Elections and Strategic Voting: Condorcet and Borda

E. Maskin
Harvard University

Indiana University
Bloomington
October 5, 2018

- voting rule

- voting rule

  method for choosing winning candidate on basis of voters’ preferences (rankings, utility functions)

- voting rule

  method for choosing winning candidate on basis of voters' preferences (rankings, utility functions)

- prominent examples

- voting rule

  method for choosing winning candidate on basis of voters' preferences (rankings, utility functions)

- prominent examples
  - Plurality Rule (MPs in Britain, members of Congress in U.S.)

- voting rule

  method for choosing winning candidate on basis of voters' preferences (rankings, utility functions)

- prominent examples
  - Plurality Rule (MPs in Britain, members of Congress in U.S.)

    choose candidate ranked first by more voters than any other

- voting rule

  method for choosing winning candidate on basis of voters' preferences (rankings, utility functions)

- prominent examples
  - Plurality Rule (MPs in Britain, members of Congress in U.S.)

    choose candidate ranked first by more voters than any other
  - Majority Rule (Condorcet Method)

- voting rule

  method for choosing winning candidate on basis of voters' preferences (rankings, utility functions)

- prominent examples
  - Plurality Rule (MPs in Britain, members of Congress in U.S.)

    choose candidate ranked first by more voters than any other
  - Majority Rule (Condorcet Method)

    choose candidate preferred by majority to each other candidate

– Run-off Voting (presidential elections in France)

– Run-off Voting (presidential elections in France)
  - choose candidate ranked first by more voters than any other, unless number of first-place rankings

    less than majority

- Run-off Voting (presidential elections in France)
  - choose candidate ranked first by more voters than any other, unless number of first-place rankings

    less than majority

      among top 2 candidates, choose alternative preferred by majority

- Run-off Voting (presidential elections in France)
  - choose candidate ranked first by more voters than any other, unless number of first-place rankings

    less than majority

    among top 2 candidates, choose alternative preferred by majority
- Rank-Order Voting (Borda Count)

- Run-off Voting (presidential elections in France)
  - choose candidate ranked first by more voters than any other, unless number of first-place rankings

    less than majority

      among top 2 candidates, choose alternative preferred by majority
- Rank-Order Voting (Borda Count)
  - candidate assigned 1 point every time some voter ranks her first, 2 points every time ranked second, etc.

– Run-off Voting (presidential elections in France)
  - choose candidate ranked first by more voters than any other, unless number of first-place rankings

    less than majority

    among top 2 candidates, choose alternative preferred by majority

– Rank-Order Voting (Borda Count)
  - candidate assigned 1 point every time some voter ranks her first, 2 points every time ranked second, etc.
  - choose candidate with lowest point total

- Run-off Voting (presidential elections in France)
  - choose candidate ranked first by more voters than any other, unless number of first-place rankings

    less than majority

      among top 2 candidates, choose alternative preferred by majority
- Rank-Order Voting (Borda Count)
  - candidate assigned 1 point every time some voter ranks her first, 2 points every time ranked second, etc.
  - choose candidate with lowest point total
- Utilitarian Principle

- Run-off Voting (presidential elections in France)
  - choose candidate ranked first by more voters than any other, unless number of first-place rankings

    less than majority

    among top 2 candidates, choose alternative preferred by majority
- Rank-Order Voting (Borda Count)
  - candidate assigned 1 point every time some voter ranks her first, 2 points every time ranked second, etc.
  - choose candidate with lowest point total
- Utilitarian Principle
  - choose candidate who maximizes sum of voters' utilities

- Which voting rule to adopt?

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy
  - see which rules best satisfy them

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy
  - see which rules best satisfy them
- One important criterion: *nonmanipulability*

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy
  - see which rules best satisfy them
- One important criterion: *nonmanipulability*
  - voters shouldn't have incentive to misrepresent preferences, i.e., vote *strategically*

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy
  - see which rules best satisfy them
- One important criterion: *nonmanipulability*
  - voters shouldn't have incentive to misrepresent preferences, i.e., vote *strategically*
  - otherwise

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy
  - see which rules best satisfy them
- One important criterion: *nonmanipulability*
  - voters shouldn't have incentive to misrepresent preferences, i.e., vote *strategically*
  - otherwise

    not implementing intended voting rule

- Which voting rule to adopt?
- Answer depends on what one wants in voting rule
  - can specify *criteria* (axioms) voting rule should satisfy
  - see which rules best satisfy them
- One important criterion: *nonmanipulability*
  - voters shouldn't have incentive to misrepresent preferences, i.e., vote *strategically*
  - otherwise

    not implementing intended voting rule

    decision problem for voters may be hard

- But basic negative result
  Gibbard-Satterthwaite (GS) theorem

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  – if 3 or more candidates, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  – if 3 or more candidates, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)

- Still, GS overly pessimistic

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  – if 3 or more candidates, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)

- Still, GS overly pessimistic
  – requires that voting rule *never* be manipulable

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  – if 3 or more candidates, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)

- Still, GS overly pessimistic
  – requires that voting rule *never* be manipulable
  – but some circumstances where manipulation can occur may be unlikely

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  – if 3 or more candidates, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)

- Still, GS overly pessimistic
  – requires that voting rule *never* be manipulable
  – but some circumstances where manipulation can occur may be unlikely

- In any case, natural question:

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  – if 3 or more candidates, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)

- Still, GS overly pessimistic
  – requires that voting rule *never* be manipulable
  – but some circumstances where manipulation can occur may be unlikely

- In any case, natural question:

  Which (reasonable) voting rule(s) nonmanipulable *most often*?

- But basic negative result

  Gibbard-Satterthwaite (GS) theorem

  – if 3 or more candidates, *no* voting rule is always nonmanipulable

    (except for dictatorial rules - - where one voter has all the power)

- Still, GS overly pessimistic
  – requires that voting rule *never* be manipulable
  – but some circumstances where manipulation can occur may be unlikely

- In any case, natural question:

  Which (reasonable) voting rule(s) nonmanipulable *most often*?

- Paper tries to answer question

- X = finite set of candidates

- X = finite set of candidates
- society consists of a continuum of voters [0,1]

- X = finite set of candidates
- society consists of a continuum of voters [0,1]
  - typical voter $i \in [0,1]$

- X = finite set of candidates
- society consists of a continuum of voters [0,1]
  - typical voter $i \in [0,1]$
  - reason for continuum clear soon

- X = finite set of candidates
- society consists of a continuum of voters [0,1]
  - typical voter $i \in [0,1]$
  - reason for continuum clear soon
- utility function for voter i $U_i : X \to \mathbb{R}$

- X = finite set of candidates
- society consists of a continuum of voters [0,1]
  - typical voter $i \in [0,1]$
  - reason for continuum clear soon
- utility function for voter i $U_i : X \to \mathbb{R}$
  - restrict attention to *strict* utility functions

    if $x \neq y$, then $U_i(x) \neq U_i(y)$

    $\mathscr{U}_X$ = set of strict utility functions

- X = finite set of candidates
- society consists of a continuum of voters [0,1]
  - typical voter $i \in [0,1]$
  - reason for continuum clear soon
- utility function for voter *i* $U_i : X \to \mathbb{R}$
  - restrict attention to *strict* utility functions

    if $x \neq y$, then $U_i(x) \neq U_i(y)$

    $\mathscr{U}_X$ = set of strict utility functions
- profile $U_{\bullet}$ - - specification of each individual's utility function

- voting rule F

$$\text{for all profiles } U_{\bullet} \text{ and all } Y \subseteq X,$$
$$F(U_{\bullet}, Y) \in Y$$

- voting rule F

$$\text{for all profiles } U_{\bullet} \text{ and all } Y \subseteq X,\quad F(U_{\bullet}, Y) \in Y$$

  - Y is *ballot*

- voting rule F

for all profiles $U_{\bullet}$ and all $Y \subseteq X$,
$$F\left(U_{\bullet}, Y\right) \in Y$$

  - Y is *ballot*
  - $F\left(U_{\bullet}, Y\right) =$ optimal candidate in Y if profile is $U_{\bullet}$

- voting rule F

  for all profiles $U_{\bullet}$ and all $Y \subseteq X$,
  $$F\left(U_{\bullet}, Y\right) \in Y$$

  - Y is *ballot*
  - $F\left(U_{\bullet}, Y\right) =$ optimal candidate in Y if profile is $U_{\bullet}$

- definition isn't quite right - - ignores ties

- voting rule F

  for all profiles $U_{\bullet}$ and all $Y \subseteq X$,
  $$F\left(U_{\bullet}, Y\right) \in Y$$

  - Y is *ballot*
  - $F\left(U_{\bullet}, Y\right) =$ optimal candidate in Y if profile is $U_{\bullet}$

- definition isn't quite right - - ignores ties
  - with plurality rule, might be two candidates who are both ranked first the most

- voting rule F

  for all profiles $U_{\bullet}$ and all $Y \subseteq X$,
  $$F\left(U_{\bullet}, Y\right) \in Y$$

  – Y is *ballot*

  – $F\left(U_{\bullet}, Y\right) =$ optimal candidate in Y if profile is $U_{\bullet}$

- definition isn't quite right - - ignores ties
  – with plurality rule, might be two candidates who are both ranked first the most
  – with rank-order voting, might be two candidates who each get lowest number of points

- voting rule F

  for all profiles $U_{\bullet}$ and all $Y \subseteq X$,
  $$F(U_{\bullet}, Y) \in Y$$

  – Y is *ballot*

  – $F(U_{\bullet}, Y) =$ optimal candidate in Y if profile is $U_{\bullet}$

- definition isn't quite right - - ignores ties
  – with plurality rule, might be two candidates who are both ranked first the most
  – with rank-order voting, might be two candidates who each get lowest number of points
- But exact ties unlikely with many voters

- voting rule F

  for all profiles $U_{\bullet}$ and all $Y \subseteq X$,
  $$F\left(U_{\bullet}, Y\right) \in Y$$

  – Y is *ballot*

  – $F\left(U_{\bullet}, Y\right) =$ optimal candidate in Y if profile is $U_{\bullet}$

- definition isn’t quite right - - ignores ties
  – with plurality rule, might be two candidates who are both ranked first the most
  – with rank-order voting, might be two candidates who each get lowest number of points
- But exact ties unlikely with many voters
  – with continuum, ties are *nongeneric*

- voting rule F

  for all profiles $U_\bullet$ and all $Y \subseteq X$,
  $$F\left(U_\bullet, Y\right) \in Y$$

  – Y is *ballot*

  – $F\left(U_\bullet, Y\right) =$ optimal candidate in Y if profile is $U_\bullet$

- definition isn't quite right - - ignores ties
  – with plurality rule, might be two candidates who are both ranked first the most
  – with rank-order voting, might be two candidates who each get lowest number of points
- But exact ties unlikely with many voters
  – with continuum, ties are *nongeneric*
- so, correct definition:

  for *generic* profile $U_\bullet$ and all $Y \subseteq X$
  $$F\left(U_\bullet, Y\right) \in Y$$

plurality rule:

plurality rule:

$$F^P\left(U_{\bullet},Y\right)=\left\{a\middle|\mu\left\{i\middle|U_i\left(a\right)\geq U_i\left(b\right)\text{ for all }b\right\}\right.$$
$$\left.\geq\mu\left\{i\middle|U_i\left(a'\right)\geq U_i\left(b\right)\text{ for all }b\right\}\text{ for all }a'\right\}$$

plurality rule:

$$F^P(U_\bullet, Y) = \Big\{a \Big| \mu\{i | U_i(a) \geq U_i(b) \text{ for all } b\}$$
$$\geq \mu\{i | U_i(a') \geq U_i(b) \text{ for all } b\} \text{ for all } a'\Big\}$$

majority rule:

plurality rule:

$$F^P(U_\bullet, Y) = \left\{ a \,\middle|\, \mu\left\{ i \,\middle|\, U_i(a) \geq U_i(b) \text{ for all } b \right\} \geq \mu\left\{ i \,\middle|\, U_i(a') \geq U_i(b) \text{ for all } b \right\} \text{ for all } a' \right\}$$

majority rule:

$$F^C(U_\bullet, Y) = \left\{ a \,\middle|\, \mu\left\{ i \,\middle|\, U_i(a) \geq U_i(b) \right\} \geq \tfrac{1}{2} \text{ for all } b \right\}$$

plurality rule:

$$F^P(U_\bullet, Y) = \left\{a \middle| \mu\left\{i \middle| U_i(a) \geq U_i(b) \text{ for all } b\right\} \geq \mu\left\{i \middle| U_i(a') \geq U_i(b) \text{ for all } b\right\} \text{ for all } a'\right\}$$

majority rule:

$$F^C(U_\bullet, Y) = \left\{a \middle| \mu\left\{i \middle| U_i(a) \geq U_i(b)\right\} \geq \tfrac{1}{2} \text{ for all } b\right\}$$

rank-order voting:

plurality rule:

$$F^{P}\left(U_{\bullet},Y\right)=\Big\{a \,\Big|\, \mu\big\{i \big| U_i\left(a\right)\geq U_i\left(b\right) \text{ for all } b\big\}$$

$$\geq \mu\big\{i \big| U_i\left(a'\right)\geq U_i\left(b\right) \text{ for all } b\big\} \text{ for all } a'\Big\}$$

majority rule:

$$F^{C}\left(U_{\bullet},Y\right)=\Big\{a \,\Big|\, \mu\big\{i \big| U_i\left(a\right)\geq U_i\left(b\right)\big\}\geq \tfrac{1}{2} \text{ for all } b\Big\}$$

rank-order voting:

$$F^{B}\left(U_{\bullet},Y\right)=\Big\{a \,\Big|\, \int r_{U_i}\left(a\right)d\mu\left(i\right)\leq\int r_{U_i}\left(b\right)d\mu\left(i\right) \text{ for all } b\Big\},$$

where $r_{U_i}\left(a\right)=\#\big\{b \big| U_i\left(b\right)\geq U_i\left(a\right)\big\}$

plurality rule:

$$F^P\left(U_\bullet,Y\right)=\left\{a\middle|\mu\left\{i\middle|U_i\left(a\right)\geq U_i\left(b\right)\text{ for all }b\right\}\right.$$
$$\left.\geq\mu\left\{i\middle|U_i\left(a'\right)\geq U_i\left(b\right)\text{ for all }b\right\}\text{ for all }a'\right\}$$

majority rule:

$$F^C\left(U_\bullet,Y\right)=\left\{a\middle|\mu\left\{i\middle|U_i\left(a\right)\geq U_i\left(b\right)\right\}\geq\tfrac{1}{2}\text{ for all }b\right\}$$

rank-order voting:

$$F^B\left(U_\bullet,Y\right)=\left\{a\middle|\int r_{U_i}\left(a\right)d\mu\left(i\right)\leq\int r_{U_i}\left(b\right)d\mu\left(i\right)\text{ for all }b\right\},$$

where $r_{U_i}\left(a\right)=\#\left\{b\middle|U_i\left(b\right)\geq U_i\left(a\right)\right\}$

utilitarian principle:

plurality rule:

$$F^P(U_\bullet, Y) = \left\{a \middle| \mu\left\{i \middle| U_i(a) \geq U_i(b) \text{ for all } b\right\} \geq \mu\left\{i \middle| U_i(a') \geq U_i(b) \text{ for all } b\right\} \text{ for all } a'\right\}$$

majority rule:

$$F^C(U_\bullet, Y) = \left\{a \middle| \mu\left\{i \middle| U_i(a) \geq U_i(b)\right\} \geq \tfrac{1}{2} \text{ for all } b\right\}$$

rank-order voting:

$$F^B(U_\bullet, Y) = \left\{a \middle| \int r_{U_i}(a)\, d\mu(i) \leq \int r_{U_i}(b)\, d\mu(i) \text{ for all } b\right\},$$

where $r_{U_i}(a) = \#\left\{b \middle| U_i(b) \geq U_i(a)\right\}$

utilitarian principle:

$$F^U(U_\bullet, Y) = \left\{a \middle| \int U_i(a)\, d\mu(i) \geq \int U_i(b)\, d\mu(i) \text{ for all } b\right\}$$

What properties should reasonable voting rule satisfy?

What properties should reasonable voting rule satisfy?

- *Pareto Property* (P): if $U_i(x) > U_i(y)$ for all i and $x \in Y$, then $y \neq F(U_\bullet, Y)$

What properties should reasonable voting rule satisfy?

- *Pareto Property* (P): if $U_i(x) > U_i(y)$ for all i and $x \in Y,$ then $y \neq F(U_\bullet, Y)$
  - if everybody prefers x to y, y should not be chosen

What properties should reasonable voting rule satisfy?

- *Pareto Property* (P): if $U_i(x) > U_i(y)$ for all i and $x \in Y$, then $y \neq F(U_\bullet, Y)$
  - if everybody prefers x to y, y should not be chosen
- *Anonymity* (A): suppose $\pi : [0,1] \to [0,1]$ measure-preserving permutation. If $U_i^\pi = U_{\pi(i)}$ for all i, then

What properties should reasonable voting rule satisfy?

- *Pareto Property* (P): if $U_i(x) > U_i(y)$ for all i and $x \in Y$, then $y \neq F(U_\bullet, Y)$
  - if everybody prefers x to y, y should not be chosen
- *Anonymity* (A): suppose $\pi : [0,1] \to [0,1]$ measure-preserving permutation. If $U_i^\pi = U_{\pi(i)}$ for all i, then $F(U_\bullet^\pi, Y) = F(U_\bullet, Y)$ for all Y

What properties should reasonable voting rule satisfy?

- *Pareto Property* (P): if $U_i(x) > U_i(y)$ for all i and $x \in Y$, then $y \neq F(U_\bullet, Y)$
  - if everybody prefers *x* to *y*, *y* should not be chosen
- *Anonymity* (A): suppose $\pi : [0,1] \to [0,1]$ measure-preserving permutation. If $U_i^\pi = U_{\pi(i)}$ for all *i*, then $F(U_\bullet^\pi, Y) = F(U_\bullet, Y)$ for all Y
  - candidate chosen depends only on voters' *preferences* and not *who* has those preferences

What properties should reasonable voting rule satisfy?

- *Pareto Property* (P): if $U_i(x) > U_i(y)$ for all i and $x \in Y$, then $y \neq F(U_\bullet, Y)$
  - if everybody prefers x to y, y should not be chosen
- *Anonymity* (A): suppose $\pi : [0,1] \to [0,1]$ measure-preserving permutation. If $U_i^\pi = U_{\pi(i)}$ for all i, then $F(U_\bullet^\pi, Y) = F(U_\bullet, Y)$ for all Y
  - candidate chosen depends only on voters' *preferences* and not *who* has those preferences
  - voters treated symmetrically

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

  then

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

  then

  $$F\left(U_{\bullet}^{\rho,Y}, Y\right) = \rho\left(F\left(U_{\bullet}, Y\right)\right).$$

- *Neutrality* (N): Suppose $\rho : Y \rightarrow Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

  then

  $$F\left(U_{\bullet}^{\rho,Y}, Y\right) = \rho\left(F\left(U_{\bullet}, Y\right)\right).$$

  - candidates treated symmetrically

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

  then

  $$F\left(U_{\bullet}^{\rho,Y}, Y\right) = \rho\left(F\left(U_{\bullet}, Y\right)\right).$$

  – candidates treated symmetrically

- All four voting rules – plurality, majority, rank-order, utilitarian – satisfy P, A, N

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

  then

  $$F\left(U_\bullet^{\rho,Y}, Y\right) = \rho\left(F\left(U_\bullet, Y\right)\right).$$

  – candidates treated symmetrically

- All four voting rules – plurality, majority, rank-order, utilitarian – satisfy P, A, N

- Next axiom most controversial

  still

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho(x)\right) > U_i^{\rho,Y}\left(\rho(y)\right) \Leftrightarrow U_i(x) > U_i(y)$ for all x, y, i,

  then

  $$F\left(U_{\bullet}^{\rho,Y}, Y\right) = \rho\left(F\left(U_{\bullet}, Y\right)\right).$$

  - candidates treated symmetrically

- All four voting rules – plurality, majority, rank-order, utilitarian – satisfy P, A, N

- Next axiom most controversial

  still

  - has quite compelling justification

- *Neutrality* (N): Suppose $\rho : Y \to Y$ permutation.

  If $U_i^{\rho,Y}\left(\rho\left(x\right)\right) > U_i^{\rho,Y}\left(\rho\left(y\right)\right) \Leftrightarrow U_i\left(x\right) > U_i\left(y\right)$ for all x, y, i,

  then

  $$F\left(U_{\bullet}^{\rho,Y}, Y\right) = \rho\left(F\left(U_{\bullet}, Y\right)\right).$$

  - candidates treated symmetrically

- All four voting rules – plurality, majority, rank-order, utilitarian – satisfy P, A, N

- Next axiom most controversial

  still

  - has quite compelling justification
  - invoked by both Arrow (1951) and Nash (1950)

- *Independence of Irrelevant Candidates* (I):

- *Independence of Irrelevant Candidates* (I):

  if $x = F\left(U_{\bullet}, Y\right)$ and $x \in Y' \subseteq Y$

- *Independence of Irrelevant Candidates* (I):

if $x = F\left(U_{\bullet}, Y\right)$ and $x \in Y' \subseteq Y$

then

- *Independence of Irrelevant Candidates* (I):

  if $x = F\left(U_{\bullet}, Y\right)$ and $x \in Y' \subseteq Y$

  then

  $$x = F\left(U_{\bullet}, Y'\right)$$

- *Independence of Irrelevant Candidates* (I):

$$\text{if } x = F\left(U_{\bullet}, Y\right) \text{ and } x \in Y' \subseteq Y$$

then

$$x = F\left(U_{\bullet}, Y'\right)$$

  – if *x* chosen and some non-chosen candidates removed, *x* still chosen

- *Independence of Irrelevant Candidates* (I):

  if $x = F\left(U_{\bullet}, Y\right)$ and $x \in Y' \subseteq Y$

  then

  $$x = F\left(U_{\bullet}, Y'\right)$$

  – if *x* chosen and some non-chosen candidates removed, *x* still chosen

  – Nash formulation (rather than Arrow)

- *Independence of Irrelevant Candidates* (I):

$$\text{if } x = F(U_{\bullet}, Y) \text{ and } x \in Y' \subseteq Y$$

then

$$x = F(U_{\bullet}, Y')$$

  - if x chosen and some non-chosen candidates removed, x still chosen
  - Nash formulation (rather than Arrow)
  - no "spoilers" (e.g. Nader in 2000 U.S. presidential election, Le Pen in 2002 French presidential election)

- Majority rule and utilitarianism satisfy I, but others don’t:

- Majority rule and utilitarianism satisfy I, but others don't:
  - plurality rule

.35	.33	.32
x	y	z
y	z	y
z	x	x

$$F^P\left(U_{\bullet},\{x,y,z\}\right)=x$$

$$F^P\left(U_{\bullet},\{x,y\}\right)=y$$

- Majority rule and utilitarianism satisfy I, but others don't:
  - plurality rule

.35	.33	.32
x	y	z
y	z	y
z	x	x

$$F^P\left(U_\bullet,\{x,y,z\}\right)=x$$

$$F^P\left(U_\bullet,\{x,y\}\right)=y$$

  - rank-order voting

.55	.45
x	y
y	z
z	x

$$F^B\left(U_\bullet,\{x,y,z\}\right)=y$$

$$F^B\left(U_\bullet,\{x,y\}\right)=x$$

Final Axiom:

Final Axiom:

- *Nonmanipulability* (NM):

Final Axiom:

- *Nonmanipulability* (NM):

if $x = F\left(U_{\bullet}, Y\right)$ and $x' = F\left(U'_{\bullet}, Y\right)$,

where $U'_j = U_j$ for all $j \notin C \subseteq [0,1]$

Final Axiom:

- *Nonmanipulability* (NM):

if $x = F\left(U_{\bullet}, Y\right)$ and $x' = F\left(U'_{\bullet}, Y\right)$,

where $U'_j = U_j$ for all $j \notin C \subseteq [0,1]$

then

Final Axiom:

- *Nonmanipulability* (NM):

if $x = F(U_{\bullet}, Y)$ and $x' = F(U'_{\bullet}, Y)$,

where $U'_j = U_j$ for all $j \notin C \subseteq [0,1]$

then

$U_i(x) > U_i(x')$ for some $i \in C$

Final Axiom:

- *Nonmanipulability* (NM):

if $x = F(U_{\bullet}, Y)$ and $x' = F(U'_{\bullet}, Y)$,

where $U'_j = U_j$ for all $j \notin C \subseteq [0,1]$

then

$U_i(x) > U_i(x')$ for some $i \in C$

– the members of coalition C can't all gain from misrepresenting utility functions as U'_i

- NM implies voting rule must be *ordinal* (no cardinal information used)

- NM implies voting rule must be *ordinal* (no cardinal information used)
- F is *ordinal* if whenever, for profiles $U_{\bullet}$ and $U'_{\bullet}$,

- NM implies voting rule must be *ordinal* (no cardinal information used)
- F is *ordinal* if whenever, for profiles $U_\cdot$ and $U'_\cdot$,
  $U_i(x) > U_i(y) \Leftrightarrow U'_i(x) > U'_i(y)$ for all i, x, y

- NM implies voting rule must be *ordinal* (no cardinal information used)
- F is *ordinal* if whenever, for profiles $U_\bullet$ and $U'_\bullet$,

  $U_i(x) > U_i(y) \Leftrightarrow U'_i(x) > U'_i(y)$ for all i, x, y

(*) $F(U_\bullet, Y) = F(U'_\bullet, Y)$ for all Y

- NM implies voting rule must be *ordinal* (no cardinal information used)
- F is *ordinal* if whenever, for profiles $U_{\bullet}$ and $U'_{\bullet}$,

  $$U_i(x) > U_i(y) \Leftrightarrow U'_i(x) > U'_i(y) \text{ for all } i, x, y$$

(*) $$F(U_{\bullet}, Y) = F(U'_{\bullet}, Y) \text{ for all } Y$$

- *Lemma*: If F satisfies NM, F ordinal

- NM implies voting rule must be *ordinal* (no cardinal information used)
- F is *ordinal* if whenever, for profiles $U_\bullet$ and $U'_\bullet$,

  $U_i(x) > U_i(y) \Leftrightarrow U'_i(x) > U'_i(y)$ for all i, x, y

(*) $F(U_\bullet, Y) = F(U'_\bullet, Y)$ for all Y

- *Lemma*: If F satisfies NM, F ordinal
- NM rules out utilitarianism

But majority rule also violates NM

But majority rule also violates NM

- F^C not even always *defined*

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^C\left(U_{\bullet},\{x,y,z\}\right)=\varnothing$$

But majority rule also violates NM

- F^C not even always *defined*

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^C\left(U_{\bullet},\{x,y,z\}\right)=\varnothing$$

  – example of *Condorcet cycle*

But majority rule also violates NM

- F^C not even always *defined*

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^C\left(U_{\bullet},\{x,y,z\}\right)=\varnothing$$

  - example of *Condorcet cycle*
  - F^C must be extended to Condorcet cycles

But majority rule also violates NM

- F^C not even always *defined*

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^C\left(U_{\bullet},\{x,y,z\}\right)=\varnothing$$

  - example of *Condorcet cycle*
  - F^C must be extended to Condorcet cycles
  - one possibility

$$F^{C/B}\left(U_{\bullet},Y\right)=\begin{cases}F^C\left(U_{\bullet},Y\right), & \text{if nonempty}\\ F^B\left(U_{\bullet},Y\right), & \text{otherwise}\end{cases}$$

(Black's method)

But majority rule also violates NM

- F^C not even always *defined*

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^C\left(U_\bullet, \{x, y, z\}\right) = \varnothing$$

  - example of *Condorcet cycle*
  - F^C must be extended to Condorcet cycles
  - one possibility

$$F^{C/B}\left(U_\bullet, Y\right) = \begin{cases} F^C\left(U_\bullet, Y\right), & \text{if nonempty} \\ F^B\left(U_\bullet, Y\right), & \text{otherwise} \end{cases} \qquad \text{(Black's method)}$$

  - extensions make F^C vulnerable to manipulation

But majority rule also violates NM

- F^C not even always *defined*

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^C\left(U_{\bullet},\{x,y,z\}\right)=\varnothing$$

  - example of *Condorcet cycle*
  - F^C must be extended to Condorcet cycles
  - one possibility

$$F^{C/B}\left(U_{\bullet},Y\right)=\begin{cases} F^C\left(U_{\bullet},Y\right), & \text{if nonempty} \\ F^B\left(U_{\bullet},Y\right), & \text{otherwise} \end{cases}$$

(Black's method)

  - extensions make F^C vulnerable to manipulation
  -

.35	.33	.32
x	y	z
y	z	x
z	x	y

$$F^{C/B}\left(U_{\bullet},\{x,y,z\}\right)=x$$

z
y
x

$$F^{C/B}\left(U'_{\bullet},\{x,y,z\}\right)=z$$

Theorem: There exists no voting rule satisfying P,A,N,I and NM

Theorem: There exists no voting rule satisfying P,A,N,I and NM

Proof: similar to that of GS

Theorem: There exists no voting rule satisfying P,A,N,I and NM

Proof: similar to that of GS

overly pessimistic - - many cases in which some rankings unlikely

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

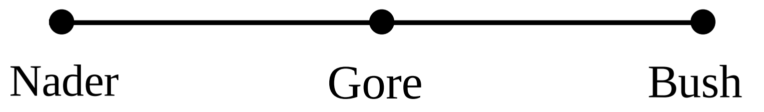
When can we rule out Condorcet cycles?

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election

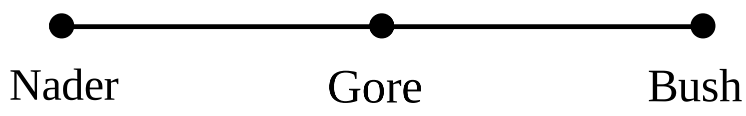


Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election



unlikely that many had ranking

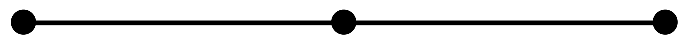
Bush		Nader
	or	
Nader		Bush
Gore		Gore

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election



unlikely that many had ranking

Bush		Nader
	or	
Nader		Bush
Gore		Gore

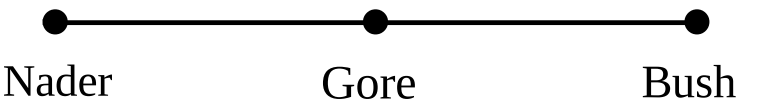
- strongly-felt candidate

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election



unlikely that many had ranking

Bush		Nader
Nader	or	Bush
Gore		Gore

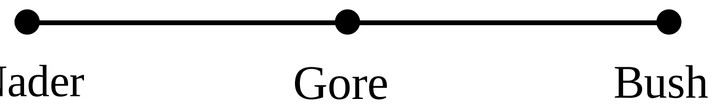
- strongly-felt candidate
  - in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election



unlikely that many had ranking

Bush	or	Nader
Nader		Bush
Gore		Gore

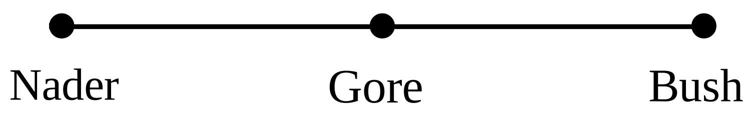
- strongly-felt candidate
  - in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen
  - voters didn’t feel strongly about Chirac and Jospin

Lemma: Majority rule satisfies all 5 properties if and only if preferences restricted to domain with no Condorcet cycles

When can we rule out Condorcet cycles?

- preferences single-peaked

2000 US election



unlikely that many had ranking

Bush		Nader
Nader	or	Bush
Gore		Gore

- strongly-felt candidate
  - in 2002 French election, 3 main candidates: Chirac, Jospin, Le Pen
  - voters didn't feel strongly about Chirac and Jospin
  - felt strongly about Le Pen (ranked him first or last)

- Voting rule F *works well* on domain $\mathscr{U}$ if satisfies P,A,N,I,NM when utility functions restricted to $\mathscr{U}$

- Voting rule F *works well* on domain $\mathscr{U}$ if satisfies P,A,N,I,NM when utility functions restricted to $\mathscr{U}$

   - e.g., F^C works well when preferences single-peaked

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

  Then if there exisits profile $U^\circ_\cdot$ on $\mathscr{U}^C$ such that

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

Then if there exisits profile $U_{\bullet}^{\circ}$ on $\mathscr{U}^C$ such that

$$F\left(U_{\bullet}^{\circ}, Y\right) \neq F^C\left(U_{\bullet}^{\circ}, Y\right) \text{ for some } Y,$$

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

  Then if there exisits profile $U_{\bullet}^{\circ}$ on $\mathscr{U}^C$ such that

$$F\left(U_{\bullet}^{\circ}, Y\right) \neq F^C\left(U_{\bullet}^{\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^C works well but F does not

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

Then if there exisits profile $U_{\bullet}^{\circ}$ on $\mathscr{U}^C$ such that

$$F\left(U_{\bullet}^{\circ},Y\right) \neq F^C\left(U_{\bullet}^{\circ},Y\right) \text{ for some } Y,$$

there exists domain $\mathscr{U}'$ on which F^C works well but F does not

Proof: From NM and I, if F works well on $\mathscr{U}$, F must be ordinal

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

  Then if there exisits profile $U_{\bullet}^{\circ}$ on $\mathscr{U}^C$ such that

  $$F\left(U_{\bullet}^{\circ}, Y\right) \neq F^C\left(U_{\bullet}^{\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^C works well but F does not

Proof: From NM and I, if F works well on $\mathscr{U}$, F must be ordinal

- Hence result follows from

  Dasgupta-Maskin (2008), *JEEA*

- *Theorem 1*: Suppose F works well on domain $\mathscr{U}$, then F^C works well on $\mathscr{U}$ too.
- Conversely, suppose that F^C works well on $\mathscr{U}^C$.

Then if there exisits profile $U_{\bullet}^{\circ}$ on $\mathscr{U}^C$ such that

$$F\left(U_{\bullet}^{\circ}, Y\right) \neq F^C\left(U_{\bullet}^{\circ}, Y\right) \text{ for some } Y,$$

there exists domain $\mathscr{U}'$ on which F^C works well but F does not

Proof: From NM and I, if F works well on $\mathscr{U}$, F must be ordinal

- Hence result follows from

  Dasgupta-Maskin (2008), *JEEA*

  - shows that Theorem 1 holds when NM replaced by ordinality

To show this D-M uses

To show this D-M uses

Lemma: F^C works well on $\mathscr{U}$ if and only if $\mathscr{U}$ has no Condorcet cycles

To show this D-M uses

Lemma: F^C works well on $\mathscr{U}$ if and only if $\mathscr{U}$ has no Condorcet cycles

- Suppose F works well on $\mathscr{U}$

To show this D-M uses

Lemma: F^C works well on $\mathscr{U}$ if and only if $\mathscr{U}$ has no Condorcet cycles

- Suppose F works well on $\mathscr{U}$
- If F^C doesn't work well on $\mathscr{U}$, Lemma implies $\mathscr{U}$ must contain Condorcet cycle $\begin{matrix} x & y & z \\ y & z & x \\ z & x & y \end{matrix}$

- Consider

- Consider

$$U_{\bullet}^{1} = \begin{array}{cccc} \underline{1} & \underline{2} & \cdots & \underline{n} \\ x & z & & z \\ z & x & & x \end{array}$$

- Consider

$$U_{\bullet}^{1} = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U_{\bullet}^{1},\{x,z\}\right)=z$

- Consider

$$U_{\bullet}^{1} = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U_{\bullet}^{1}, \{x, z\}\right) = z$

- $$U_{\bullet}^{2} = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & \underline{n} \\ x & y & z & z \\ y & z & x & x \\ z & x & y & y \end{array}$$

- Consider

$$U^1_{\bullet} = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U^1_{\bullet}, \{x, z\}\right) = z$

- $$U^2_{\bullet} = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & \underline{n} \\ x & y & z & z \\ y & z & x & x \\ z & x & y & y \end{array}$$

$F\left(U^2_{\bullet}, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U^2_{\bullet}, \{x, z\}\right) = x$, contradicts (*)

- Consider

$$U^1_\bullet = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U^1_\bullet, \{x, z\}\right) = z$

- $$U^2_\bullet = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & \underline{n} \\ x & y & z & z \\ y & z & x & x \\ z & x & y & y \end{array}$$

$F\left(U^2_\bullet, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U^2_\bullet, \{x, z\}\right) = x$, contradicts (*)

$F\left(U^2_\bullet, \{x, y, z\}\right) = y \quad \Rightarrow$ (from I) $F\left(U^2_\bullet, \{x, y\}\right) = y$, contradicts (*) (A,N)

- Consider

$$U^1_{\cdot} = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U^1_{\cdot}, \{x, z\}\right) = z$

- $$U^2_{\cdot} = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & \underline{n} \\ x & y & z & z \\ y & z & x & x \\ z & x & y & y \end{array}$$

$F\left(U^2_{\cdot}, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U^2_{\cdot}, \{x, z\}\right) = x$, contradicts (*)

so $F\left(U^2_{\cdot}, \{x, y, z\}\right) = y \quad \Rightarrow$ (from I) $F\left(U^2_{\cdot}, \{x, y\}\right) = y$, contradicts (*) (A,N)

- Consider

$$U_{\bullet}^{1} = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U_{\bullet}^{1}, \{x, z\}\right) = z$

- $$U_{\bullet}^{2} = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & & \underline{n} \\ x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{array}$$

$F\left(U_{\bullet}^{2}, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2}, \{x, z\}\right) = x$, contradicts (*)

$F\left(U_{\bullet}^{2}, \{x, y, z\}\right) = y \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2}, \{x, y\}\right) = y$, contradicts (*) (A,N)

so

$F\left(U_{\bullet}^{2}, \{x, y, z\}\right) = z$

- Consider

$$U_\bullet^1 = \begin{matrix} \underline{1} & \underline{2} & \ldots & \underline{n} \\ x & z & & z \\ z & x & & x \end{matrix}$$

(*) Suppose $F\left(U_\bullet^1, \{x, z\}\right) = z$

- $$U_\bullet^2 = \begin{matrix} \underline{1} & \underline{2} & \underline{3} & & \underline{n} \\ x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{matrix}$$

$F\left(U_\bullet^2, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U_\bullet^2, \{x, z\}\right) = x$, contradicts (*)

$F\left(U_\bullet^2, \{x, y, z\}\right) = y \quad \Rightarrow$ (from I) $F\left(U_\bullet^2, \{x, y\}\right) = y$, contradicts (*) (A,N)

so

$F\left(U_\bullet^2, \{x, y, z\}\right) = z$

- so $F\left(U_\bullet^2, \{y, z\}\right) = z$ (I)

- Consider

$$U^1_\bullet = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U^1_\bullet, \{x, z\}\right) = z$

- $$U^2_\bullet = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & \underline{n} \\ x & y & z & z \\ y & z & x & x \\ z & x & y & y \end{array}$$

$F\left(U^2_\bullet, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U^2_\bullet, \{x, z\}\right) = x$, contradicts (*)

$F\left(U^2_\bullet, \{x, y, z\}\right) = y \quad \Rightarrow$ (from I) $F\left(U^2_\bullet, \{x, y\}\right) = y$, contradicts (*) (A,N)

so

$F\left(U^2_\bullet, \{x, y, z\}\right) = z$

- so $F\left(U^2_\bullet, \{y, z\}\right) = z \quad$ (I)
- so for

- Consider

$$U_{\bullet}^{1} = \begin{matrix} \underline{1} & \underline{2} & \cdots & \underline{n} \\ x & z & & z \\ z & x & & x \end{matrix}$$

(*) Suppose $F\left(U_{\bullet}^{1},\{x,z\}\right)=z$

- $$U_{\bullet}^{2} = \begin{matrix} \underline{1} & \underline{2} & \underline{3} & & \underline{n} \\ x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{matrix}$$

$F\left(U_{\bullet}^{2},\{x,y,z\}\right)=x \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2},\{x,z\}\right)=x$, contradicts (*)

$F\left(U_{\bullet}^{2},\{x,y,z\}\right)=y \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2},\{x,y\}\right)=y$, contradicts (*) (A,N)

so

$F\left(U_{\bullet}^{2},\{x,y,z\}\right)=z$

- so $F\left(U_{\bullet}^{2},\{y,z\}\right)=z$ (I)
- so for

$$U_{\bullet}^{3} = \begin{matrix} \underline{1} & \underline{2} & \underline{3} & \cdots & \underline{n} \\ x & x & z & & z \\ z & z & x & & x \end{matrix}$$

- Consider

$$U_{\bullet}^{1} = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U_{\bullet}^{1}, \{x, z\}\right) = z$

- $$U_{\bullet}^{2} = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & & \underline{n} \\ x & y & z & & z \\ y & z & x & & x \\ z & x & y & & y \end{array}$$

$F\left(U_{\bullet}^{2}, \{x, y, z\}\right) = x \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2}, \{x, z\}\right) = x$, contradicts (*)

$F\left(U_{\bullet}^{2}, \{x, y, z\}\right) = y \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2}, \{x, y\}\right) = y$, contradicts (*) (A,N)

so

$F\left(U_{\bullet}^{2}, \{x, y, z\}\right) = z$

- so $F\left(U_{\bullet}^{2}, \{y, z\}\right) = z$ (I)
- so for

$$U_{\bullet}^{3} = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \ldots & \underline{n} \\ x & x & z & & z \\ z & z & x & & x \end{array}$$

$F\left(U_{\bullet}^{3}, \{x, z\}\right) = z$ (N)

- Consider

$$U^1_\bullet = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U^1_\bullet, \{x, z\}\right) = z$

- $$U^2_\bullet = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & \underline{n} \\ x & y & z & z \\ y & z & x & x \\ z & x & y & y \end{array}$$

$$F\left(U^2_\bullet, \{x, y, z\}\right) = x \quad \Rightarrow \text{(from I) } F\left(U^2_\bullet, \{x, z\}\right) = x, \text{ contradicts (*)}$$

$$F\left(U^2_\bullet, \{x, y, z\}\right) = y \quad \Rightarrow \text{(from I) } F\left(U^2_\bullet, \{x, y\}\right) = y, \text{ contradicts (*) (A,N)}$$

so

$$F\left(U^2_\bullet, \{x, y, z\}\right) = z$$

- so $F\left(U^2_\bullet, \{y, z\}\right) = z$ (I)
- so for

$$U^3_\bullet = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} \ldots & \underline{n} \\ x & x & z & z \\ z & z & x & x \end{array}$$

$$F\left(U^3_\bullet, \{x, z\}\right) = z \quad \text{(N)}$$

- Continuing in the same way, let $U^4_\bullet = \begin{array}{ccc} \underline{1} \ldots & \underline{n-1} & \underline{n} \\ x & x & z \\ z & z & x \end{array}$

- Consider

$$U_{\bullet}^{1} = \begin{array}{ccc} \underline{1} & \underline{2} \ldots & \underline{n} \\ x & z & z \\ z & x & x \end{array}$$

(*) Suppose $F\left(U_{\bullet}^{1},\{x,z\}\right)=z$

- $$U_{\bullet}^{2} = \begin{array}{cccc} \underline{1} & \underline{2} & \underline{3} & \underline{n} \\ x & y & z & z \\ y & z & x & x \\ z & x & y & y \end{array}$$

$F\left(U_{\bullet}^{2},\{x,y,z\}\right)=x \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2},\{x,z\}\right)=x$, contradicts (*)

$F\left(U_{\bullet}^{2},\{x,y,z\}\right)=y \quad \Rightarrow$ (from I) $F\left(U_{\bullet}^{2},\{x,y\}\right)=y$, contradicts (*) (A,N)

so

$F\left(U_{\bullet}^{2},\{x,y,z\}\right)=z$

- so $F\left(U_{\bullet}^{2},\{y,z\}\right)=z$ (I)
- so for

$$U_{\bullet}^{3} = \begin{array}{ccccc} \underline{1} & \underline{2} & \underline{3} & \ldots & \underline{n} \\ x & x & z & & z \\ z & z & x & & x \end{array}$$

$F\left(U_{\bullet}^{3},\{x,z\}\right)=z$ (N)

- Continuing in the same way, let $U_{\bullet}^{4} = \begin{array}{ccc} \underline{1} \ldots & \underline{n-1} & \underline{n} \\ x & x & z \\ z & z & x \end{array}$

$F\left(U_{\bullet}^{4},\{x,z\}\right)=z$, contradicts (*)

- So F can't work well on $\mathscr{U}$ with Condorcet cycle

- So F can't work well on $\mathscr{U}$ with Condorcet cycle
- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

- So F can't work well on $\mathscr{U}$ with Condorcet cycle
- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

  $F\left(U_{\bullet}^{\circ}, Y\right) \neq F^C\left(U_{\bullet}^{\circ}, Y\right)$ for some $U_{\bullet}^{\circ}$ and Y

- So F can't work well on $\mathscr{U}$ with Condorcet cycle
- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

  $F\left(U_{\bullet}^{\circ}, Y\right) \neq F^C\left(U_{\bullet}^{\circ}, Y\right)$ for some $U_{\bullet}^{\circ}$ and Y
- Then there exist α with $1-\alpha > \alpha$ and

- So F can't work well on $\mathscr{U}$ with Condorcet cycle

- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

$$F\left(U^{\circ}_{\bullet}, Y\right) \neq F^C\left(U^{\circ}_{\bullet}, Y\right) \text{ for some } U^{\circ}_{\bullet} \text{ and } Y$$

- Then there exist α with $1-\alpha > \alpha$ and

$$U^{\circ}_{\bullet} = \begin{array}{cc} 1-\alpha & \alpha \\ \hline x & y \\ y & x \end{array}$$

- So F can't work well on $\mathscr{U}$ with Condorcet cycle

- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

  $$F\left(U_\bullet^\circ, Y\right) \neq F^C\left(U_\bullet^\circ, Y\right)$$ for some $U_\bullet^\circ$ and Y

- Then there exist α with $1-\alpha > \alpha$ and

  $$U_\bullet^\circ = \begin{array}{cc} \dfrac{1-\alpha}{x} & \dfrac{\alpha}{y} \\ y & x \end{array}$$

  such that

- So F can't work well on $\mathscr{U}$ with Condorcet cycle

- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

  $F\left(U^{\circ}_{\bullet}, Y\right) \neq F^C\left(U^{\circ}_{\bullet}, Y\right)$ for some $U^{\circ}_{\bullet}$ and Y

- Then there exist α with $1-\alpha > \alpha$ and

$$U^{\circ}_{\bullet} = \begin{array}{cc} 1-\alpha & \alpha \\ \hline x & y \\ y & x \end{array}$$

  such that

$$x = F^C\left(U^{\circ}_{\bullet}, \{x, y\}\right) \text{ and } y = F\left(U^{\circ}_{\bullet}, \{x, y\}\right)$$

- So F can't work well on $\mathscr{U}$ with Condorcet cycle

- Conversely, suppose that F^C works well on $\mathscr{U}^C$ and

  $F\left(U_{\bullet}^{\circ}, Y\right) \neq F^C\left(U_{\bullet}^{\circ}, Y\right)$ for some $U_{\bullet}^{\circ}$ and Y

- Then there exist α with $1-\alpha > \alpha$ and

$$U_{\bullet}^{\circ} = \begin{array}{cc} \frac{1-\alpha}{} & \frac{\alpha}{} \\ x & y \\ y & x \end{array}$$

  such that

$$x = F^C\left(U_{\bullet}^{\circ}, \{x, y\}\right) \text{ and } y = F\left(U_{\bullet}^{\circ}, \{x, y\}\right)$$

- But not hard to show that F^C unique voting rule satisfying P,A,N, and NM when $|X| = 2$ - - contradiction

- Let’s drop I

- Let’s drop I
  - most controversial

- Let's drop I
  - most controversial

- *no* voting rule satisfies P,A,N,NM on $\mathscr{U}_X$

- Let's drop I
  - most controversial

- *no* voting rule satisfies P,A,N,NM on $\mathscr{U}_X$
  - GS again

- Let's drop I
  - most controversial

- *no* voting rule satisfies P,A,N,NM on $\mathscr{U}_X$
  - GS again

- *F works nicely* on $\mathscr{U}$ if satisfies P,A,N,NM on $\mathscr{U}$

Theorem 2:

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^* works nicely on $\mathscr{U}^*$, where $F^* = F^C$ or F^B.

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^*works nicely on $\mathscr{U}^*$, where $F^* = F^C$or F^B.

  Then, if there exisits profile $U_\bullet^{\circ\circ}$ on $\mathscr{U}^*$ such that

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^* works nicely on $\mathscr{U}^*$, where $F^* = F^C$ or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

$$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^*works nicely on $\mathscr{U}^*$, where $F^* = F^C$or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

$$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^*works nicely on $\mathscr{U}^*$, where $F^* = F^C$or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

$$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^*works nicely on $\mathscr{U}^*$, where $F^* = F^C$or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

$$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

- F^C works nicely on any Condorcet-cycle-free domain

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^*works nicely on $\mathscr{U}^*$, where $F^* = F^C$or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

  $$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

- F^C works nicely on any Condorcet-cycle-free domain

- F^B works nicely only when $\mathscr{U}$ is subset of Condorcet cycle

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^* works nicely on $\mathscr{U}^*$, where $F^* = F^C$ or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

  $$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

- F^C works nicely on any Condorcet-cycle-free domain

- F^B works nicely only when $\mathscr{U}$ is subset of Condorcet cycle

- so F^C and F^B complement each other

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^* works nicely on $\mathscr{U}^*$, where $F^* = F^C$ or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

$$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

- F^C works nicely on any Condorcet-cycle-free domain
- F^B works nicely only when $\mathscr{U}$ is subset of Condorcet cycle
- so F^C and F^B complement each other
  - if F works nicely on $\mathscr{U}$ and $\mathscr{U}$ doesn't contain Condorcet cycle, F^C works nicely too

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^* works nicely on $\mathscr{U}^*$, where $F^* = F^C$ or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

  $$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

- F^C works nicely on any Condorcet-cycle-free domain

- F^B works nicely only when $\mathscr{U}$ is subset of Condorcet cycle

- so F^C and F^B complement each other
  - if F works nicely on $\mathscr{U}$ and $\mathscr{U}$ doesn't contain Condorcet cycle, F^C works nicely too
  - if F works nicely on $\mathscr{U}$ and $\mathscr{U}$ contains Condorcet cycle, then $\mathscr{U}$ can't contain any other ranking (otherwise *no* voting rule works nicely)

Theorem 2:

- Suppose F works nicely on $\mathscr{U}$, then F^C or F^B works nicely on $\mathscr{U}$ too.

- Conversely suppose F^* works nicely on $\mathscr{U}^*$, where $F^* = F^C$ or F^B.

  Then, if there exisits profile $U_{\bullet}^{\circ\circ}$ on $\mathscr{U}^*$ such that

  $$F\left(U_{\bullet}^{\circ\circ}, Y\right) \neq F^*\left(U_{\bullet}^{\circ\circ}, Y\right) \text{ for some } Y,$$

  there exists domain $\mathscr{U}'$ on which F^* works nicely but F does not

Proof:

- F^C works nicely on any Condorcet-cycle-free domain

- F^B works nicely only when $\mathscr{U}$ is subset of Condorcet cycle

- so F^C and F^B complement each other
  - if F works nicely on $\mathscr{U}$ and $\mathscr{U}$ doesn't contain Condorcet cycle, F^C works nicely too
  - if F works nicely on $\mathscr{U}$ and $\mathscr{U}$ contains Condorcet cycle, then $\mathscr{U}$ can't contain any other ranking (otherwise *no* voting rule works nicely)
  - so F^B works nicely on $\mathscr{U}$.

Striking that the 2 longest-studied voting rules (Condorcet and Borda) are also

Striking that the 2 longest-studied voting rules (Condorcet and Borda) are also

- *only two* that work nicely on maximal domains