) $W(x, \bullet)$ is concave at 0 for each $x \geq 0$, that is

$$W(x, \mu y) \geq \mu W(x, y) + (1 - \mu) W(x, 0)$$

for each $\mu \in [0, 1]$ and each $(x, y) \in \mathbb{R}^2_+$;

(T4) $W(x, 0) > 0$ for each $x > 0$.

Additional MM Aggregator Assumptions

MM assume Thompson aggregators satisfy (T5) &(T6) below.

(T5) W is $\gamma-$ subhomogeneous — there is some $\gamma > 0$ such that:

$$W\left(\mu^{\gamma}x, \mu y\right) \geq \mu W\left(x, y\right)$$

for each $\mu \in (0, 1]$ and each $(x, y) \in \mathbb{R}^2_+$.

- Thompson KDW satisfies (T5) with $\gamma = b^{-1}$;
- Thompson CES aggregators satisfy (T5) with $\gamma = 1$.

MM also assume:

(T6) W satisfies the *MM-Limit Condition*: for each $\alpha \geq 1$ and $\gamma > 0$,

$$\lim_{t \to \infty} \frac{W(1,t)}{t} < \alpha^{-1/\gamma}, \tag{11}$$

with $t > 0$.

- The parameter α in $(T6)$ is the economy's maximum long-run possible consumption growth factor.
- We set $\alpha = 1$ (no long-run growth) today and note (T6) holds whenever the LHS of 11 is less than 1.
- Parameter γ is taken from (T5).
- KDW, CES and Quasi-Linear aggregators satisfy (T6).

Annex 3: The Commodity Space Setup

- Commodity Space: ℓ_∞^+ with sup norm with $\|C\|_\infty = \sup_t |c_t|$ whenever $C \in \ell_\infty$. The zero sequence is 0.
- Commodity spaces admitting exponential growth are admissible in the paper's more general setup. Focus on this special case of bounded growth in today's talk.
- Imagine "bounded" production possibilities – e.g. diminishing returns to capital & one-sector model.
- The commodity space ℓ_∞ is a Banach lattice with the usual pointwise partial order. The positive cone is ℓ_∞^+.
- It is also an *AM - space with unit* given by $e = (1, 1, \ldots)$. This fact implies: the sup norm interior of ℓ_∞^+ is non-empty. Denote the positive cone's interior by ℓ_∞^{++}; thus, ℓ_∞^+ is a *solid cone*. Clearly $e \in \ell_\infty^{++}$.

Annex 4: Industrial Strength Version of Blackwell's Theorem

The basic Blackwell aggregator theory in Becker and Boyd (1997) rests on the following mathematical theorem:

Monotone Contraction Mapping Theorem for Ideals: *Let A_ω be a principal Riesz ideal of the Riesz space* $\mathbf{E}$ *that is complete in the associated lattice norm. Suppose that $T : A_\omega \to \mathbf{E}$ obeys:*

1. $Tx \leq Ty$ *whenever* $x \leq y$;
2. $T\theta \in A_\omega$;
3. $T(x + \lambda\omega) \leq Tx + \lambda\delta\omega$ *with* $0 \leq \delta < 1$ and $\lambda > 0$.

Then T is a strict contraction and has a unique fixed point.

FOCUS TODAY: T is a monotone operator.

Fix the vector $\omega \in \mathbf{E}$ — an **order unit** in the norm interior of $\mathbf{E}$.
A_ω is the subset of $\mathbf{E}$ defined by:

$$x \in A_\omega \text{ iff } |x| \leq \lambda\omega \text{ for some scalar } \lambda \geq 0.$$

Here: $|x| = \sup(x, -x)$.
Lattice Norm: $|x| \leq y \Rightarrow \|x\| \leq \|y\|$ where
$\|x\| = \inf\{\lambda > 0 : |x| \leq \lambda\omega\}$ for $x \in E$ & $\omega > 0$ an order unit.
Example: Let $\mathbf{E}$ be the space of real sequences & $\omega = (1, 1, 1, \ldots)$. Then $A_\omega = \ell_\infty$.
Example: Suppose for $\alpha \geq 1$, $\omega = (\alpha, \alpha^2, \alpha^3, \ldots)$. A_ω contains sequences that grow exponentially.
Example: The space B introduced below is a Principal Ideal: The order unit is the weight function φ_γ.

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