Elections and Strategic Voting: Condorcet and Borda

E. Maskin
Harvard University

Indiana University
Bloomington
October 5, 2018
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    choose candidate preferred by majority to each other candidate
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  - choose candidate who maximizes sum of voters’ utilities
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decision problem for voters may be hard
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• Paper tries to answer question
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    \( \forall_X \) = set of strict utility functions
• profile \( U \). --- specification of each individual's utility function
- voting rule $F$
  
  for all profiles $U$, and all $Y \subseteq X$,
  
  $F(U., Y) \in Y$
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• so, correct definition:

  for generic profile $U$, and all $Y \subseteq X$
  \[ F(U, Y) \in Y \]
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\[ F^p(U, Y) = \{ a \mid \mu \{ i \mid U_i(a) \geq U_i(b) \text{ for all } b \} \geq \mu \{ i \mid U_i(a') \geq U_i(b) \text{ for all } b \} \text{ for all } a' \} \]
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where \( r_{U_i}(a) = \# \{ b \mid U_i(b) \geq U_i(a) \} \)
plurality rule:
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utilitarian principle:
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  - voters treated symmetrically
• *Neutrality* (N): Suppose $\rho : Y \rightarrow Y$ permutation.
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$$F\left(U_{i}^{\rho,Y}, Y\right) = \rho\left(F\left(U_{i}, Y\right)\right).$$
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• All four voting rules – plurality, majority, rank-order, utilitarian – satisfy P, A, N
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  • invoked by both Arrow (1951) and Nash (1950)
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– Nash formulation (rather than Arrow)
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  – Nash formulation (rather than Arrow)

  – no “spoilers” (e.g. Nader in 2000 U.S. presidential election, Le Pen in 2002 French presidential election)
• Majority rule and utilitarianism satisfy I, but others don’t:
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  – plurality rule

\[
\begin{array}{ccc}
.35 & .33 & .32 \\
x & y & z \\
y & z & y \\
z & x & x \\
\end{array}
\]

\[F^P (U, \{x, y, z\}) = x\]
\[F^P (U, \{x, y\}) = y\]
• Majority rule and utilitarianism satisfy I, but others don’t:
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  – rank-order voting

\[
\begin{array}{cc}
0.55 & 0.45 \\
x & y \\
y & z \\
z & x \\
\end{array}
\]

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- **Nonmanipulability (NM):**

  if \( x = F(U, Y) \) and \( x' = F(U', Y) \),

  where \( U'_j = U_j \) for all \( j \notin C \subseteq [0,1] \)

  then

  \( U_i(x) > U_i(x') \) for some \( i \in C \)
Final Axiom:

- **Nonmanipulability (NM):**

  if \( x = F(U, Y) \) and \( x' = F(U', Y) \),

  where \( U'_j = U_j \) for all \( j \notin C \subseteq [0,1] \)

  then

  \( U_i(x) > U_i(x') \) for some \( i \in C \)

  - the members of coalition \( C \) can’t all gain from misrepresenting

    utility functions as \( U'_i \)
• NM implies voting rule must be *ordinal* (no cardinal information used)
• NM implies voting rule must be \textit{ordinal} (no cardinal information used)

• \( F \) is \textit{ordinal} if whenever, for profiles \( U \) and \( U' \),
• NM implies voting rule must be *ordinal* (no cardinal information used)

• *F* is *ordinal* if whenever, for profiles *U* and *U*′, 
  
  \[ U_i(x) > U_i(y) \iff U_i'(x) > U_i'(y) \]
  
  for all *i*, *x*, *y*
• NM implies voting rule must be *ordinal* (no cardinal information used)

• *F* is *ordinal* if whenever, for profiles *U* and *U'*,
  \[ U_i(x) > U_i(y) \iff U'_i(x) > U'_i(y) \]  
  for all *i, x, y*

(*) \[ F(U_, Y) = F(U', Y) \]  
for all *Y*
• NM implies voting rule must be ordinal (no cardinal information used)

• $F$ is ordinal if whenever, for profiles $U_i$ and $U'_i$,
  $U_i(x) > U_i(y) \iff U'_i(x) > U'_i(y)$ for all $i, x, y$

(*) $F(U_i, Y) = F(U'_i, Y)$ for all $Y$

• Lemma: If $F$ satisfies NM, $F$ ordinal
• NM implies voting rule must be *ordinal* (no cardinal information used)

• *F* is *ordinal* if whenever, for profiles *U_* and *U'*,

\[ U_i(x) > U_i(y) \iff U'_i(x) > U'_i(y) \text{ for all } i, x, y \]

(*) \[ F(U, Y) = F(U', Y) \text{ for all } Y \]

• **Lemma:** If *F* satisfies NM, *F* ordinal

• NM rules out utilitarianism
But majority rule also violates NM
But majority rule also violates NM

- $F^C$ not even always *defined*

\[
\begin{array}{ccc}
0.35 & 0.33 & 0.32 \\
x & y & z \\
y & z & x \\
z & x & y \\
\end{array}
\]

$F^C(U, \{x, y, z\}) = \emptyset$
But majority rule also violates NM

- $F^C$ not even always *defined*

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$$F^C(U,\{x, y, z\}) = \emptyset$$

- example of *Condorcet cycle*
But majority rule also violates NM

- $F^C$ not even always *defined*

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$F^C(U, \{x, y, z\}) = \emptyset$

- example of *Condorcet cycle*
- $F^C$ must be extended to Condorcet cycles
But majority rule also violates NM

- $F^C$ not even always defined

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\[ F^C \left(U, \{x, y, z\}\right) = \emptyset \]

- example of Condorcet cycle
- $F^C$ must be extended to Condorcet cycles
- one possibility

\[
F^{C/B}(U,Y) = \begin{cases} 
F^C(U,Y), & \text{if nonempty} \\
F^B(U,Y), & \text{otherwise} 
\end{cases} \quad \text{(Black's method)}
\]
But majority rule also violates NM

- $F^C$ not even always defined

\[
\begin{array}{ccc}
0.35 & 0.33 & 0.32 \\
x & y & z \\
y & z & x \\
z & x & y \\
\end{array}
\]

$F^C(U, \{x, y, z\}) = \emptyset$

- example of Condorcet cycle
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F^{C/B}(U, Y) = \begin{cases} 
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F^B(U, Y), & \text{otherwise}
\end{cases}
\]

(Black's method)

- extensions make $F^C$ vulnerable to manipulation
But majority rule also violates NM

- $F^C$ not even always defined

\[
\begin{array}{c|c|c}
.35 & .33 & .32 \\
\hline
x & y & z \\
y & z & x \\
z & x & y
\end{array}
\]

$F^C (U, \{x, y, z\}) = \emptyset$

- example of Condorcet cycle
- $F^C$ must be extended to Condorcet cycles
- one possibility

\[
F^{C/B} (U, Y) = \begin{cases} 
F^C (U, Y), \text{ if nonempty} \\
F^B (U, Y), \text{ otherwise}
\end{cases}
\] (Black's method)

- extensions make $F^C$ vulnerable to manipulation

\[
\begin{array}{c|c|c}
.35 & .33 & .32 \\
\hline
x & y & z \\
y & z & x \\
z & x & y
\end{array}
\]

$F^{C/B} (U, \{x, y, z\}) = x$

\[
\begin{array}{c}
z \\
y \\
x
\end{array}
\]

$F^{C/B} (U', \{x, y, z\}) = z$
Theorem: There exists no voting rule satisfying P, A, N, I and NM
Theorem: There exists no voting rule satisfying P, A, N, I and NM

Proof: similar to that of GS
Theorem: There exists no voting rule satisfying P, A, N, I and NM

Proof: similar to that of GS

overly pessimistic -- many cases in which some rankings unlikely
Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles
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When can we rule out Condorcet cycles?
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When can we rule out Condorcet cycles?

• preferences single-peaked

2000 US election

Nader  Gore  Bush
**Lemma:** Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles.

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election

- unlikely that many had ranking
  - Bush or Nader
  - Nader or Bush
  - Gore or Gore
**Lemma**: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles.

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election

```
Nader ─────── Gore ─────── Bush
```

unlikely that many had ranking

```
Bush or Nader
Nader or Bush
Gore or Gore
```

- strongly-felt candidate
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```
Nader       Gore       Bush
```

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```
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Nader or Bush
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```

- strongly-felt candidate
  - in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen
Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

• preferences single-peaked

2000 US election

\[
\begin{array}{ccc}
Nader & \text{Gore} & \text{Bush} \\
\end{array}
\]

unlikely that many had ranking Bush or Nader
Nader or Bush
Gore or Gore

• strongly-felt candidate
  – in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen
  – voters didn’t feel strongly about Chirac and Jospin
Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

• preferences single-peaked

2000 US election

unlikely that many had ranking Bush Nader or Nader Bush Gore

• strongly-felt candidate
  – in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen
  – voters didn’t feel strongly about Chirac and Jospin
  – felt strongly about Le Pen (ranked him first or last)
• Voting rule $F$ works well on domain $\mathcal{U}$ if satisfies P,A,N,I,NM when utility functions restricted to $\mathcal{U}$
• Voting rule $F$ works well on domain $\mathcal{U}$ if satisfies $P,A,N,I,NM$ when utility functions restricted to $\mathcal{U}$

  – e.g., $F^C$ works well when preferences single-peaked
• *Theorem 1*: Suppose $F$ works well on domain $\mathcal{U}$, then $F^C$ works well on $\mathcal{U}$ too.
• **Theorem 1**: Suppose $F$ works well on domain $\mathcal{U}$, then $F^c$ works well on $\mathcal{U}$ too.

• Conversely, suppose that $F^c$ works well on $\mathcal{U}^c$. 
• **Theorem 1**: Suppose $F$ works well on domain $\mathcal{U}$, then $F^C$ works well on $\mathcal{U}$ too.

• Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$.

Then if there exists profile $U^*$ on $\mathcal{U}^C$ such that...
• **Theorem 1**: Suppose $F$ works well on domain $\mathcal{U}$, then $F^C$ works well on $\mathcal{U}$ too. Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$.

Then if there exists profile $U^\circ$ on $\mathcal{U}^C$ such that

$$F(U^\circ, Y) \neq F^C(U^\circ, Y)$$

for some $Y$. 

• Theorem 1: Suppose $F$ works well on domain $\mathcal{U}$, then $F^C$ works well on $\mathcal{U}$ too. Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$.

Then if there exists profile $U^\circ$ on $\mathcal{U}^C$ such that

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there exists domain $\mathcal{U}'$ on which $F^C$ works well but $F$ does not
• Theorem 1: Suppose $F$ works well on domain $\mathcal{U}$, then $F^C$ works well on $\mathcal{U}$ too.
• Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$.

Then if there exists profile $U^\circ$ on $\mathcal{U}^C$ such that

$$F(U^\circ, Y) \neq F^C(U^\circ, Y)$$

for some $Y$,

there exists domain $\mathcal{U}'$ on which $F^C$ works well but $F$ does not

Proof: From NM and I, if $F$ works well on $\mathcal{U}$, $F$ must be ordinal
Theorem 1: Suppose \( F \) works well on domain \( \mathcal{U} \), then \( F^C \) works well on \( \mathcal{U} \) too.

Conversely, suppose that \( F^C \) works well on \( \mathcal{U}^C \).

Then if there exists profile \( U^\circ \) on \( \mathcal{U}^C \) such that
\[
F\left(U^\circ, Y\right) \neq F^C\left(U^\circ, Y\right)
\]
for some \( Y \),

there exists domain \( \mathcal{U}' \) on which \( F^C \) works well but \( F \) does not

**Proof:** From NM and I, if \( F \) works well on \( \mathcal{U} \), \( F \) must be ordinal

Hence result follows from

Dasgupta-Maskin (2008), *JEEA*
• **Theorem 1**: Suppose $F$ works well on domain $\mathcal{U}$, then $F^C$ works well on $\mathcal{U}$ too.

• Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$.

Then if there exists profile $U^\circ$ on $\mathcal{U}^C$ such that

$$F(U^\circ, Y) \neq F^C(U^\circ, Y)$$ for some $Y$,

there exists domain $\mathcal{U}'$ on which $F^C$ works well but $F$ does not

**Proof**: From NM and I, if $F$ works well on $\mathcal{U}$, $F$ must be ordinal

• Hence result follows from

  Dasgupta-Maskin (2008), *JEEA*

  – shows that Theorem 1 holds when NM replaced by ordinality
To show this D-M uses
To show this D-M uses

Lemma: $F^C$ works well on $\not\exists$ if and only if $\not\exists$ has no Condorcet cycles
To show this D-M uses

Lemma: $F^C$ works well on $\mathcal{U}$ if and only if $\mathcal{U}$ has no Condorcet cycles

- Suppose $F$ works well on $\mathcal{U}$
To show this D-M uses

Lemma: $F^C$ works well on $U$ if and only if $U$ has no Condorcet cycles

• Suppose $F$ works well on $U$

• If $F^C$ doesn't work well on $U$, Lemma implies $U$ must contain Condorcet cycle $x\ y\ z$
  
  $\ y\ z\ x$
  
  $z\ x\ y$
• Consider
Consider

\[
\begin{bmatrix}
1 & 2 & \ldots & n \\
\end{bmatrix}
\]

\[
U_1^1 = \begin{bmatrix}
x & z & z \\
z & x & x \\
z & x & x \\
\end{bmatrix}
\]
• Consider

\[
U_1 = \begin{array}{ccc}
1 & 2 & \ldots & n \\
 x & z & z \\
 z & x & x \\
\end{array}
\]

(*) Suppose \( F(U_1, \{x, z\}) = z \)
• Consider

\[
U^1 = \begin{array}{ccc}
1 & 2 & \ldots & n \\
\downarrow & \downarrow &{}& \downarrow \\
x & z & z \\
z & x & x \\
\end{array}
\]

(*) Suppose \( F(U^1, \{x, z\}) = z \)

• \( U^2 = \begin{array}{cccc}
1 & 2 & 3 & n \\
\downarrow & \downarrow & \downarrow & \downarrow \\
x & y & z & z \\
y & z & x & x \\
z & x & y & y \\
\end{array} \)
• Consider

\[
U_1 = \begin{array}{ccc}
1 & 2 & \ldots n \\
x & z & z \\
z & x & x \\
\end{array}
\]

(*) Suppose \( F(U_1, \{x, z\}) = z \)

• \( U_2 = \begin{array}{ccc}
1 & 2 & 3 & n \\
x & y & z & z \\
y & z & x & x \\
z & x & y & y \\
\end{array} \)

\[
F(U_2, \{x, y, z\}) = x \quad \Rightarrow \quad (\text{from I}) \quad F(U_2, \{x, z\}) = x, \text{ contradicts (*)}
\]
• Consider

\[
U_1^1 = \begin{array}{cccc}
1 & 2 & \ldots & n \\
x & z & z \\
\end{array}
\]

\[
\begin{array}{cccc}
z & x & x \\
\end{array}
\]

(*) Suppose \( F(U_1^1, \{x, z\}) = z \)

\cdot \quad U_2^1 = \begin{array}{cccc}
1 & 2 & \ldots & n \\
x & y & z & z \\
\end{array}
\]

\[
\begin{array}{cccc}
y & z & x & x \\
z & x & y & y \\
\end{array}
\]

\[
F(U_2^1, \{x, y, z\}) = x \quad \Rightarrow \text{(from I)} \quad F(U_2^1, \{x, z\}) = x, \text{ contradicts (*)}
\]

\[
F(U_2^1, \{x, y, z\}) = y \quad \Rightarrow \text{(from I)} \quad F(U_2^1, \{x, y\}) = y, \text{ contradicts (*) (A,N)}
\]
Consider

\[
\begin{array}{cccc}
1 & 2 & \ldots & n \\
\ U^1_1 = & x & z & z \\
& z & x & x
\end{array}
\]

(*) Suppose \( F(U^1_1, \{x, z\}) = z \)

\[
\begin{array}{cccc}
1 & 2 & \ldots & n \\
\ U^2_1 = & x & y & z \\
& z & x & x \\
& y & z & x \\
& z & x & y \\
\end{array}
\]

\[
F(U^2_1, \{x, y, z\}) = x \implies \text{(from I)} F(U^2_1, \{x, z\}) = x, \text{ contradicts (*)}
\]

\[
F(U^2_1, \{x, y, z\}) = y \implies \text{(from I)} F(U^2_1, \{x, y\}) = y, \text{ contradicts (*) (A,N)}
\]

so
Consider
\[ U^1_1 = \begin{array}{ccc} 1 & 2 & \ldots n \\ x & z & z \\ z & x & x \end{array} \]

(*) Suppose \( F\left(U^1_1, \{x,z\}\right) = z \)

\[ U^2 = \begin{array}{ccc} 1 & 2 & 3 & \ldots n \\ x & y & z & \vdots \\ y & z & x & x \\ z & x & y & y \end{array} \]

\( F\left(U^2_1, \{x,y,z\}\right) = x \) \( \Rightarrow \) (from I) \( F\left(U^2_1, \{x,z\}\right) = x \), contradicts (*)

\( F\left(U^2_1, \{x,y,z\}\right) = y \) \( \Rightarrow \) (from I) \( F\left(U^2_1, \{x,y\}\right) = y \), contradicts (*) (A,N)

so

\( F\left(U^2_1, \{x,y,z\}\right) = z \)
• Consider

\[
U_1 = \begin{array}{ccc}
1 & 2 & \ldots \ n \\
1 & 2 & \ldots \ n \\
\end{array} \\
\begin{array}{ccc}
x & z & z \\
z & x & x \\
\end{array}
\]

(*) Suppose \( F\left( U_1, \{x, z\} \right) = z \)

• \( U_2 = \begin{array}{cccc}
1 & 2 & 3 & \ldots \ n \\
1 & 2 & 3 & \ldots \ n \\
\end{array} \\
\begin{array}{cccc}
x & y & z & z \\
y & z & x & x \\
z & x & y & y \\
\end{array}
\]

\[
F\left( U_2, \{x, y, z\} \right) = x \quad \Rightarrow \text{(from I) } F\left( U_2, \{x, z\} \right) = x, \text{ contradicts (*)}
\]

\[
F\left( U_2, \{x, y, z\} \right) = y \quad \Rightarrow \text{(from I) } F\left( U_2, \{x, y\} \right) = y, \text{ contradicts (*) (A,N)}
\]

so

\[
F\left( U_2, \{x, y, z\} \right) = z
\]

• so \( F\left( U_2, \{y, z\} \right) = z \quad \text{(I)} \)
Consider

\[ U_1 = \begin{array}{ccc}
1 & 2 & \cdots n \\
x & z & z \\
z & x & x
\end{array} \]

\( (\ast) \) Suppose \( F\left(U_1, \{x, z\}\right) = z \)

\[ U_2 = \begin{array}{cccc}
1 & 2 & 3 & n \\
x & y & z & z \\
y & z & x & x \\
z & x & y & y
\end{array} \]

\[ F\left(U_2, \{x, y, z\}\right) = x \implies (\text{from I}) F\left(U_2, \{x, z\}\right) = x, \text{ contradicts } (\ast) \]

\[ F\left(U_2, \{x, y, z\}\right) = y \implies (\text{from I}) F\left(U_2, \{x, y\}\right) = y, \text{ contradicts } (\ast) \quad (A, N) \]

so

\[ F\left(U_2, \{x, y, z\}\right) = z \]

so \( F\left(U_2, \{x, y\}\right) = z \quad (I) \)

so for
Consider

\[ U^1 = \begin{array}{cccc}
1 & 2 & \ldots & n \\
x & z & z \\
z & x & x \\
\end{array} \]

so

\[ U^1 = \begin{array}{cccc}
1 & 2 & \ldots & n \\
x & z & z \\
z & x & x \\
\end{array} \]

\( (*) \) Suppose \( F(U^1, \{x, z\}) = z \)

\[ U^2 = \begin{array}{cccc}
1 & 2 & 3 & n \\
x & y & z & z \\
y & z & x & x \\
z & x & y & y \\
\end{array} \]

\[ F(U^2, \{x, y, z\}) = x \implies (\text{from I}) \ F(U^2, \{x, z\}) = x, \text{ contradicts (}) \]

\[ F(U^2, \{x, y, z\}) = y \implies (\text{from I}) \ F(U^2, \{x, y\}) = y, \text{ contradicts (}) \ (A,N) \]

so

\[ F(U^2, \{x, y, z\}) = z \]

\[ \text{so} F(U^2, \{y, z\}) = z \quad (\text{I}) \]

\[ \text{so for} \]

\[ U^3 = \begin{array}{cccc}
1 & 2 & 3 & \ldots & n \\
x & x & z & z \\
z & z & x & x \\
\end{array} \]
• Consider 

\[ U^1 = \begin{array}{ccc}
1 & 2 & \ldots \ n \\
\frac{1}{2} & \frac{3}{2} & \ldots \frac{n}{2} \\
\end{array} \]

\[ = x \quad z \quad z \\
\frac{z}{2} \quad \frac{x}{2} \quad \frac{x}{2} \]

\( (*) \) Suppose \( F\left(U^1, \{x, z\}\right) = z \)

• \[ U^2 = \begin{array}{cccc}
1 & 2 & 3 & \ldots n \\
x & y & z & z \\
y & z & x & x \\
z & x & y & y \\
\end{array} \]

\[ F\left(U^2, \{x, y, z\}\right) = x \Rightarrow (\text{from I}) \ F\left(U^2, \{x, z\}\right) = x, \text{ contradicts (*)} \]

\[ F\left(U^2, \{x, y, z\}\right) = y \Rightarrow (\text{from I}) \ F\left(U^2, \{x, y\}\right) = y, \text{ contradicts (*) (A,N)} \]

so \[ F\left(U^2, \{x, y, z\}\right) = z \]

• so \( F\left(U^2, \{y, z\}\right) = z \) (I)

• so for

\[ U^3 = \begin{array}{cccc}
1 & 2 & 3 & \ldots n \\
x & x & z & z \\
\frac{z}{2} & \frac{z}{2} & \frac{x}{2} \quad \frac{x}{2} \]

\[ F\left(U^3, \{x, z\}\right) = z \text{ (N)} \]
Consider
\[
\begin{array}{llll}
1 & 2 & \cdots & n \\
U^1_1 &=& x & z \\
& &=& z \\
& &=& x \\
\end{array}
\]

\[
\begin{array}{llll}
1 & 2 & 3 & n \\
U^2_1 &=& x & y \\
& &=& z & z \\
& &=& x & x \\
& &=& z & y \\
\end{array}
\]

\[
\begin{array}{llll}
1 & 2 & 3 & \cdots n \\
U^3_1 &=& x & x \\
& &=& z & z \\
& &=& x & x \\
\end{array}
\]

\[
\begin{array}{llll}
1 & \cdots & n-1 & n \\
U^4_1 &=& x & x \\
& &=& z & z \\
& &=& x & x \\
\end{array}
\]

(*) Suppose \( F\left(U^1_1, \{x, z\}\right) = z \)

\[
\begin{array}{llll}
1 & 2 & 3 & n \\
U^2_1 &=& x & y \\
& &=& z & z \\
& &=& x & x \\
& &=& z & y \\
\end{array}
\]

\[
\begin{array}{llll}
1 & 2 & 3 & \cdots n \\
U^3_1 &=& x & x \\
& &=& z & z \\
& &=& x & x \\
\end{array}
\]

\[
\begin{array}{llll}
1 & \cdots & n-1 & n \\
U^4_1 &=& x & x \\
& &=& z & z \\
& &=& x & x \\
\end{array}
\]

Continuing in the same way, let \( U^4_1 = \frac{1 \cdots n-1}{x} \frac{n}{z} \frac{z}{z} \frac{x}{x} \)
Consider
\[
\begin{array}{cccc}
1 & 2 & \ldots & n \\
U^1 & = & x & z \\
z & & x & x \\
\end{array}
\]

(*) Suppose \( F(U^1, \{x, z\}) = z \)

\[
\begin{array}{cccc}
1 & 2 & 3 & n \\
U^2 & = & x & y \\
y & z & x & x \\
z & x & y & y \\
\end{array}
\]

\( F(U^2, \{x, y, z\}) = x \) \( \Rightarrow \) (from I) \( F(U^2, \{x, z\}) = x \), contradicts (*)

\( F(U^2, \{x, y, z\}) = y \) \( \Rightarrow \) (from I) \( F(U^2, \{x, y\}) = y \), contradicts (*) (A,N)

so \( F(U^2, \{x, y, z\}) = z \)

so \( F(U^2, \{y, z\}) = z \) \( \) (I)

so for
\[
\begin{array}{cccc}
1 & 2 & 3 & \ldots & n \\
U^3 & = & x & x \\
x & z & z & z \\
z & z & x & x \\
\end{array}
\]

\( F(U^3, \{x, z\}) = z \) \( \) (N)

Continuing in the same way, let \( U^4 = \)
\[
\begin{array}{cccc}
1 & \ldots & n-1 & n \\
x & x & z & z \\
z & z & x & x \\
\end{array}
\]

\( F(U^4, \{x, z\}) = z \), contradicts (*)
• So $F$ can’t work well on $\mathcal{U}$ with Condorcet cycle
• So $F$ can’t work well on $\mathcal{U}$ with Condorcet cycle

• Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$ and
• So $F$ can’t work well on $\mathcal{U}$ with Condorcet cycle

• Conversely, suppose that $F^c$ works well on $\mathcal{U}^c$ and

$$F\left(U^\circ, Y\right) \neq F^c\left(U^\circ, Y\right)$$

for some $U^\circ$ and $Y$
• So $F$ can’t work well on $\mathcal{U}$ with Condorcet cycle

• Conversely, suppose that $F^c$ works well on $\mathcal{U}^c$ and

$$F(U^*, Y) \neq F^c(U^*, Y)$$

for some $U^*$ and $Y$

• Then there exist $\alpha$ with $1 - \alpha > \alpha$ and
• So $F$ can’t work well on $\mathcal{U}$ with Condorcet cycle

• Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$ and

$$F(U^\circ, Y) \neq F^C(U^\circ, Y)$$

for some $U^\circ$ and $Y$

• Then there exist $\alpha$ with $1 - \alpha > \alpha$ and

$$U^\circ = \frac{1 - \alpha}{x} \begin{pmatrix} \alpha \\ x \\ y \\ x \end{pmatrix}$$
• So $F$ can’t work well on $\mathcal{U}$ with Condorcet cycle

• Conversely, suppose that $F^c$ works well on $\mathcal{U}^c$ and

$$F(U^\circ, Y) \neq F^c(U^\circ, Y)$$ for some $U^\circ$ and $Y$

• Then there exist $\alpha$ with $1 - \alpha > \alpha$ and

$$U^\circ = \frac{1-\alpha}{x} \frac{\alpha}{y}$$

such that
• So \( F \) can’t work well on \( \mathcal{U} \) with Condorcet cycle

• Conversely, suppose that \( F^C \) works well on \( \mathcal{U}^C \) and

\[
F\left(U^\circ, Y\right) \neq F^C\left(U^\circ, Y\right)
\]

for some \( U^\circ \) and \( Y \)

• Then there exist \( \alpha \) with \( 1 - \alpha > \alpha \) and

\[
U^\circ = \begin{pmatrix}
1 - \alpha \\
\alpha
\end{pmatrix}
\]

such that

\[
x = F^C\left(U^\circ, \{x, y\}\right) \quad \text{and} \quad y = F\left(U^\circ, \{x, y\}\right)
\]
• So $F$ can’t work well on $\mathcal{U}$ with Condorcet cycle

• Conversely, suppose that $F^C$ works well on $\mathcal{U}^C$ and

$$F(U^\circ, Y) \neq F^C(U^\circ, Y)$$

for some $U^\circ$ and $Y$

• Then there exist $\alpha$ with $1 - \alpha > \alpha$ and

$$U^\circ = \frac{1 - \alpha}{x} \frac{\alpha}{y} = \frac{\alpha}{x} \frac{1 - \alpha}{y}$$

such that

$$x = F^C(U^\circ, \{x, y\})$$

and

$$y = F(U^\circ, \{x, y\})$$

• But not hard to show that $F^C$ unique voting rule satisfying P, A, N, and NM when $|X| = 2$ - - contradiction
• Let’s drop I
• Let’s drop I
  – most controversial
• Let’s drop I
  – most controversial

• *no* voting rule satisfies P,A,N,NM on $\forall_X$
• Let’s drop I
  – most controversial

• *no* voting rule satisfies P,A,N,NM on $\forall_X$
  – GS again
• Let’s drop I
  – most controversial

• *no* voting rule satisfies P, A, N, NM on $\mathcal{X}$
  – GS again

• *F works nicely* on $\mathcal{Y}$ if satisfies P, A, N, NM on $\mathcal{Y}$
Theorem 2:
Theorem 2:

- Suppose $F$ works nicely on $\mathcal{H}$, then $F^C$ or $F^B$ works nicely on $\mathcal{H}$ too.
Theorem 2:

• Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

• Conversely suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.  

Theorem 2:

- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

- Conversely, suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.

Then, if there exists profile $U^{\infty}$ on $\mathcal{U}^*$ such that
Theorem 2:

- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

- Conversely suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.
  
  Then, if there exists profile $U^\infty$ on $\mathcal{U}^*$ such that
  
  $$F(U^\infty, Y) \neq F^*(U^\infty, Y)$$
  
  for some $Y$,
Theorem 2:

- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

- Conversely, suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.

  Then, if there exists profile $U^\circ$ on $\mathcal{U}^*$ such that

  $$F\left(U^\circ, Y\right) \neq F^*\left(U^\circ, Y\right)$$

  for some $Y$,

  there exists domain $\mathcal{U}'$ on which $F^*$ works nicely but $F$ does not.
Theorem 2:

- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

- Conversely suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.
  
  Then, if there exists profile $U^*$ on $\mathcal{U}^*$ such that
  
  $$F(U^*, Y) \neq F^*(U^*, Y)$$
  
  for some $Y$,
  
  there exists domain $\mathcal{U}'$ on which $F^*$ works nicely but $F$ does not

Proof:
Theorem 2:
- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.
- Conversely, suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.
  Then, if there exists profile $U^*$ on $\mathcal{U}^*$ such that
  \[ F(U^*, Y) \neq F^*(U^*, Y) \] for some $Y$,
  there exists domain $\mathcal{U}'$ on which $F^*$ works nicely but $F$ does not work.

Proof:
- $F^C$ works nicely on any Condorcet-cycle-free domain
Theorem 2:

- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

- Conversely suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.

Then, if there exists profile $U^\infty$ on $\mathcal{U}^*$ such that

$$F(U^\infty, Y) \neq F^*(U^\infty, Y)$$

for some $Y$,

there exists domain $\mathcal{U}'$ on which $F^*$ works nicely but $F$ does not work nicely only when $\mathcal{U}'$ is subset of Condorcet cycle

Proof:

- $F^C$ works nicely on any Condorcet-cycle-free domain

- $F^B$ works nicely only when $\mathcal{U}$ is subset of Condorcet cycle
Theorem 2:
• Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

• Conversely suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.

Then, if there exists profile $U^\infty$ on $\mathcal{U}^*$ such that

$$F\left(U^\infty, Y\right) \neq F^*\left(U^\infty, Y\right)$$

for some $Y$,

there exists domain $\mathcal{U}'$ on which $F^*$ works nicely but $F$ does not work nicely.

Proof:
• $F^C$ works nicely on any Condorcet-cycle-free domain
• $F^B$ works nicely only when $\mathcal{U}$ is subset of Condorcet cycle

so $F^C$ and $F^B$ complement each other
Theorem 2:

- Suppose \( F \) works nicely on \( \forall \), then \( F^C \) or \( F^B \) works nicely on \( \forall \) too.

- Conversely suppose \( F^* \) works nicely on \( \forall^* \), where \( F^* = F^C \) or \( F^B \).

  Then, if there exists profile \( U^\infty \) on \( \forall^* \) such that

  \[
  F\left(U^\infty, Y\right) \neq F^*\left(U^\infty, Y\right)
  \]

  for some \( Y \), there exists domain \( \forall' \) on which \( F^* \) works nicely but \( F \) does not

Proof:

- \( F^C \) works nicely on any Condorcet-cycle-free domain

- \( F^B \) works nicely only when \( \forall \) is subset of Condorcet cycle

- so \( F^C \) and \( F^B \) complement each other
  - if \( F \) works nicely on \( \forall \) and \( \forall \) doesn't contain Condorcet cycle, \( F^C \) works nicely too
Theorem 2:

- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

- Conversely, suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.

  Then, if there exists profile $U^*$ on $\mathcal{U}^*$ such that
  \[ F(U^*, Y) \neq F^*(U^*, Y) \]
  for some $Y$,
  there exists domain $\mathcal{U}'$ on which $F^*$ works nicely but $F$ does not work nicely on $\mathcal{U}'$.

Proof:

- $F^C$ works nicely on any Condorcet-cycle-free domain.

- $F^B$ works nicely only when $\mathcal{U}$ is subset of Condorcet cycle.

- so $F^C$ and $F^B$ complement each other
  - if $F$ works nicely on $\mathcal{U}$ and $\mathcal{U}$ doesn't contain Condorcet cycle, $F^C$ works nicely too.
  - if $F$ works nicely on $\mathcal{U}$ and $\mathcal{U}$ contains Condorcet cycle, then $\mathcal{U}$ can't contain any other ranking (otherwise no voting rule works nicely).
Theorem 2:

- Suppose $F$ works nicely on $\mathcal{U}$, then $F^C$ or $F^B$ works nicely on $\mathcal{U}$ too.

- Conversely suppose $F^*$ works nicely on $\mathcal{U}^*$, where $F^* = F^C$ or $F^B$.

Then, if there exists profile $U^\infty$ on $\mathcal{U}^*$ such that

$$F(U^\infty, Y) \neq F^*(U^\infty, Y)$$

for some $Y$,

there exists domain $\mathcal{U}'$ on which $F^*$ works nicely but $F$ does not

Proof:

- $F^C$ works nicely on any Condorcet-cycle-free domain

- $F^B$ works nicely only when $\mathcal{U}$ is subset of Condorcet cycle

- so $F^C$ and $F^B$ complement each other

  - if $F$ works nicely on $\mathcal{U}$ and $\mathcal{U}$ doesn't contain Condorcet cycle, $F^C$ works nicely too

  - if $F$ works nicely on $\mathcal{U}$ and $\mathcal{U}$ contains Condorcet cycle, then $\mathcal{U}$ can't contain any other ranking (otherwise no voting rule works nicely)

  - so $F^B$ works nicely on $\mathcal{U}$. 

Striking that the 2 longest-studied voting rules (Condorcet and Borda) are also
Striking that the 2 longest-studied voting rules (Condorcet and Borda) are also
• *only two* that work nicely on maximal domains