Taxing Wealth and Capital Income When Returns are Heterogeneous *

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Abstract

When is a wealth tax preferable to a capital income tax? We study this question theoretically in an infinite-horizon model with entrepreneurs and workers, in which entrepreneurial firms differ in their productivity and are subject to collateral constraints. The model features heterogeneous returns and misallocation of capital in steady state. We show analytically that increasing the wealth tax increases aggregate productivity. The gains result from the *use-it-or-lose-it* effect, which causes a reallocation of capital from entrepreneurs with low productivity to those with high productivity. Furthermore, if the capital income tax is adjusted to balance the government's budget, aggregate capital, output, and wages increase. We provide necessary and sufficient conditions for a switch to a wealth tax to imply higher average welfare, which amount to a lower bound on the capital-elasticity of output, α—around $1/3$ for most parameter combinations. We then study the optimal tax mix when both instruments can be used to maximize welfare. Optimal policy depends on two thresholds. When α is sufficiently high, optimal policy involves a positive wealth tax and a negative capital income tax (a subsidy); the sign flips when α is sufficiently low, and both taxes are positive between these two thresholds. Endogenizing productivity through innovation or managerial effort strengthens these results as wealth taxes incentivize entrepreneurs to exert more effort. Finally, we show that these results carry through to an economy where entrepreneurs are subject to idiosyncratic productivity shocks if and only if entrepreneurial productivity is positively auto-correlated.

JEL Codes: E21, E22, E62, H21.

Keywords: Wealth tax, capital income tax, optimal taxation, rate of return heterogeneity, wealth inequality.

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1 Introduction

When is a wealth tax preferable to a capital income tax? When is the opposite true? What is the optimal mix of capital income and wealth taxes when it is feasible to use both? While these and related questions dominate policy debates, some standard frameworks used by economists to study capital taxation are largely silent on them. This is because capital income and wealth taxation are equivalent under the standard assumption that all individuals earn the same rate of return on wealth. However, a growing body of empirical work documents large and persistent heterogeneity in returns across individuals, which challenges this assumption and opens the door for differences in the aggregate and distributional outcomes of these two forms of taxation.[1]

In this paper, we study capital income and wealth taxation when returns are heterogeneous across individuals. We establish conditions under which replacing capital income taxes with wealth taxes generates productivity and welfare gains. We also study the more general problem of the optimal mix of wealth and capital income taxes that maximizes average welfare. The framework we employ is fairly standard: an analytical model with entrepreneurs and workers subject to mortality risk, in which entrepreneurial firms differ in their productivity and are subject to collateral constraints, similar to the setup in Moll (2014) and many others.

Entrepreneurs produce a final good that is sold to consumers in a perfectly competitive market, using a common constant-returns-to-scale technology that combines capital and labor but they differ in their productivity. There is a bond market, with zero net supply, through which entrepreneurs can borrow from each other subject to a collateral constraint. Entrepreneurs with high productivity borrow to invest in their own firms, while those with low productivity lend at least part of their wealth. Workers are hand-to-mouth and consume their wages, so all the wealth is held by entrepreneurs. Upon death, entrepreneurs and workers are replaced by newborn individuals, who each inherit the same amount of wealth.

We show four main results. First, we demonstrate that under plausible parameter values, there exists a unique steady state equilibrium that is inefficient and exhibits capital misallocation. In this equilibrium, collateral constraints bind for more-productive entrepreneurs, who then earn higher returns on wealth than less productive ones. Return

[1]See Fagereng, Guiso, Malacrino and Pistaferri (2020), Bach, Calvet and Sodini (2020), Campbell, Ramadorai and Ranish (2019), and Smith, Zidar and Zwick (2021) for empirical evidence on persistent return heterogeneity. See Chari and Kehoe (1999), Golosov, Tsyvinski and Werning (2006), and Stantcheva (2020) for reviews of the literature on capital taxation.

heterogeneity makes this the empirically relevant equilibrium. Furthermore, it emerges as the only equilibrium in the economy (regardless of parameter values) when we endogenize productivity by introducing costly innovation effort by newborn entrepreneurs.

Second, we show that an increase in wealth taxes increases aggregate productivity (TFP). We focus on the reallocation effect of wealth taxes coming from the *use-it-or-lose-it* effect studied quantitatively in Guvenen, Kambourov, Kuruscu, Ocampo and Chen (2023). All the allocative effects of wealth taxes come from the change in after-tax returns as there is no behavioral response in the present model—saving rates are (endogenously) constant, thanks to the log utility assumption. A wealth tax triggers a reallocation of wealth because it places a similar tax burden on entrepreneurs with similar wealth levels regardless of their productivity, unlike a capital income tax that places a higher tax burden on more productive entrepreneurs.

We show that the use-it-or-lose-it effect operates by decreasing after-tax returns of entrepreneurs who earn below the wealth-weighted average return in the economy, while increasing the returns for those who earn above it. This increase in return dispersion allows high-productivity entrepreneurs to accumulate more wealth. Moreover, when capital income taxes decrease in response to the increase in wealth taxes, capital, output and wages increase.

Third, we study the welfare implications of a marginal increase in wealth taxes (matched by an adjustment in capital income taxes) and provide necessary and sufficient conditions for the change in taxes to increase welfare. Workers unambiguously benefit from higher wealth taxes through the rise in wages, while entrepreneurs on average have welfare losses because higher wealth taxes imply higher dispersion and lower expected value of returns. Thus, the aggregate welfare gains of an increase in wealth taxes depend on the strength of the increase in wages, which is determined by the output elasticity with respect to capital. We show that the conditions for welfare gains amount to a lower bound on this elasticity, which is close to one-third for most parameter combinations.

We also study how the welfare of entrepreneurs varies with their productivity. A higher wealth tax increases wealth accumulation, ameliorating (and potentially overturning) the welfare loss of entrepreneurs. High-productivity entrepreneurs benefit unambiguously from the increase in wealth taxes but low-productivity entrepreneurs generally lose.

Fourth, we study the optimal combination of capital income and wealth taxes. We derive an optimal tax formula for the wealth tax as a function of the output elasticity with respect to capital, α. The optimal tax weighs the benefit to workers and entrepreneurs from the increase in wages and capital against the cost to entrepreneurs from higher dispersion and lower expected value of returns. A larger value of α implies a larger response of wages and capital to increases in TFP coming from the increase in wealth taxes, resulting on a higher optimal wealth tax. Accordingly, we characterize optimal taxes as functions of a lower bound and an upper bound on the elasticity. If the elasticity is above the upper bound, the optimal wealth tax is positive and the capital income tax is negative (a subsidy), the signs flip when the elasticity is below the lower bound, and both taxes are positive in the narrow range between the thresholds.

We then study the effects of taxation on the extensive and intensive margin of entrepreneurial activity. First, to study the extensive margin, we endogenize entrepreneurial productivity as the outcome of a costly and risky innovation process, in which innovation effort at the outset can lead to either a high- or low-productivity technology. Optimal innovation effort depends on the dispersion of returns, with higher dispersion providing more incentives to exert effort. Consequently, an increase in wealth taxes increases innovation effort and, through it, the share of high-productivity entrepreneurs in the economy. This effect of wealth taxes on the extensive margin (more high-productivity entrepreneurs) increases the optimal wealth tax level.

To understand the effect on the intensive margin, we incorporate entrepreneurial effort in the entrepreneurs' production function. A capital income tax reduces the incentives for higher effort by taxing away the profits generated by the entrepreneur, while the wealth tax is independent of the entrepreneurs' production choices. As a result, increasing the wealth tax rate and reducing the capital income tax rate increases productivity and output.

Finally, we study the role of the persistence in entrepreneurial productivity for our results. We do this in an alternative model where infinitely-lived entrepreneurs are subject to idiosyncratic productivity shocks following a first-order Markov process. We find that all of our results go through if and only if entrepreneurial productivity is positively autocorrelated. When entrepreneurial productivity is persistent, the wealth share of more productive entrepreneurs increases over time when their returns increase, increasing aggregate productivity. This is precisely what happens when the wealth tax increases.[2]

Related literature. An important common element in most of the previous studies on capital taxation is the assumption of homogenous returns across the population. Because

[2]Our results build on those of Moll (2014) on the role of the persistence of entrepreneurial productivity in determining aggregate productivity. Higher persistence allows productive entrepreneurs to save and relax their financial constraints, increasing efficiency. We show that wealth taxes can achieve the same objectives through their heterogeneous effects across entrepreneurs (for a given degree of persistence).

capital income and wealth taxes are equivalent under this assumption, an analysis of the differences between the two taxes is naturally absent from this earlier literature, which focuses on capital income taxation (a short list includes Judd 1985; Chamley 1986; Aiyagari, 1995; Imrohoroglu, 1998; Erosa and Gervais, 2002; Garriga, 2003; Conesa, Kitao and Krueger, 2009; Kitao, 2010; Saez and Stantcheva, 2018; Straub and Werning, 2020). That said, a series of recent empirical papers analyze the behavioral savings response to changes in wealth taxes (Seim, 2017; Jakobsen, Jakobsen, Kleven and Zucman, 2019; Londoño-Vélez and Ávila-Mahecha, 2021; Ring, 2021; Brulhart, Gruber, Krapf and Schmidheiny, 2022).

By contrast, there have been few theoretical studies of wealth taxation (and its comparison to capital income taxation) when returns are heterogeneous and, to our knowledge, no analysis of the use-it-or-lose it effect of wealth taxes until very recently Guvenen et al. (2023).[3] Allais (1977) and Piketty (2014) are partial exceptions, they describe the use-it-or-lose-it mechanism but do not study it. Guvenen et al. (2023) build a rich overlapping generations model that matches the distribution of cross-sectional and lifetime rates of returns, as well as the extreme concentration and the Pareto tail of the wealth distribution. They show quantitatively that there are large efficiency and distributional welfare gains from using wealth taxes instead of capital income taxes.

Relative to Guvenen et al. (2023), here we consider an analytical framework—an infinite-horizon entrepreneur-worker model with heterogeneous/stochastic productivities, which has been widely used in the literature.[4] We use this set up to establish theoretically the conditions under which a wealth tax generates higher aggregate efficiency and welfare than a capital income tax, and vice versa. We also study the optimal mix of the two taxes, which is not studied in Guvenen et al. (2023). Overall, we show that efficiency and welfare gains from wealth taxation arise as a robust outcome under reasonable and large range of parameter values when there is return heterogeneity, and the optimal combination of capital income and wealth taxes depend on the elasticity of output with respect to capital.

Our focus on (persistent) return heterogeneity is motivated by strong empirical evidence for it (that we mentioned earlier) and theoretical work showing the importance of return heterogeneity for generating the dynamics and the Pareto tail of the wealth distribution

[3] Scheuer and Slemrod (2021) provide an excellent survey of the debate on the implementation and optimality of wealth taxes.

[4] This framework is closest to Moll (2014), Buera and Moll (2015) and Itskhoki and Moll (2019). Some quantitative papers that feature similar entrepreneurial heterogeneity and financial frictions are Quadrini (2000), Cagetti and De Nardi (2006), Buera, Kaboski and Shin (2011), and Boar and Midrigan (2020).

observed in the data (Benhabib, Bisin and Zhu 2011, 2013, 2014; Gabaix, Lasry, Lions and Moll 2016; Jones and Kim 2018; Stachurski and Toda 2019, among others).

2 Benchmark Model

Time is discrete. There are two types of agents: homogenous workers of size L and heterogenous entrepreneurs of size 1. Both are subject to mortality risk and die with a constant probability $1-\delta$. Workers' and entrepreneurs' preferences take the form

$$E_0 \sum_{t=1}^{\infty} (\beta\delta)^{t-1} \log(c_t), \tag{1}$$

where $\beta, \delta \in (0,1)$. Workers supply one unit of labor inelastically, behave as hand-to-mouth agents, and hold no wealth. Entrepreneurs have a permanent productivity type z that differs across agents. We assume that the wealth of the entrepreneurs that die is bequeathed equally among all newborn entrepreneurs. This makes the bequest, denoted by $\overline{a}$, equal to the average wealth in the economy because mortality risk is independent of the age, wealth, and productivity of entrepreneurs.

Each entrepreneur produces a homogenous good combining capital, k, and labor, n, using a constant-returns-to-scale technology

$$y = (zk)^{\alpha} n^{1-\alpha}. \tag{2}$$

We assume that capital does not depreciate.

Entrepreneurs hire labor at wage rate w and can borrow through a bond market at interest rate r to invest in their firm over and above their own wealth a. Both markets are perfectly competitive. The same bonds, which are in zero net supply, can be used as a savings device, which will be optimal for entrepreneurs whose return is lower than the interest rate r. Thus, k can be greater or smaller than a. However, entrepreneurs' borrowing is subject to a collateral constraint that depends on beginning-of-period wealth (a), so that

$$k \leq \lambda a, \tag{3}$$

where $\lambda \geq 1$. If $\lambda = 1$ an entrepreneur can use only their wealth in production.[5]

[5]This specification of collateral constraints is analytically tractable and can be motivated as resulting

The government taxes capital income at a rate τ_k and (beginning-of-period) wealth at a rate τ_a to finance an exogenous expenditure G.

2.1 Entrepreneur's Problem

Entrepreneurs choose capital and labor to maximize entrepreneurial income every period taking prices as given. The problem for an entrepreneur with productivity z and wealth a is

$$\Pi^{\star}(z, a) = \max_{k \leq \lambda a, n \geq 0} (zk)^{\alpha} n^{1-\alpha} - rk - wn. \tag{4}$$

which yields the following labor demand function:

$$n^{\star}(z, k) = \left(\frac{1-\alpha}{w}\right)^{1/\alpha} zk. \tag{5}$$

Substituting (5) into the profit, the optimal capital choice problem is

$$k^{\star}(z, a) = \operatorname*{argmax}_{0 \leq k \leq \lambda a} \left[\alpha \left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z - r\right] k. \tag{6}$$

The constant-return-to-scale technology with which the entrepreneur produces implies that entrepreneurs whose marginal return to capital $\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z$ is greater than the interest rate r borrow up to their limit and set $k^{\star} = \lambda a$, while those whose return is below the interest rate do not produce and instead earn the return r in the bond market on their wealth a. Consequently, the optimal entrepreneurial income can be written as $\Pi^{\star}(z, a) = \pi^{\star}(z) a$, where

$$\pi^{\star}(z) \equiv \begin{cases} \left(\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z - r\right)\lambda & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z > r \\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z \leq r \end{cases} \tag{7}$$

is the excess return an entrepreneur earns above the interest rate r.

We now turn to the entrepreneurs' optimal savings problem that takes as given the income generated from entrepreneurial activities and participation in the bond market.

from an underlying limited commitment problem. For other papers that use the same specification, see Banerjee and Newman (2003), Buera and Shin (2013), and Moll (2014), among others.

The problem is described by the following Bellman equation:

$$V(a,z) = \max_{a'} \log(c) + \beta\delta V(a',z) \tag{8}$$
$$\text{s.t. } c + a' = R(z)\,a,$$

where $R(z) \equiv (1-\tau_a) + (1-\tau_k)(r + \pi^\star(z))$ is the after-tax gross return on savings. In solving this problem, we take as given time-invariant taxes τ_a and τ_k, and prices r and w. Importantly, wealth taxes only take into account the beginning-of-period wealth and do not affect the income flow during the period (generated from lending in the bond market or from production). The solution is the following optimal savings rule

$$a' = \beta\delta R(z)\,a. \tag{9}$$

Importantly, the saving rate of entrepreneurs is constant and independent of their productivity. In particular, saving rates do not respond to policy, such as changes in wealth taxes, implying that all the reallocation effects of changes in taxation operate through their effect on returns. We present the details of these derivations in Appendix A.

2.2 Entrepreneurial Productivity

Entrepreneurial productivity can be high, z_h, or low z_ℓ. A share μ of entrepreneurs have high productivity and a share $1-\mu$ have low productivity. We start our analysis by taking the share of high-productivity entrepreneurs as given. Holding μ constant makes the implications of return heterogeneity for wealth and capital income taxation clear and tractable, as we show in Sections 3 and 4 below where we establish the effects of wealth taxation for productivity and welfare and characterize the optimal combination of wealth and capital income taxes.

The same mechanisms are present when we endogenize the distribution of productivity. In Section 4 we model entrepreneurial productivity as the outcome of a costly and risky innovation process, in which innovation effort can lead either to a high-productivity, z_h, or low-productivity, z_ℓ, technology. The probability of each depends on the innovation effort of entrepreneurs. In this way, the innovation effort choices of entrepreneurs determine the composition of the entrepreneurs in the economy. In equilibrium, there is a share μ of high-productivity entrepreneurs and a share $1-\mu$ of low-productivity entrepreneurs. These shares are taken as given by individuals when choosing production, savings, and effort, and so we are able to build upon the results of Sections 3 and 4 to establish the effects of taxation on innovation and aggregate productivity.

2.3 Bond Market Equilibrium

We now focus on the bond market in which entrepreneurs trade funds for production. The key variable for this purpose is the interest rate on bonds. For the capital market to clear, the equilibrium interest rate must be between the marginal return to capital of the low- and high-productivity entrepreneurs, i.e.,

$$\alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_\ell \quad \leq r \leq \quad \alpha\left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}} z_h. \tag{10}$$

The market has high-productivity entrepreneurs (weakly) demanding funds and low-productivity (weakly) supply them. The amount of funds demanded by high-productivity entrepreneurs is at most $(\lambda-1)\,\mu A_h$ and the amount of funds supplied by the low-productivity entrepreneurs is at most $(1-\mu)\,A_\ell$.[6]

When $(\lambda-1)\,\mu A_h < (1-\mu)\,A_\ell$, low-productivity entrepreneurs supply more funds than can be demanded by high-productivity entrepreneurs and therefore bid down the equilibrium interest rate to their marginal product (the lower bound in equation 10). High-productivity entrepreneurs are constrained, and their average capital is $K_h = \lambda A_h$. Low-productivity entrepreneurs are indifferent between using their assets in their firm and lending them in the bond market. Their average capital is $K_\ell = \frac{(1-\mu)A_\ell - (\lambda-1)\mu A_h}{1-\mu} > 0$. More importantly, returns are heterogeneous across entrepreneurs, with $R_h > R_\ell$, and there is capital misallocation as not all capital is being used by the high-productivity entrepreneurs.

We provide a necessary and sufficient condition for $(\lambda-1)\,\mu A_h < (1-\mu)\,A_\ell$ to hold in steady state in Proposition 1 and show that it is satisfied under a wide range of parameter values. Defining $s_h \equiv \frac{\mu A_h}{\mu A_h + (1-\mu)A_\ell}$ as the wealth share of high-productivity entrepreneurs, this condition can be stated as $s_h < {}^1/_\lambda$. We show in Section 4 that this is the only case under which an equilibrium can arise when we endogenize the share of high-productivity entrepreneurs. This is because return heterogeneity is necessary for incentivizing

[6]We can equivalently introduce a corporate sector that faces no collateral constraints and provides entrepreneurs with an alternative use for their wealth. The corporate sector's technology is $Y_c = (z_c K_c)^\alpha L_c^{1-\alpha}$. The marginal return of capital in the corporate sector imposes a lower bound on the equilibrium interest rate: $r \geq \alpha z_c \left(\frac{1-\alpha}{w}\right)^{\frac{1-\alpha}{\alpha}}$. If $z_\ell < z_c < z_h$ the corporate sector and the high-productivity entrepreneurs operate in equilibrium, while the low-productivity entrepreneurs do not produce and instead lend all of their assets. This delivers the same result as our benchmark model with z_c taking the role of z_ℓ.

entrepreneurs to exert innovation effort.[7]

2.4 Equilibrium Values and Aggregation

In equilibrium, the aggregate productivity of capital is endogenous and depends on the wealth distribution. We define the wealth-weighted productivity as

$$Z \equiv s_h z_\lambda + (1 - s_h)\, z_\ell, \tag{11}$$

where $z_\lambda \equiv z_h + (\lambda - 1)(z_h - z_\ell)$ denotes the *effective productivity of the wealth of high-productivity entrepreneurs* (i.e., the return they earn from their own wealth, captured by z_h, and the excess return from borrowed capital, $(\lambda - 1)(z_h - z_\ell)$). By contrast, high-productivity entrepreneurs use all the capital and $Z = Z^\star = z_h$ in an efficient allocation. [8]

We express all other aggregate variables as functions of aggregate capital $K \equiv \mu A_h + (1 - \mu) A_\ell$ and productivity Z (as in Moll, 2014). We report this result in Lemma 1.

Lemma 1. ***(Aggregate Variables in Equilibrium)*** *In the heterogenous-return equilibrium* $((\lambda - 1)\mu A_h < (1 - \mu) A_\ell)$*, output, wages, interest rate, and gross returns are:*

$$Y = (ZK)^\alpha L^{1-\alpha} \tag{12}$$

$$w = (1 - \alpha)\left({}^{ZK}/{}_{L}\right)^\alpha \tag{13}$$

$$r = \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} z_\ell \tag{14}$$

$$R_\ell = (1 - \tau_a) + (1 - \tau_k)\, \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} z_\ell \tag{15}$$

$$R_h = (1 - \tau_a) + (1 - \tau_k)\, \alpha \left({}^{ZK}/{}_{L}\right)^{\alpha-1} z_\lambda. \tag{16}$$

Remark. Defining "effective capital" as $Q \equiv \mu z_h K_h + (1 - \mu) z_\ell K_\ell$, aggregate output can be rewritten in the familiar Cobb-Douglas form as $Y = Q^\alpha L^{1-\alpha}$ as in Guvenen et al. (2023).

[7] When μ is taken as given it is also possible to study the case in which $(\lambda - 1)\mu A_h > (1 - \mu) A_\ell$. There, high-productivity entrepreneurs demand more funds than can be supplied by low-productivity entrepreneurs and therefore bid up the equilibrium interest rate to their marginal product (the upper bound in equation 10). In this case, all entrepreneurs would earn the same rate of return and the equilibrium aggregates coincide with the (efficient) complete markets allocation. This makes the value of high- and low-productivity technologies equal for entrepreneurs at birth, as they have the same initial wealth and returns, eliminating the incentives for innovation effort.

[8] We can measure the TFP loss due to the collateral constraints and the resulting misallocation of capital as the ratio $\frac{TFP^\star}{TFP} = \left(\frac{z_h}{s_h z_\lambda + (1 - s_h) z_\ell}\right)^\alpha$, which declines with the wealth share of high-productivity entrepreneurs, s_h, and the borrowing limit, λ, and increases with the productivity gap, $\frac{z_h}{z_\ell}$.

Evolution of aggregates. The evolution of each group's average wealth is given by

$$A_i^{'} = \delta^2 \beta R_i A_i + (1-\delta)\,\overline{a}, \tag{17}$$

where $\overline{a} \equiv K = (1-\mu)\, A_\ell + \mu A_h$ denotes the wealth endowment of a newborn entrepreneur, equal to the total (average) wealth in the economy. We later show that $\beta\delta R_\ell < 1 < \beta\delta R_h < {}^1\!/\!_\delta$, so that in equilibrium low types dissave and high types save, but not at rate that prevents the existence of a stationary equilibrium for the economy (Lemma 3).

From the evolution of the assets of low- and high-productivity entrepreneurs, we obtain the low of motion for aggregate capital:

$$\frac{K^{'}}{K} = \delta^2 \beta \left(s_h R_h + (1-s_h)\, R_\ell\right) + (1-\delta)\,, \tag{18}$$

which shows that the growth rate of aggregate capital depends on the wealth-weighted return.

2.5 Characterizing the Steady State

Steady state K: Imposing steady state on the law of motion for aggregate capital and using the expressions for R_ℓ and R_h from Lemma 1 gives a condition analogous to the steady state condition for the Neoclassical Growth Model, where the (after-tax) marginal product of capital is determined by the rate of inter-temporal discount:

$$(1-\tau_a) + (1-\tau_k)\,\alpha Z^\alpha \left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta}. \tag{19}$$

This condition characterizes the level of steady-state capital K given the level of aggregate productivity, Z, which is endogenous, so equation (19) is not enough on its own to characterize the steady state (see Moll, 2014). The implications of equation (19) are far-reaching: it pins down the wealth-weighted average return in the economy (i.e., the marginal return on capital), here given by $(1-\tau_a) + (1-\tau_k)\,\alpha Z^\alpha \left(\frac{K}{L}\right)^{\alpha-1}$, and through it the returns of individual entrepreneurs. The following lemma formalizes this point.

Lemma 2. *The steady-state after-tax returns are independent of the steady-state level of capital and of capital income taxes and satisfy*

$$R_\ell = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\ell}{Z} \quad \textit{and} \quad R_h = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z_\lambda}{Z}. \tag{20}$$

Moreover, the wealth-weighted return depends only on the entrepreneurial saving rate, $s_h R_h + (1 - s_h) R_\ell = {}^1\!/\!_{\beta\delta}$.

Crucially, the previous lemma implies that steady-state returns depend on the wealth tax rate τ_a and not on the capital income tax rate τ_k. This is because the level of capital adjusts in steady state so that the after-tax marginal product of capital is ${}^1\!/\!_{\beta\delta} - (1 - \tau_a)$, neutralizing the effects of τ_k beyond capital accumulation and ensuring that entrepreneurial returns are unaltered. By contrast, τ_a affects the steady state level of the marginal product of capital and through it the level and dispersion of entrepreneurial returns in steady state. In this sense, τ_k has no steady state distributional effects (in our linear, CRS, economy) while τ_a does.[9]

Steady state Z. We are now ready to find the steady state level of productivity Z. For this, we impose the steady state condition on the aggregate wealth of each type of entrepreneur in (17), which gives

$$A_i = \frac{1-\delta}{1-\delta^2\beta R_i}\overline{a}. \tag{21}$$

Substituting in the definition of $\overline{a} = (1-\mu) A_\ell + \mu A_h$, we obtain

$$1 = (1-\delta)\frac{1 - \delta^2\beta\left((1-\mu) R_h + \mu R_\ell\right)}{\left(1-\delta^2\beta R_\ell\right)\left(1-\delta^2\beta R_h\right)}. \tag{22}$$

We then use the steady state value of returns from (20) to obtain an equation determining steady state productivity:

$$\begin{aligned}\left(1 - \delta^2\beta\left(1-\tau_a\right)\right) Z^2 - \left[(1-\delta)\left(\mu z_\lambda + (1-\mu) z_\ell\right) + \delta\left(1 - \delta\beta\left(1-\tau_a\right)\right)\left(z_\lambda + z_\ell\right)\right] Z \\ + \delta\left(1-\delta\beta\left(1-\tau_a\right)\right) z_\ell z_\lambda = 0.\end{aligned} \tag{23}$$

The solution to this quadratic equation determines the steady state of the economy as

[9] These results reflect the fact that τ_k affects the marginal returns of capital for high- and low-productivity entrepreneurs proportionally, while τ_a affects gross-returns linearly and therefore has a disproportionate effect on returns of low-productivity entrepreneurs. We study these effects further in Lemma 4.

well as the upper bound on the collateral constraint parameter λ that ensures that the economy is in the heterogeneous returns equilibrium. Existence and uniqueness follow from analyzing the roots of equation (23). There is only one root (the largest) that satisfies $z_\ell < Z < z_\lambda$. Plugging this root into equation (19) pins down the steady state level of K. Hence, there is always a unique equilibrium.

For this equilibrium to feature return heterogeneity, with $R_h > R_\ell$, there must be misallocation, therefore Z is below its efficient level z_h. We obtain an upper bound on the collateral constraint parameter, $\overline{\lambda}$, that guarantees that $Z < z_h$. This upper bound turns out to be not only sufficient but also necessary for the result. The proofs for these and all other results are presented in the Appendix.

Proposition 1. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state productivity level, $Z \in (z_\ell, z_h)$, that features heterogenous returns ($R_h > R_\ell$) if and only if* $\lambda < \overline{\lambda} \equiv 1 + \frac{(1-\delta)(1-\mu)}{(1-\delta)\mu+\delta(1-\delta\beta(1-\tau_a))\left(1-\frac{z_\ell}{z_h}\right)}$.

Corollary 1. *The condition for the steady state to feature heterogeneous returns can be restated as an upper bound on wealth taxes for a given value of λ:*

$$\lambda < \overline{\lambda} \longleftrightarrow \tau_a < \overline{\tau}_a = 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu}{(\lambda-1)\left(1-\frac{z_\ell}{z_h}\right)}\right). \tag{24}$$

Proposition 1 provides an upper bound on λ given a wealth tax rate τ_a for the economy to be in the heterogeneous-return equilibrium. The same condition is equivalent to an upper bound on τ_a given λ (Corollary 1). Intuitively, neither λ nor τ_a can be too high for there to be heterogeneous returns because both make it harder for the condition $s_h < 1/\lambda$ to be satisfied. A higher λ increases funds demanded by high-productivity entrepreneurs, directly pushing the boundary of this constraint. It also increases z_λ, which in turn increases R_h, and (through equation 21) increases the wealth share of high-productivity entrepreneurs, s_h. An increase in τ_a increases the dispersion of (after-tax) returns and hence s_h, as we show in Proposition 2 below. Finally, a higher μ also increases s_h, thereby constraining both the maximum λ and τ_a that can support the heterogenous return equilibrium. In Section 4 where we study optimal taxation with endogenous innovation, the fraction μ will be an equilibrium object, which increases with the wealth tax, further tightening the range of wealth tax that can support the heterogenous-return equilibrium with innovation.

In the rest of the analysis, we assume that the economy we study has a λ value that satisfies the condition in (24) so that the steady state equilibrium without a wealth tax features return heterogeneity. Given this λ, we study the effects of wealth taxes on the steady state equilibrium of the economy for $\tau_a < \overline{\tau}_a$. Formally,

Assumption 1. *The value of λ is such that $\lambda < \overline{\lambda}\left(\tau_a = 0\right) = 1 + \frac{(1-\delta)(1-\mu)}{(1-\delta)\mu + \delta(1-\delta\beta)\left(1-\frac{z_\ell}{z_h}\right)}$.*

It is useful to calculate the maximum level of borrowing that is possible for a given value of $\overline{\lambda}$. One way to gauge this is to compare the entrepreneurial debt-to-GDP ratio from the model when we set $\lambda = \overline{\lambda}\left(\tau_a = 0\right)$ to the corresponding ratio in the data. Guvenen et al. (2023) calculate the aggregate business debt-to-GDP ratio for the US to be approximately 1.5. For comparison, in Figure 1, we plot $\overline{\lambda}\left(\tau_a = 0\right)$ and the debt-to-GDP ratio $\left({(\overline{\lambda}-1)\mu A_h}/{Y}\right)$in the model for different values of β and productivity dispersion, z_ℓ/z_h.[10] As can be seen here, the debt-to-GDP ratio associated with the $\overline{\lambda}$ limit is typically quite a bit higher than the data counterpart of 1.5 as long as households are not too impatient, implying that the λ values that deliver a heterogeneous-return equilibrium is not restrictive. For example, for $\beta = 0.98$, $\overline{\lambda} = 1.54$ and the debt-to-GDP ratio is 2.52.[11]

Steady state saving rates and the wealth distribution. Finally, we characterize gross saving rates in equilibrium and the implied wealth distribution among entrepreneurs. We show that low-productivity entrepreneurs dissave and high-productivity entrepreneurs save in equilibrium, but they do so at a rate lower than the rate at which they die (and their wealth is re-set) which ensures the existence of a stationary wealth distribution. This also implies that high-productivity entrepreneurs hold a disproportionate share of the wealth, $s_h > \mu$.

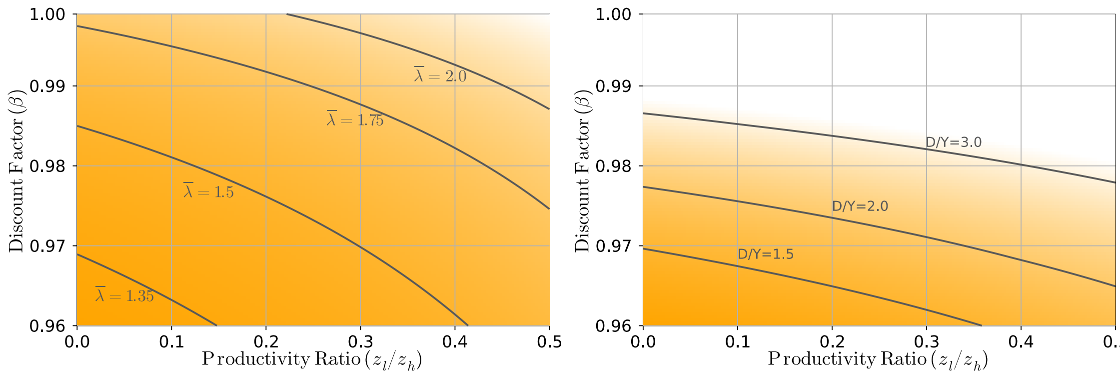
Lemma 3. ***(Saving and Dissaving in Steady State)*** *In steady state, the rates of return of low- and high-productivity entrepreneurs satisfy the following inequalities: $\beta\delta R_\ell < 1 < \beta\delta R_h < {1}/{\delta}$. Moreover, the wealth share of high-productivity entrepreneurs is higher than their population share: $s_h > \mu$.*

[10] $\overline{\lambda}$ increases with savings $\left({d\overline{\lambda}}/{d\beta} > 0\right)$ and decreases if the productivity gap between types widens $\left({d\overline{\lambda}}/{d(z_h/z_\ell)} < 0\right)$. We consider in Appendix F the behavior of $\overline{\tau}_a$: it gets tighter whenλ increases and gets looser when ${z_h}/{z_\ell}$ decreases.

[11] These results are not driven by unreasonable dispersion in returns. The return gap, $R_h - R_\ell$, is between 3 and 8 percentage points for the relevant combinations of parameters, see Figure 10, Appendix F. For our numerical examples, we first set parameters so that the share of entrepreneurs in the economy is 15 percent, the share of high-productivity entrepreneurs is 10 percent, the return gap is 5 percent, the debt-to-GDP ratio is 1.5, the average life-span of entrepreneurs is 50 years, and the effective discount rate is $\beta\delta = 0.96$.

Figure 1: Conditions for Steady State with Heterogeneous Returns

(a) Threshold $\overline{\lambda}$ (b) Debt-to-Output Ratio $\left(\lambda = \overline{\lambda}\right)$



Note: *Figure 1a reports the value of $\overline{\lambda}$ found in Proposition 1 for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). Figure 1b reports the debt-to-output ratio when $\lambda = \overline{\lambda}$ computed as $(\overline{\lambda}-1)A_h/Y$. In both figures we set the remaining parameters as follows: $\delta = 49/50$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

To derive the stationary distribution of assets, recall that all entrepreneurs are born with the same level of wealth $\overline{a}$ and save at a constant rate during their lifetimes. From the previous lemma, we know that high-productivity entrepreneurs save at a (gross) rate $\beta\delta R_h > 1$, and low-productivity entrepreneurs dissave at a (gross) rate $\beta\delta R_\ell < 1$. So, in the stationary equilibrium the wealth distribution of high-types is discrete and has support in the interval $[\overline{a}, \infty)$, with endogenous mass points at $\left\{\overline{a}, \beta\delta R_h\overline{a}, \left(\beta\delta R_h\right)^2\overline{a}, \ldots\right\}$, and the distribution of low-types has support in the interval $(0, \overline{a}]$, with mass points at $\left\{\overline{a}, \beta\delta R_\ell\overline{a}, \left(\beta\delta R_\ell\right)^2\overline{a}, \ldots\right\}$.

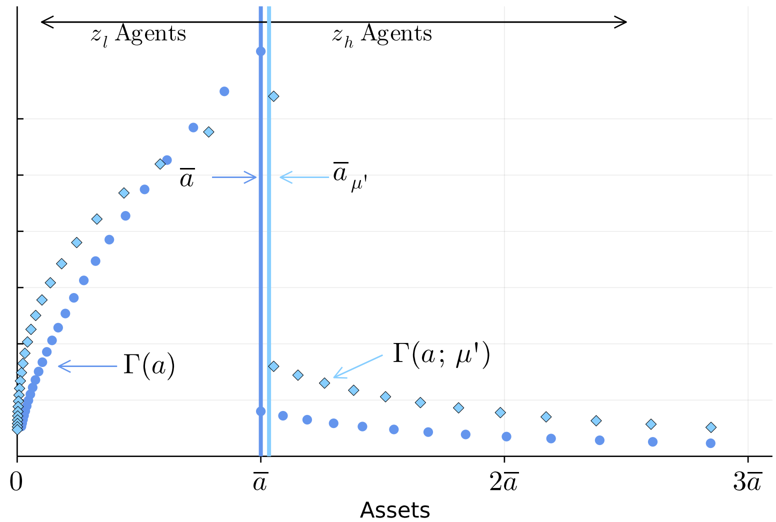
The share of entrepreneurs of type i with wealth $a = \left(\beta\delta R_i\right)^t\overline{a}$ is given by the share of agents who have lived exactly t periods:

$$\Gamma_i\left(\left(\beta\delta R_i\right)^t\overline{a}\right) = \Pr\left(z = z_i\right)\Pr\left(\text{age} = t\right) = \Pr\left(z = z_i\right)\delta^t\left(1-\delta\right). \tag{25}$$

So, the distribution of wealth is a geometric distribution with parameter δ.[12] Figure 2 illustrates the resulting stationary wealth distribution and what happens to it as μ increases. All entrepreneurs start with wealth $\overline{a}$. The two tails of the distribution depend on the returns of low- and high-productivity entrepreneurs. As μ increases, the average wealth in the economy increases, raising $\overline{a}$ and shifting the distribution to the right. This shifts all of the mass points and the initial mass of high-productivity entrepreneurs, the mass at

[12]The characterization of the stationary distribution of assets mimics the derivations in Jones (2015) adapted to the discrete time setting.

Figure 2: Stationary Wealth Distribution



Note: *The figure shows the shape of the stationary wealth distribution for a given level of μ in circles. The vertical lines mark the level of $\overline{a}$. The wealth distribution of low-productivity entrepreneurs is to the left of $\overline{a}$ as they dissave and the distribution of high-productivity entrepreneurs is to the right. The figure also shows the distribution under a higher level of μ in diamonds. A higher level of μ increases the initial asset level, $\overline{a}_{\mu'}$ and shifts the distribution.*

all other points is proportional to it. The increase in μ also reduces returns as seen the expression for returns in Lemma (4) below. This means that low-productivity entrepreneurs de-accumulate assets faster (increasing the distance between mass points) and that high-productivity entrepreneurs accumulate assets slower (decreasing the distance between mass points) with a higher μ.

The characterization of the stationary distribution allows us to define a convenient measure of wealth concentration. Because wealth is determined by productivity and age, we can define the top wealth share as the fraction of wealth held by high-productivity entrepreneurs above an age t. This corresponds to the wealth share of the top $100 \times (1-\delta)\sum_{s=t}^{\infty} \delta^s = 100 \times \delta^t$ percent. Their total wealth is given by

$$A_{h,t} \equiv (1-\delta) \sum_{s=t}^{\infty} \left(\beta\delta^2 R_h\right)^s \mu\overline{a} = \left(\beta\delta^2 R_h\right)^t \mu A_h. \tag{26}$$

Then the top wealth shares are given by

$$s_{h,t} \equiv \frac{\left(\beta\delta^2 R_h\right)^t \mu A_h}{K} = \left(\beta\delta^2 R_h\right)^t s_h. \tag{27}$$

3 Steady State Effects of Wealth Taxes

In this section, we consider the effect of increasing the wealth tax on the steady state values of key variables. The results here are global in nature—they hold for any starting level of $\tau_a < \overline{\tau}_a$. We study the optimal combination of capital income and wealth taxes that maximizes average welfare in Section 4. We abstract from other taxes and transfers to focus on the trade-offs between these two taxes.

3.1 The Effect of Wealth Taxes on Aggregate Productivity and Returns

Because the capital income tax τ_k does not affect Z, we can study the effect of changing τ_a on Z without needing to specify the government budget. As mentioned earlier, we set $\lambda < \overline{\lambda}\,(\tau_a = 0)$ so that the benchmark economy without wealth taxes features heterogeneous returns and analyze the effect of increasing the wealth tax for $\tau_a < \overline{\tau}_a$. We show that an increase in wealth taxes raises steady-state productivity Z. The proof follows from studying the properties of the quadratic equation (23) that determines the steady state value of productivity. [13]

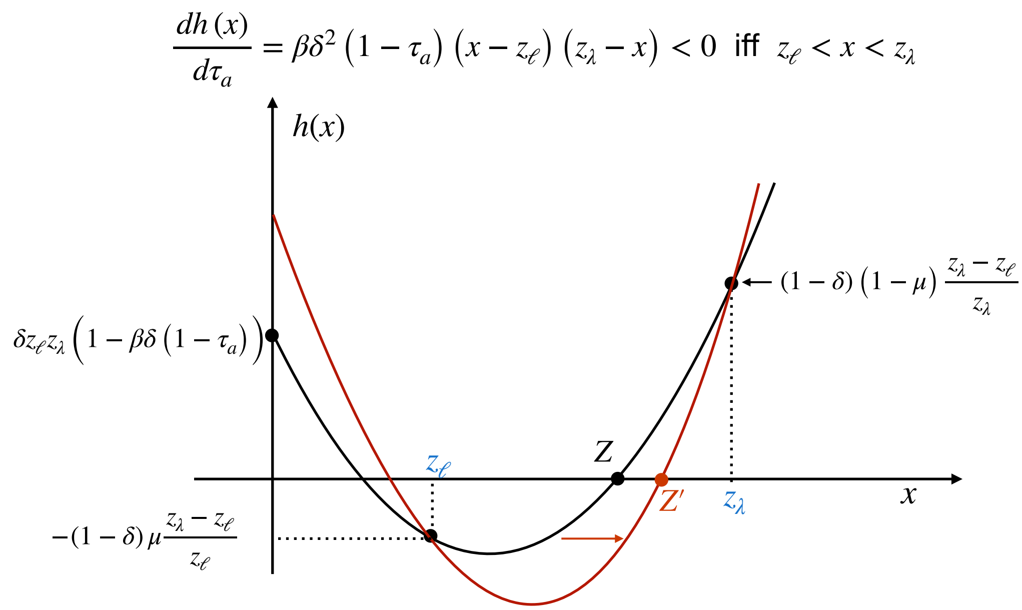
Proposition 2. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, an increase in wealth taxes (τ_a) increases productivity, $\frac{dZ}{d\tau_a} > 0$.*

We revisit this result in Section 4 where the share of high-productivity entrepreneurs also responds to taxes. We show that innovation effort also increases in response to an increase in wealth taxes, increasing μ and strengthening the response of aggregate productivity.

Higher productivity is necessarily a consequence of the reallocation of wealth towards high-productivity entrepreneurs, that is, an increase in s_h (see equation 11). Thus, the increase in Z implies that s_h must have increased. In the model, this reallocation is the consequence of wealth taxes changing entrepreneurial returns. However, the ex-ante direction of the change in after-tax returns is not immediate because it involves two effects: a direct use-it-or-lose-it effect due to the rise in the wealth tax and a general equilibrium effect triggered by an increase in the effective capital stock $Q = ZK$ (for given

[13] As Figure 3 shows, an increase in τ_a shifts the steady state value of Z to the right, marked by the largest root of equation (23). Geometrically, an increase in τ_a increases the y-intercept of h, defined in Figure 3. The values of the parabola are fixed at z_ℓ and z_λ, forcing the x-intercepts (the roots of h) to shift right.

Figure 3: Efficiency Gains from Wealth Taxation



Note: *The figure plots* $h(x) = \left(1 - \delta^2\beta\left(1 - \tau_a\right)\right)x^2 - \left[(1 - \delta)\left(\mu z_\lambda + (1 - \mu)\, z_\ell\right) + \delta\left(1 - \delta\beta\left(1 - \tau_a\right)\right)\left(z_\lambda + z_\ell\right)\right]x + \delta\left(1 - \delta\beta\left(1 - \tau_a\right)\right) z_\ell z_\lambda = 0$ *for two levels of wealth tax. The black line represents the initial value and the red line a higher value of the tax. The steady state productivity corresponds to the larger root of* h*, marked with a circle on the horizontal axis.*

K), which reduces returns because of decreasing marginal returns to capital. We can decompose the change in after-tax returns into the two effects using equation (20):

$$\frac{dR(z)}{d\tau_a} = \underbrace{\left(\frac{z}{Z} - 1\right)}_{\text{use-it-}\ell\text{ose-it}\lesseqgtr 0} \underbrace{-\left(\frac{1}{\beta} - \left(1 - \tau_a\right)\right)\frac{z}{Z^2}\frac{dZ}{d\tau_a}}_{\text{G.E. effect}<0} \tag{28}$$

While the general equilibrium effect is always negative, the sign of the direct use-it-or-lose-it effect depends on entrepreneurial productivity z.

We show that the use-it-or-lose-it effect is positive for high-productivity entrepreneurs and that their returns increase in response to higher wealth taxes. By contrast, the use-it-or-lose-it effect is negative for low-productivity entrepreneurs and their returns decrease. These changes increase the dispersion of returns as we show in Lemma 4 below. The higher dispersion in returns translates into higher wealth accumulation by the high-productivity entrepreneurs, so the top wealth shares, $s_{h,t} = \left(\beta\delta^2 R_h\right)^t s_h$, also increase. Importantly, these results do not depend on how (or whether) the government's budget is balanced. This is because the level of capital adjusts in steady state according to equation (19) in such a way that returns depend only on productivity and the wealth tax (Lemma 2).

We also show that these changes necessarily decrease the (population-weighted) average return among entrepreneurs, also making the average elasticity of returns with respect to productivity negative. These changes in returns shape the welfare consequences of an increase in wealth taxes by lowering the welfare of newborns entrepreneurs, as we show in Section 4.

Lemma 4. ***(Wealth Shares and Return Dispersion in Steady State)*** *The steady-state wealth share of high-productivity entrepreneurs and entrepreneurial returns satisfy*

$$s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} > \mu \qquad \frac{ds_h}{dZ} > 0 \tag{29}$$

$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\,\mu}{s_h}\right) \qquad \xi^Z_{R_h} \equiv \frac{d\log R_h}{d\log Z} > 0 \tag{30}$$

$$R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\,(1-\mu)}{(1-s_h)}\right) \qquad \xi^Z_{R_\ell} \equiv \frac{d\log R_\ell}{d\log Z} < 0. \tag{31}$$

Moreover, (population-weighted) average returns decrease with the wealth tax; the average elasticity of returns with respect to Z (and hence, with respect to the wealth tax) is negative,

$$\frac{d\left(\mu R_\ell + (1-\mu)\,R_h\right)}{d\tau_a} < 0 \qquad \textit{and} \qquad \mu\xi^Z_{R_h} + (1-\mu)\,\xi^Z_{R_\ell} < 0; \tag{32}$$

and the top wealth shares $s_{h,t}$ defined in (27) increase.

Next, we turn to the response of aggregate variables to changes in equilibrium Z (and hence to τ_a).

3.2 The Effect of Wealth taxes on Aggregate Variables

In order to study the effects of the wealth tax on aggregate variables, we need to specify the government budget and how it is balanced with the capital incomes tax.

Government Budget

The government uses capital income and wealth tax revenues to finance non-productive government expenditures G.

$$G \;=\; \tau_k\alpha Y + \tau_a K \;=\; \left(\tau_k + \tau_a\frac{\beta\delta\,(1-\tau_k)}{1-\beta\delta\,(1-\tau_a)}\right)\alpha Y. \tag{33}$$

Next, we make an assumption that greatly simplifies our upcoming analysis.

Assumption 2. *G is a constant fraction $\theta\alpha$ of aggregate output: $G = \theta\alpha Y$.*

Under Assumption 2, equation (33) implies a tight link between τ_k and τ_a:[14]

$$\frac{1-\theta}{1-\beta\delta} = \frac{1-\tau_k}{1-\beta\delta\,(1-\tau_a)}. \tag{34}$$

Although Assumption 2 requires the tax revenue to increase with the size of the economy, we show that increasing the wealth tax still delivers a higher output while meeting the increased revenue requirements. The output gains we find would have likely been higher if we had imposed revenue neutrality. A special case worth highlighting is when $\theta = 0$: there are no revenue requirements and taxation only serves to redistribute or increase productivity. In this case, it must be that either $\tau_k \geq 0$ and $\tau_a \leq 0$ or $\tau_k \leq 0$ and $\tau_a \geq 0$, with no taxation being feasible, $\tau_k = \tau_a = 0$.

We can now turn to the characterization of the response of aggregates to changes in the wealth tax. We do this indirectly, by first obtaining the responses to changes in productivity and then relating to wealth taxes thanks to Proposition 2. As expected, higher productivity increases capital accumulation, output, and wages. This holds true after taking into account the balancing of the government budget as in Assumption 2, so that the capital income tax adjusts in response to changes in the wealth tax. This implies that the increase in productivity and aggregates happens even as the total revenue collected by the government increases. Finally, we show that the wealth of high-productivity entrepreneurs increases, while the effect on low-productivity entrepreneurs is ambiguous (as they are born with higher wealth but dissave at a higher rate).

Lemma 5. ***(Aggregate Variables in Steady State)*** *If $\tau_a < \overline{\tau}_a$ and under Assumption 2, the steady-state level of aggregate capital is*

$$K = \left(\alpha \frac{\beta\delta\,(1-\theta)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}} L \tag{35}$$

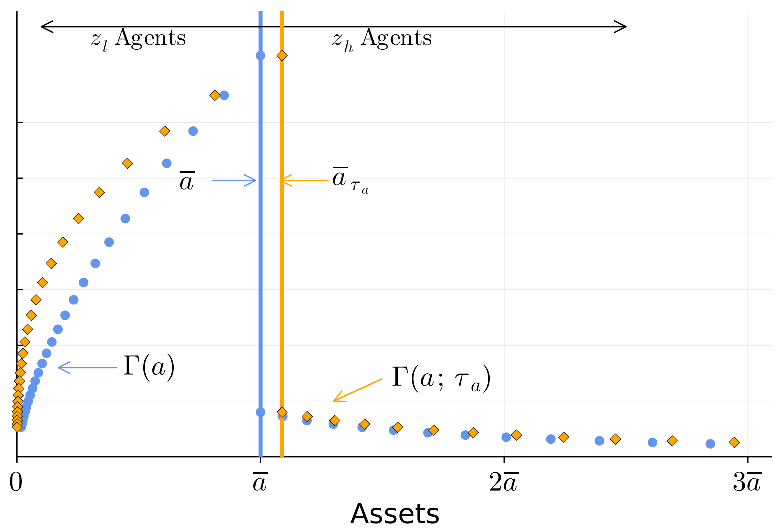
and the steady-state elasticities of aggregate variables with respect to productivity are

$$\xi_K^Z = \xi_Y^Z = \xi_w^Z = \xi \equiv \frac{\alpha}{1-\alpha}, \tag{36}$$

where $\xi_x^Z \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z.

[14] $\tau_k = \theta$ if there is only a capital income tax ($\tau_a = 0$), and $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$ if there is only wealth tax ($\tau_k = 0$).

Figure 4: Stationary Wealth Distribution and Wealth Taxes



Note: *The figure shows the shape of the stationary wealth distribution for an economy without wealth taxes in blue circles and an economy with wealth taxes (τ_a) in orange diamonds. The vertical lines mark the level of $\overline{a}$ in the respective economy. The wealth distribution of low-productivity entrepreneurs is to the left of $\overline{a}$ as they dissave and the distribution of high-productivity entrepreneurs is to the right. The wealth tax economy has a higher level of $\overline{a}$ and different mass-points for the distribution as a result. The share of high-productivity entrepreneurs, μ is held constant.*

Moreover, the wealth level of each entrepreneurial type in steady state is

$$A_h = \frac{1}{\mu}\frac{Z - z_\ell}{z_\lambda - z_\ell}K \qquad \frac{dA_h}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(Z - \alpha z_\ell\right) > 0 \tag{37}$$

$$A_\ell = \frac{1}{1-\mu}\frac{z_\lambda - Z}{z_\lambda - z_\ell}K \qquad \frac{dA_\ell}{dZ} \propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(\alpha z_\lambda - Z\right), \tag{38}$$

where $\frac{dA_\ell}{dZ} < 0$ *if and only if* $\alpha z_\lambda < Z$.

Remark. We can interpret the condition $\alpha z_\lambda < Z$ as a threshold level for α because the steady-state level of Z is independent of α (see equation 23).

The previous results also imply that the whole wealth distribution shifts after an increase in the wealth tax, reflecting the increase in aggregate wealth and the change in returns. The change in $\overline{a}\,(= K)$ impacts all mass points (which are proportional to $\overline{a}$), shifting them to the right. They are further affected by the compounding effect of returns—see Lemma 4. The resulting shift is shown in Figure 4. When we take into account the changes in innovation effort there is additional change in the distribution following an increase in wealth taxes as the share of high-productivity entrepreneurs increases, shifting the mass of the distribution towards them as in Figure 2.

3.3 The Effect of Wealth Taxes on Individual Welfare

An increase in the wealth tax increases the welfare of both workers and high-productivity entrepreneurs, as the higher wealth tax increases wages, average wealth, and the returns of high-productivity entrepreneurs. The effects on the welfare of low-productivity entrepreneurs and on entrepreneurs as a group depend on the relative strength of two countervailing forces: the increase in the initial wealth of entrepreneurs versus the decline in their returns. The results follow from the value of workers,

$$V_w = \frac{1}{1-\beta\delta} \log w, \tag{39}$$

and the solution to the entrepreneurs' problem in (9), which imply that the value of an entrepreneur with productivity z_i, $i \in \{\ell, h\}$, and assets a is give by

$$V_i(a) = \frac{1}{1-\beta\delta} \log(a) + \frac{1}{(1-\beta\delta)^2} \left[\log(\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta} + \log R_i\right]. \tag{40}$$

This gives rise to the following result characterizing the conditions for welfare gains after an increase in wealth taxes (holding μ fixed).

Proposition 3. *For all $\tau_a < \overline{\tau}_a$ and $\mu \in (0,1)$, an increase in the wealth tax increases the welfare of workers, ${}^{dV_w}\!/_{d\tau_a} > 0$, and the welfare of high-productivity entrepreneurs, ${}^{dV_h(\overline{a})}\!/_{d\tau_a} > 0$. Moreover, it increases the welfare of low-productivity entrepreneurs, ${}^{dV_\ell(\overline{a})}\!/_{d\tau_a} > 0$, if $\xi_K > \frac{-1}{1-\beta\delta}\xi_{R_\ell}$, and the ex ante welfare of newborn entrepreneurs, ${}^{d(\mu V_h(\overline{a})+(1-\mu)V_\ell(\overline{a}))}\!/_{d\tau_a} > 0$, if $\xi_K > \frac{-1}{1-\beta\delta}\left(\mu\xi_{R_h} + (1-\mu)\xi_{R_\ell}\right)$, where $\xi_K = {}^{\alpha}\!/_{1-\alpha}$ and ξ_{R_i} are the elasticities of capital and returns with respect to productivity.*

The welfare change of low-productivity entrepreneurs and entrepreneurs as a group are ambiguous because of two countervailing forces that are already apparent in (40): an increase in wealth taxes increases the initial wealth of entrepreneurs, lemma 5, but decreases the returns of low-productivity entrepreneurs as well as the average returns of entrepreneurs, Lemma 4. Both effects operate through the pass-through of productivity, Z, to assets and returns, captured by the elasticities ξ_K, ξ_{R_ℓ}, and ξ_{R_h}. If the pass-through to capital is sufficiently high, entrepreneurs overall benefit from the increase in wealth taxes, despite the decrease in returns, that is, if α is sufficiently high. In this way, the conditions established in the previous proposition imply threshold values for α above which the welfare of entrepreneurs increases. However, for a range of plausible parameter values, these thresholds turn out to be too high, with α having to be above 0.7. See Figure 11 in Appendix F.

4 Optimal Taxation

The government is benevolent and its objective is to maximize the steady-state utilitarian welfare of the newborns, $\mathcal{W}$, by choosing the optimal combination of capital income and wealth taxes, subject to the government's budget constraint:

$$\max_{\tau_k,\tau_a} \mathcal{W} \qquad \text{s.t. (33)} \tag{41}$$

Letting $n_w \equiv L/(1+L)$ represent the fraction of workers in the population, the government's objective function is

$$\mathcal{W} \equiv n_w V_w(w) + (1-n_w)(\mu V_h(\overline{a}) + (1-\mu) V_\ell(\overline{a})). \tag{42}$$

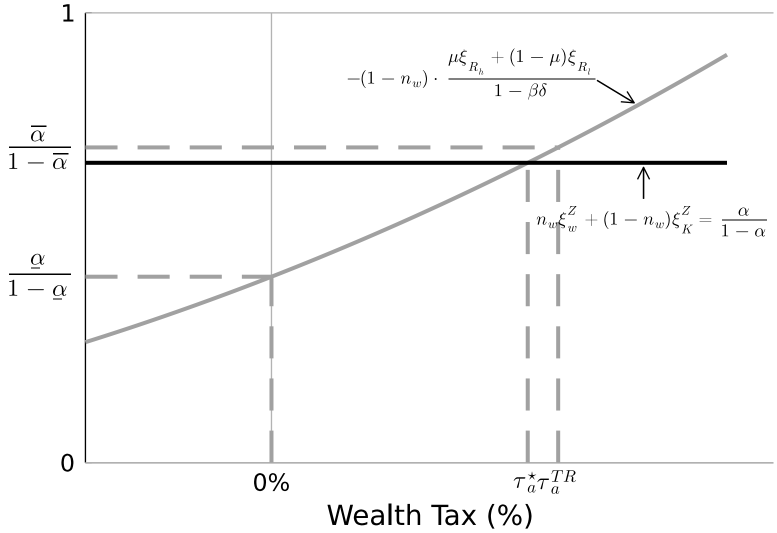
We can make the trade-off faced by the government clearer by substituting in the value functions of workers and entrepreneurs from (39) and (40):

$$\mathcal{W} = \frac{1}{1-\beta\delta}(n_w \log w + (1-n_w)\log \overline{a}) + \frac{1-n_w}{(1-\beta\delta)^2}(\mu \log R_h + (1-\mu)\log R_\ell) + v, \tag{43}$$

where $v \equiv \frac{1-n_w}{(1-\beta\delta)^2}\log(\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta}$ is a constant. Increasing the wealth tax (while simultaneously reducing the capital income tax (as in 34) affects aggregates through its effect on aggregate productivity and leads to higher wages and wealth (lemma 5). We call this the level effect of wealth taxation, but it also results in lower average log returns as noted above (Lemma 4), the return dispersion effect. An interior solution balances these trade-offs and satisfies ${}^{d\mathcal{W}}/_{d\tau_a} = 0$, where ${}^{d\mathcal{W}}/_{d\tau_a}$ depends on the elasticities of aggregates with respect to Z.

Figure 5 illustrates the forces at play. The elasticities of wages and wealth with respect to productivity $(n_w\xi_w + (1-n_w)\xi_K)$ give the (percentage) gain in workers' and entrepreneurs' welfare as the wealth tax increases (raising productivity). These elasticities are constant in our economy as shown in Lemma 5 and are both equal to ${}^{\alpha}/_{(1-\alpha)}$. At the same time, the (negative) average elasticity of returns $\left(-\left(\mu\xi^Z_{R_h}(\tau_a) + (1-\mu)\xi^Z_{R_\ell}(\tau_a)\right)\right)$ is increasing in τ_a, reflecting the widening gap between low and high returns as the wealth tax increases. The intersection of the two lines pins down the optimal wealth tax. We formalize this in the

Figure 5: Optimal Wealth Tax



Note: *The figure shows the conditions satisfied by the optimal wealth tax solving (41). The horizontal line is the (population) average of the elasticity of wages and capital with respect to productivity, ξ_w and ξ_K respectively. The increasing line is proportional to the negative of the average elasticity of returns with respect to productivity (ξ_R). $\tau_a^\star$ denotes the optimal wealth tax. $\tau_a^{TR} = {}^{\theta(1-\beta\delta)}/_{\beta\delta(1-\theta)}$ denotes the tax reform tax, the level at which $\tau_k = 0$. The remaining parameters are as follows: $\delta = {}^{49}/_{50}$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

following proposition.

Proposition 4. ***(Optimal Taxes)*** *Under Assumption 2, there exists a unique tax combination $(\tau_a^\star, \tau_k^\star)$ that maximizes the utilitarian welfare. An interior solution $\tau_a^\star < \overline{\tau}_a$ is the solution to:*

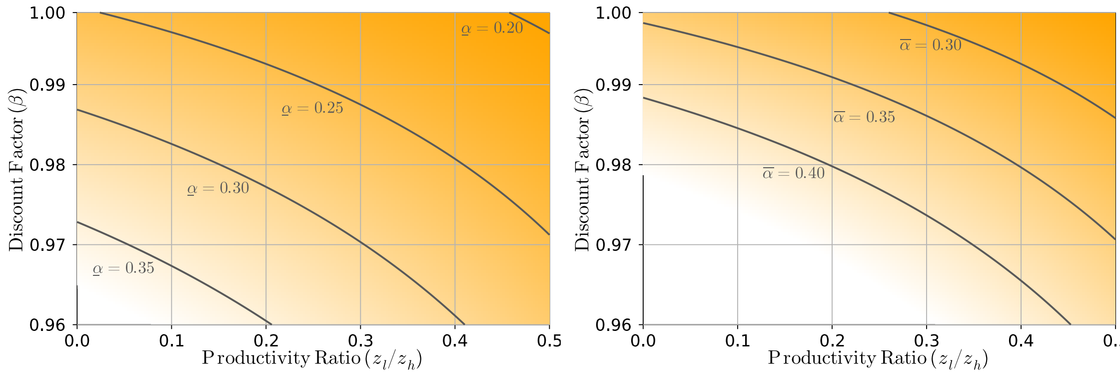
$$0 = \left[\underbrace{n_w\xi_w^Z + (1-n_w)\,\xi_K^Z}_{(+)} + \frac{1-n_w}{1-\beta\delta}\left(\underbrace{\mu\xi_{R_h}^Z(\tau_a) + (1-\mu)\,\xi_{R_\ell}^Z(\tau_a)}_{(-)}\right)\right]\underbrace{\frac{d\log Z}{d\tau_a}}_{(>0)} \tag{44}$$

where $\xi_x^Z \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there are two cutoff values for α, $\underline{\alpha}$ and $\overline{\alpha}$, such that $(\tau_a^\star, \tau_k^\star)$ has the following properties:

$$\begin{aligned}
\tau_a^\star &\in \left[1-\frac{1}{\beta\delta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha}\\
\tau_a^\star &\in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0,\theta] && \text{if } \underline{\alpha} \le \alpha \le \bar{\alpha}\\
\tau_a^\star &\in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha}
\end{aligned}$$

Figure 6: α Thresholds for Optimal Wealth Taxes

(a) Lower Threshold $\underline{\alpha}$ for $\tau_a^\star > 0$ (b) Upper Threshold $\overline{\alpha}$ for $\tau_k^\star < 0$



Note: *The figures report the threshold value of α for the optimal wealth taxes to be positive (left) and capital income taxes to be positive (right) for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

where $\tau_a^{\max} \geq 1$, $\underline{\alpha}$ and $\overline{\alpha}$ are the solutions to equation (44) with $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$, respectively.

Figure 5 also clarifies the roles of the thresholds $\underline{\alpha}$ and $\overline{\alpha}$.[15] The lower threshold $\underline{\alpha}$ marks the level of $n_w\xi_w + (1-n_w)\,\xi_K$ for which $\tau_a = 0$ is optimal. Any $\alpha > \underline{\alpha}$ implies a higher scope for wages and capital to rise with the wealth tax and thus a positive optimal wealth tax. The upper threshold $\overline{\alpha}$ is similarly defined by the level of $n_w\xi_w + (1-n_w)\,\xi_K$ for which

$$\tau_a = \tau_a^{TR} \equiv {}^{\theta(1-\beta\delta)}\!/_{\beta\delta(1-\theta)}$$

is optimal. At that level, the wealth tax finances all government spending, so $\tau_k = 0$. Consequently, any $\alpha > \overline{\alpha}$ implies that the optimal tax combination is one of wealth taxes and capital income subsidies. Finally, the upper bound on the wealth tax ($\tau_a^{\max}$) ensures that R_ℓ remains positive.

Figure 6 shows how the thresholds for α vary with parameters. In Figure 12 in Appendix F, we report how and the optimal wealth tax level changes parameters.

Finally, taking the economy with $\tau_a = 0$ as our benchmark, we would like to compare equilibrium outcomes under the optimal tax economy to those in the benchmark economy for different α values. When $\alpha < \underline{\alpha}$, it optimal to set a capital income tax that is higher than

[15] The value of the thresholds depend on Z, which is endogenous but independent of α (equation 23), so they can be used to define the threshold.

the benchmark level. The revenue raised with the optimal capital income tax is enough to finance government spending and to implement a subsidy on wealth. In this case, workers' wages are lower than in the benchmark economy. Therefore, they suffer welfare losses. At the same time, the dispersion in returns is lower and mean returns are higher than their corresponding benchmark values. As a result, entrepreneurs benefit under the optimal tax system relative to the benchmark. When $\alpha \geq \underline{\alpha}$, the optimal wealth tax is positive, and the optimal capital income tax is lower than in the benchmark. Workers' welfare is higher than in the benchmark while entrepreneurs' welfare is lower. For higher α values, these effects are even more pronounced.Wealth Taxation with Innovation

We now endogenize the productivity of entrepreneurs as the outcome of costly and risky innovation process. We assume that newborn innovators come up with new ideas for production. The quality of these ideas is captured by the productivity, z, of the technology they describe. Once an idea is generated, the innovator uses it to produce and has access to it for the rest of their lifetime (akin to having a perpetual patent), just as in the model presented in Section 2.

Innovation requires costly effort. Crucially, innovation is a risky endeavor, and the innovators' effort is not guaranteed to grant them success. Instead, effort affects the probability that the innovator's idea is successful. Specifically, an idea can turn into either a high-productivity or low-productivity technology. The probability of a high-productivity technology, $p(e)$, is therefore a function of the effort, e put in by an innovator.

Therefore, the innovators' problem is

$$\max_{e} \; p(e) V_h(\overline{a}) + (1 - p(e)) V_\ell(\overline{a}) - \frac{1}{(1-\beta\delta)^2}\Lambda(e), \tag{45}$$

where Λ is a strictly increasing, strictly convex, and twice continuously differentiable cost function for effort with $\Lambda(0) = 0$ and $\Lambda'(0) = 0$. The resulting value of an idea is given by $V_i(a) = m_i + n\log(a)$, where $n = \frac{1}{1-\beta\delta}$ and

$$m_i = \frac{1}{(1-\beta\delta)^2}\left[\beta\delta\log\beta\delta + (1-\beta\delta)\log(1-\beta\delta) + \log R_i\right].$$

To simplify, we set $p(e) = e$ without loss of generality. An interior effort choice is characterized by[16]

$$\Lambda'(e) = (1-\beta\delta)^2 \left(V_h(\overline{a}) - V_\ell(\overline{a})\right) = \log R_h - \log R_\ell. \tag{46}$$

After replacing the value of entrepreneurs from equation (40), we find that the effort choice depends on the dispersion of (log) returns, with returns as in equation (20). Because the returns are a function of aggregate productivity Z given a wealth tax rate τ_a, the optimal effort decision rule will be a function of Z.

4.1 Steady State with Innovation

All innovators are identical (at birth) and therefore make the same choices. This makes $\mu \equiv p(e) = e$ the share of high-productivity entrepreneurs in the economy. The rest of the economy is characterized as in Section 2. The equilibrium can be stated as a fixed point in μ. Given μ, the equilibrium productivity Z is determined by the same quadratic equation (23) above. When making their innovation effort choice, individual innovators take the share of high-productivity entrepreneurs, μ, as well as the associated Z and the implied returns R_h and R_ℓ as given. We next present the equilibrium definition.

Definition 1. (Steady State $\mu^\star$) For a given wealth tax level $\tau_a \leq \overline{\tau}_a$, the steady state share of high-productivity entrepreneurs, $\mu^\star$, is determined by the solution to

$$\mu^\star = e\left(Z\left(\mu^\star\right)\right), \tag{47}$$

where

i. $Z(\mu)$ gives the steady state productivity given μ, that is $Z(\mu)$ is the solution to equation (23), for a given $\mu \in (0,1)$; and

ii. $e(Z)$ gives the optimal innovation effort given steady state productivity Z, that is $e(Z)$ solves equation (46) for a given $Z \in (z_\ell, z_h]$, where equilibrium returns are given by equation (20).

A couple of remarks are in order. First, there are no equilibria with innovation in which returns are homogenous. This is because when the dispersion in returns is zero, the optimal innovation effort and the share of high-productivity entrepreneurs are also zero, as implied

[16]We assume that the cost function Λ is such that a corner solution is never optimal. This is done by evaluating the equation at $Z \in \{z_\ell, z_h\}$ and ensuring that the solution is interior in both cases.

by equation (46). This, in turn, implies that there is no demand for funds coming from high-productivity entrepreneurs, which would push the economy to the heterogeneous-returns equilibrium, leading to a contradiction Thus, all equilibria must feature return heterogeneity.

Second, for the economy to be in equilibrium, the condition in Proposition 1 must be satisfied. The condition can be restated as an upper bound on τ_a given the value of the model's parameters. However, this upper bound depends on the share of high-productivity entrepreneurs, which is now endogenous. As τ_a is higher, entrepreneurs exert more effort, increasing the share of high-productivity entrepreneurs and overall productivity, as we will show below in Propositions 6 and 7. As μ increases so does s_h, making it harder to guarantee that the demand for funds from the high-productivity entrepreneurs is met by the wealth held by the low-productivity entrepreneurs, $s_h < 1/\lambda$, which is a requirement for the heterogeneous-return equilibrium to arise.

To show that an equilibrium exists, we establish the existence of a unique fixed point on innovation effort (equivalently on the share of high-productivity entrepreneurs), where effort implies productivity that implies itself the original level of effort. This is captured by a mapping $\varphi : \mathcal{M} \to \mathcal{M}$ that takes as an input a share of high-productivity entrepreneurs, $\mu \in \mathcal{M}$ and provides the implied level of effort, and hence $\varphi(\mu) \equiv e(Z(\mu)) \in \mathcal{M}$. The existence of the fixed point for φ follows from standard fixed point arguments relying on Cellina's and Brouwer's fixed point theorems (Border, 1985, Thm. 15.1, 16.1).

The uniqueness of the equilibrium follows from the monotonicity of the equilibrium mapping φ and standard comparative statics results for fixed points. To see this, it is instructive to first think of the mapping $Z(\mu)$ from μ to equilibrium Z and then on how Z affects innovation effort as in $e(Z)$. We first state a series of intermediate results as lemmas and then join them to prove our main result. These results are not directly about the behavior of equilibrium quantities, but of the functions that determine the equilibrium.

We start by inspecting equation (23) and show that equilibrium productivity is increasing in the share of high-productivity entrepreneurs.

Lemma 6. *The steady state level of productivity, $Z(\mu)$, is increasing in the share of high-productivity entrepreneurs, μ.*

Then, from Lemma 2 we can see that steady-state returns are decreasing in Z, for *arbitrary* changes in productivity for a given wealth tax rate τ_a, in such a way that return dispersion declines with productivity. As a result, innovation effort declines in Z.

Lemma 7. *Innovation effort, $e(Z)$, is decreasing in the level of steady state productivity Z.*

Remark. These results describe the mapping from an arbitrary level of Z to returns and innovation. This is the relevant mapping for constructing the fixed point that constitutes an equilibrium, holding τ_a and all the models' parameters fixed. It is this mapping from productivity to return dispersion that is decreasing in productivity. This is different from the result established in Lemma 4 that takes into account the equilibrium conditions of the economy (that is, taking into account that Z and s_h adjust to satisfy equation (23) when τ_a changes). See the proof of the proposition in Appendix C for details.

Taken together, these results imply that the mapping from φ is decreasing in μ, guaranteeing that the fixed point is unique.

Lemma 8. *Innovation effort, $\varphi(\mu) = e(Z(\mu))$, is decreasing in the share of high-productivity entrepreneurs, μ.*

We can now state the main result of this section.

Proposition 5. ***(Existence of Unique Steady State in Innovation Effort)*** *There exists an upper bound for the wealth tax $\overline{\tau}_a^{\mu}$ such that for $\tau_a < \overline{\tau}_a^{\mu}$, there exists a unique steady state that features heterogeneous returns. That is, there is a unique level of the share of high-productivity entrepreneurs, $\mu^{\star}$, such that the optimal level of effort exerted by innovators satisfies equation (46), $\mu^{\star} = e(Z(\mu^{\star}))$, and $Z(\mu^{\star}) \in (z_{\ell}, z_h)$ satisfies equation (23). The upper bound for the wealth tax satisfies*

$$\overline{\tau}_a^{\mu} = 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu^{\star}\left(\overline{\tau}_a^{\mu}\right)}{(\lambda-1)\left(1-\frac{z_{\ell}}{z_h}\right)}\right), \tag{48}$$

where we make the dependence of $\mu^{\star}$ on τ_a explicit.

In Section 2, μ was given as a primitive of the model. As we stated following Corollary 1, a higher μ increases the fraction of wealth held by high-productivity entrepreneurs, tightening the range of τ_a for which heterogenous return equilibrium arises. As we show below, the equilibrium $\mu^{\star}$ in fact increases with τ_a and hence tightens this range. $\overline{\tau}_a^{\mu}$ is the maximum τ_a that supports the heterogenous return equilibrium taking the equilibrium response of $\mu^{\star}$ into account.

4.2 The Effect of Wealth Taxes in Steady State Equilibrium

Innovation. We now show that innovation increases with the wealth tax. For this, we study the mapping φ that defines the equilibrium innovation effort but indexing it to the level of taxes: $\varphi(\cdot;\tau_a)$. The wealth tax increases the *equilibrium* dispersion of returns for any given μ, as implied by Proposition 2 and Lemma 4 from Section 3. This increase in return dispersion provides incentives for increasing the innovation effort, as the returns to a high-productivity idea are higher and the returns to a low-productivity one are lower. The result is an increase in the equilibrium level of innovation effort and hence in the share of high-productivity entrepreneurs. The proof of this result builds on standard comparative static results for fixed points found in Villas-Boas (1997).

Proposition 6. ***(Innovation Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a^{\mu}$, an increase in wealth taxes (τ_a) increases the equilibrium share of high-productivity entrepreneurs, $\mu^\star$.*

Productivity. Having established that innovation effort is increasing in the wealth tax, we can also prove that equilibrium productivity increases in τ_a after taking into account the changes in μ. The proof follows from the fact that the solution to equation (23) is increasing in both μ and τ_a. This, together with Proposition 6 ensures that productivity rises with the wealth tax.

Proposition 7. ***(Productivity Gains from Wealth Taxation with Innovation)*** *For all $\tau_a < \overline{\tau}_a^{\mu}$, an increase in wealth taxes (τ_a) increases the equilibrium level of productivity, $Z^\star$.*

Similar to what we found in Section 2, this result implies that the wealth share of high-productivity entrepreneurs increases with the wealth tax. The fact that the equilibrium level of μ increases implies that the dispersion in returns, measured as $\log R_h - \log R_\ell$, must have increased as well. We can also show that $dR_\ell/d\tau_a < 0$; however, the direction of the change in R_h cannot be signed without putting further restrictions on the cost function Λ.

Aggregates. Proposition 7 also implies that capital, output, and wages increase in response to an increase in the wealth tax. This follows directly from Lemma 5 as the steady-state value of these variables does not depend on μ directly. This result also implies that the value of a worker increases with the wealth tax, as it depends only on wages,

$$\frac{dV_w}{d\tau_a} = \frac{1}{1-\beta\delta}\xi_w \frac{d\log Z}{d\tau_a} > 0.$$

Remark. We want to highlight the neutrality of the capital income tax for innovation effort in our environment. The result is immediate and follows from the capital income tax not having an effect on steady-state productivity or returns. However, we also note that this is not a general result about capital income taxes, as it depends on the constant-returns-to-scale assumption, which makes the level of assets irrelevant for determining equilibrium returns and productivity.

4.3 Optimal Taxes with Innovation

We now turn to the choice of optimal tax rates. As in Section 4, we choose taxes to maximize the steady-state welfare of newborns, which now includes the cost of innovation effort

$$\mathcal{W} \equiv n_w V_w\left(w\right) + \left(1 - n_w\right)\left(\mu V_h\left(\overline{a}\right) + \left(1-\mu\right) V_\ell\left(\overline{a}\right) - \frac{\Lambda\left(\mu\right)}{\left(1-\beta\delta\right)^2}\right),$$

where we incorporated the fact that $\mu = e$ in equilibrium.

The optimal tax combination is obtained similarly to the the result established in Proposition 8, balancing the increase in welfare coming from the level effect on higher wages and wealth accumulation, with the decrease in returns that accompanies the increase in productivity. However, there is now a new margin coming from the change in returns in response to an increase in innovation. More innovation increases the share of high-productivity entrepreneurs, which in turn increases average returns. Crucially, this is a change in the level of returns, separate from the change in the population weights, which has no welfare effect as μ is being chosen optimally by the entrepreneurs. We show that this effect increases returns and hence implies a higher optimal wealth tax (relative to that in Proposition 8).

The following proposition characterizes the optimal tax rates when entrepreneurs exert innovation effort.

Proposition 8. *Under Assumption 2, an interior solution* ($\tau^{\star}_{a,\mu} < \overline{\tau}^{\mu}_{a}$) *to the optimal tax combination* $\left(\tau^{\star}_{a,\mu}, \tau^{\star}_{k,\mu}\right)$ *that maximizes the welfare of newborns,* $\mathcal{W}$*, is the solution to the* *following equation:*

$$0 = \Bigg(\underbrace{n_w \xi_w^Z + (1-n_w)\,\xi_K^Z}_{\textit{Productity Effect on } K \textit{ and } w\ (+)} + \underbrace{\frac{1-n_w}{1-\beta\delta}\left(\mu \xi_{R_h}^Z + (1-\mu)\,\xi_{R_\ell}^Z\right)}_{\textit{Productivity Effect on Returns (-)}} \Bigg) \frac{d\log Z}{d\tau_a} \tag{49}$$
$$+ \underbrace{\frac{1-n_w}{1-\beta\delta}\left(\mu \xi_{R_h}^\mu + (1-\mu)\,\xi_{R_\ell}^\mu\right)\frac{d\mu}{d\tau_a}}_{\textit{Innovation Effect on Returns } (+)}$$

where $\xi_x^Z \equiv \frac{\partial \log x}{\partial \log Z}$ *is the elasticity of variable* x *with respect to* Z *and* $\xi_x^\mu \equiv \frac{\partial \log x}{\partial \mu}$*. Recall from Lemma 5 that* $\xi_w^Z = \xi_K^Z = {}^{\alpha}/_{1-\alpha}$*.*

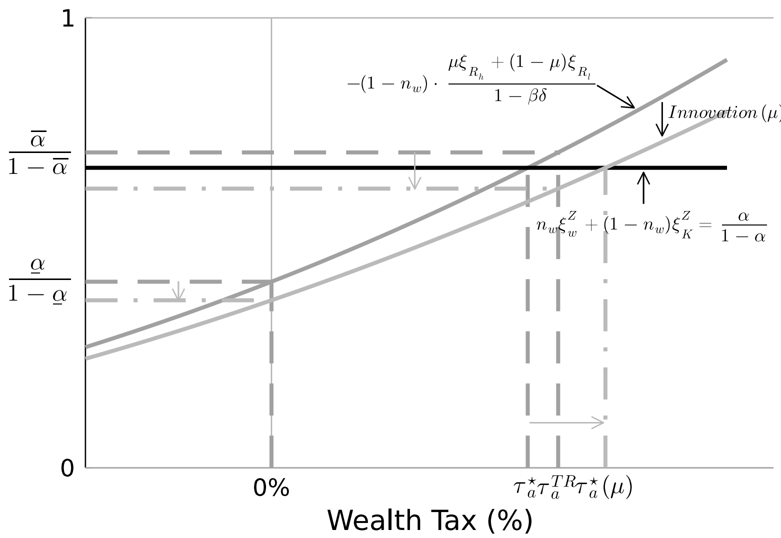
Furthermore, there are two cutoff values for α*,* $\underline{\alpha}_\mu$ *and* $\overline{\alpha}_\mu$*, such that* $\left(\tau_{a,\mu}^\star, \tau_{k,\mu}^\star\right)$ *has the following properties:*

$$\begin{aligned}
\tau_{a,\mu}^\star &\in \left[1-\frac{1}{\beta\delta}, 0\right) \textit{ and } \tau_{k,\mu}^\star > \theta && \textit{if } \alpha < \underline{\alpha}_\mu \\
\tau_{a,\mu}^\star &\in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \textit{ and } \tau_{k,\mu}^\star \in [0,\theta] && \textit{if } \underline{\alpha}_\mu \le \alpha \le \overline{\alpha}_\mu \\
\tau_{a,\mu}^\star &\in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau_a^{\max}\right) \textit{ and } \tau_{k,\mu}^\star < 0 && \textit{if } \alpha > \overline{\alpha}_\mu
\end{aligned}$$

where $\tau_a^{\max} \ge 1$ *is such that* $R_\ell \ge 0$*, and* $\underline{\alpha}_\mu$ *and* $\overline{\alpha}_\mu$ *are the solutions to equation ((49)) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$*, respectively. When* $\theta = 0$ *and there are no revenue needs, so* $\underline{\alpha}_{,\mu} = \overline{\alpha}_{,\mu}$*.*

A couple of remarks are in order. First, we cannot establish whether there is a unique solution to equation (49) without putting further restrictions on the cost function Λ. However, an interior optimum should always satisfy equation (49) as well as the fact that the second derivative of the expression above to the wealth tax is negative. Otherwise, the optimal wealth tax would not be interior. As a result, we can compare the optimal wealth tax $\tau_{a,\mu}^\star$ (of this section) to $\tau_a^\star$ (the one obtained in Section 2). In particular, the sum of productivity effects on K, w, and returns in equation (49) evaluated at $\tau_a^\star$ are zero since these terms are the same as the ones in equation (44). But since the innovation effect is positive, the derivative is positive, which implies that $\tau_{a,\mu}^\star$ is greater than $\tau_a^\star$. Figure 7 illustrates the effect of incorporating the changes in innovation into the optimal tax choice. The level effect on wages and wealth remains the same, but the decrease in returns is not as pronounced as it was with a fixed level of μ. This results in a higher value for the

Figure 7: Optimal Wealth Tax with Endogenous Innovation



Note: *The figure shows the conditions satisfied by the optimal wealth tax solving (41) and (49). The horizontal line is the (population) average of the elasticity of wages and capital with respect to productivity, ξ_w and ξ_K respectively. The increasing lines are proportional to the negative of the average elasticity of returns with respect to productivity (ξ_R) when μ is fixed and to the elasticity of returns taking into account changes in innovation, the lighter gray line. $\tau^\star_a$ denotes the optimal wealth tax. $\tau^{TR}_a = \theta(1-\beta\delta)/\beta\delta(1-\theta)$ denotes the tax reform tax, the level at which $\tau_k = 0$. The remaining parameters are as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

optimal wealth tax, $\tau^\star_a < \tau^\star_{a,\mu}$.

Similarly, the threshold values of α, for which the optimal wealth or capital income taxes would have been zero, are also lower ($\underline{\alpha}_\mu < \underline{\alpha}$ and $\overline{\alpha}_\mu < \overline{\alpha}$). This can be seen by rewriting equation (49) as $\alpha/1-\alpha = -\frac{1-n_w}{1-\beta\delta}\left(\mu\xi^Z_{R_h} + (1-\mu)\,\xi^Z_{R_\ell}\right) - \frac{1-n_w}{1-\beta\delta}\left(\mu\xi^\mu_{R_h} + (1-\mu)\,\xi^\mu_{R_\ell}\right)\frac{d\mu}{d\tau_a}/\frac{d\log Z}{d\tau_a}$. in Section 2, we obtained $\underline{\alpha}$ and $\overline{\alpha}$ by evaluating the first term in this equation at $\tau_a = 0$ or $\tau_k = 0$, respectively. Since the second term is always negative, the corresponding α thresholds with innovation are lower.

5 Extensions

5.1 Entrepreneurial Effort: Intensive Margin

We now consider the role of the intensive margin of entrepreneurial effort in shaping the productivity of private enterprises, as well as the role of the tax system in affecting entrepreneurs' incentives to exert effort. We show that capital income and wealth taxes have different effects on the effort choice of entrepreneurs. While both taxes affect capital accumulation as shown in Section 2.5, capital income taxes directly distort the effort choice of entrepreneurs by reducing their marginal benefit from exerting effort. This channel, that we spell out shortly, introduces a new channel by which replacing capital income taxes with wealth taxes increases output and welfare.

We introduce effort in a tractable manner that allows us to identify its core implications for wealth taxation. Effort, e, affects production according to

$$y = (zk)^{\alpha} e^{\gamma} n^{1-\alpha-\gamma}, \tag{50}$$

where $0 \leq \gamma < 1 - \alpha$. Exerting effort has a utility cost that we capture by modifying the utility function to

$$u(c, e) = \log(c - h(e)), \tag{51}$$

where $h(e) = \psi e$ and $\psi > 0$.[17] Tractability depends on preserving the constant-returns-to-scale in production and abstracting from income effects in the effort choice as in Greenwood, Hercowitz and Huffman (1988).[18] Together, these properties allow us to solve the model analytically as we show in Appendix D. The solution inherits the properties of our benchmark model after a suitable change of variables. We define consumption and profits net of effort as $\hat{c} = c - h(e)$ and

$$\hat{\pi}(z, k) = \max_{n,e} y - wn - rk - \frac{1}{1-\tau_k} h(e). \tag{52}$$

Crucially, the capital income tax has a direct effect on effort. Labor and capital rental costs can be deducted from taxes, while effort costs are paid privately by the entrepreneur and are not deductible. Because of this, the effective cost per unit of effort is $h'(e)/1-\tau_k$.

We obtain closed-form expressions for equilibrium quantities as a function of aggregate capital, K, and productivity, Z, paralleling the results of Lemma 1. The main difference is of course the introduction of effort. Aggregate effort is

$$E = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}}. \tag{53}$$

[17] In general, we can let effort affect production according to an increasing function $g(e)$, and we only require that the ratio $h'(e)/g'(e)$ is constant. See Appendix D.

[18] Abstracting from the income effect on the entrepreneur's effort choice potentially leads to an overstatement of the response of effort to wealth taxes. Wealth taxes increase returns and incentivize effort, but wealthier entrepreneurs may want to exert less effort in the presence of income effects.

There are two different, but related, forces shaping aggregate entrepreneurial effort. First, effort is increasing in effective capital, ZK, because it raises the marginal product of effort. Second, effort is disincentivized by capital income taxes, that reduce the after-tax marginal product of effort, effectively making effort more costly. Consequently, capital income taxes also reduce aggregate output and wages, through their effect on effort

$$Y = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}}\, L^{\frac{1-\alpha-\gamma}{1-\gamma}}, \tag{54}$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} \left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}}. \tag{55}$$

By contrast, wealth taxes do not directly affect the effort choice because they do not affect the fraction of profits retained by the entrepreneur.

The steady state behavior of aggregate productivity remains unchanged (equation 23). Even though the relationship between productivity, taxes, and steady state capital in equation (19) changes, the relationship between productivity and the after-tax return net of effort costs, $\hat{R}(z)$, is the same as in Lemma 2,

$$\hat{R}(z) = (1-\tau_a) + \left(\frac{1}{\beta\delta} - (1-\tau_a)\right)\frac{z}{Z} \tag{56}$$

This is because the steady state level of capital adjusts so that its marginal after-tax product is equal to $\frac{1}{\beta\delta} - (1-\tau_a)$ as in equation (19). Consequently, the results of our benchmark model regarding the existence of the steady state and the efficiency gains from wealth taxation (Propositions 1 and 2) remain unchanged.

Proposition 9. *A steady state equilibrium with heterogeneous returns exists if and only if $\lambda < \overline{\lambda}$, and an increase in wealth taxes in such an equilibrium increases productivity Z.*

Nevertheless, introducing entrepreneurial effort does change the response of aggregates to wealth taxes and the optimal tax combination. As wealth taxes increase, productivity rises, along with capital, output, and wages as described in Lemma 5. But, higher wealth taxes also reduce the level of capital income taxes (equation 34), incentivizing entrepreneurial effort and, trough it, increasing aggregate output, capital, and wages further, as equations (53) and (55) make clear.

Lemma 9. *If $\tau < \overline{\tau}_a$ and under Assumption 2, an increase in wealth taxes (τ_a) increases aggregate entrepreneurial effort, capital, output, and wages, $\frac{dE}{d\tau_a}, \frac{dK}{d\tau_a}, \frac{dY}{d\tau_a}, \frac{dw}{d\tau_a} > 0$. It also* *increases the wealth share of high-productivity entrepreneurs,* $\frac{ds_h}{d\tau_a} > 0$*, and the after-tax return net of effort costs of high-productivity entrepreneurs,* $\frac{d\hat{R}_h}{d\tau_a} > 0$*, while the after-tax returns net of effort costs of low-productivity entrepreneurs decreases,* $\frac{d\hat{R}_\ell}{d\tau_a} < 0$*.*

As for the optimal tax choice, the reduction of distortions on effort adds a motive for replacing capital income taxes with wealth taxes. Just as in Proposition 8, the optimal tax combination balances the gains to workers and entrepreneurs from a higher wages and wealth with the reduction in average after-tax returns (now net of effort costs). The response of the after-tax returns net of effort cost to taxes is not affected by effort, as implied by equation (56), but the increase in wages and wealth is now augmented via an increase in entrepreneurial effort. Because of this, the optimal tax combination now involves higher wealth taxes and lower capital income taxes.

Proposition 10. *Under Assumption 2, there exists a unique tax combination* $(\tau_a^\star, \tau_k^\star)$ *that maximizes the utilitarian welfare. An interior solution* $\tau_a^\star < \overline{\tau}_a$ *is the solution to:*

$$\frac{\gamma}{1-\alpha-\gamma}\frac{\beta\delta}{\frac{\log Z}{d\tau_a}} + \frac{\alpha}{1-\alpha-\gamma} = -\frac{1-n_w}{1-\beta\delta}\left(\mu\xi^Z_{\hat{R}_h} + (1-\mu)\,\xi^Z_{\hat{R}_\ell}\right) \tag{57}$$

where $\xi^Z_x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. An interior solution is higher than a solution to taxes in Proposition 8 and is equal if and only if* $\gamma = 0$*.*

5.2 Persistence of Entrepreneurial Productivity

We now consider the role of persistence of entrepreneurial productivity in shaping our results on productivity gains from wealth taxation. The models studied so far assume that entrepreneurial productivity is constant (in an entrepreneur's lifetime). An increase in wealth taxes increases the dispersion of returns, benefiting high-productivity entrepreneurs whose returns increase permanently. This also guarantees that it is high-productivity entrepreneurs who accumulate assets following an increase in wealth taxes, leading to an increase in productivity. However, if productivity is not permanent, it is not clear that the effects of wealth taxation are reflected in higher overall productivity, as there is misallocation coming from the changes in individual productivity levels. However, we show that productivity needs only be persistent (as in having a positive autocorrelation) in order to preserve our main results.

To study the role of productivity persistence, we now put forth a model economy similar in construction to the one in Section 2, but where entrepreneurial productivity follows a first order Markov process and entrepreneurs are infinitely lived. This model remains tractable, despite the complications introduced by the changes in individual productivity and provides a clear cut answer to the conditions under which wealth taxes increase productivity and welfare.[19] We provide a summary of the model and the results here and a detailed derivation of results in Appendix E.

As in Section 2 there are two types agents, homogenous workers of size L and heterogenous entrepreneurs of size 1, but they are now of infinitely-lived. This amounts to setting $\delta = 1$. Preferences are as before as is the behavior of workers. Entrepreneurs produce according to the technology in (2) and are subject to the same collateral constraint as in (3) and their labor and capital choices solve (4) as in Section 2.1. This also implies that we can aggregate as in Lemma 1.

Individual entrepreneurs differ in their productivity $z \in \{z_\ell, z_h\}$, where $0 \leq z_\ell < z_h$. However, we now assume that individual productivity follows a Markov process with transition matrix

$$\mathbb{P} = \begin{bmatrix} p & 1-p \\ 1-p & p \end{bmatrix}, \tag{58}$$

where $0 < p < 1$ is the probability that an entrepreneur retains their productivity across periods. The autocorrelation coefficient of the productivity process is $\rho \equiv 2p - 1$, so that if $p > 1/2$ ($\rho > 0$) productivity is persistent across time. The symmetry in transition probabilities ensures that half of the entrepreneurs are high types at a point in time.

The dynamic problem of the entrepreneurs is then

$$V(a, z) = \max_{a'} \log\left(R(z)\, a - a'\right) + \beta \sum_{z'} \mathbb{P}(z' \mid z)\, V(a', z'), \tag{59}$$

where $R(z)$ is defined as in Section 2.1. The solution to this problem is a saving rule

$$a' = \beta R(z)\, a. \tag{60}$$

We show in Appendix E that this leads to steady state conditions paralleling those in (19), (20), and (23), where capital adjusts to make the wealth weighted returns equal to $1/\beta$

[19] The model does not admit a stationary wealth distribution but remains tractable by focusing on the behavior of aggregates and wealth shares across entrepreneurial types, similar to the results of Moll (2014).

and productivity is endogenous and determined by a quadratic equation that is now

$$0 = \left(1-\rho\beta\left(1-\tau_a\right)\right)Z^2 - \left(1+\rho\left(1-2\beta\left(1-\tau_a\right)\right)\right)\frac{z_h+z_\ell}{2}Z + \rho\left(1-\beta\left(1-\tau_a\right)\right)z_h z_\ell = 0. \tag{61}$$

This equation shows that productivity depends now on ρ, the persistence of the productivity process, that now interacts with wealth taxes.[20] Just as in Section 2.5, we can prove that there exists a unique steady state with heterogeneous returns, provided that the collateral constraint is sufficiently tight (or equivalently, that wealth taxes are sufficiently low).

Proposition 11. *There exists a unique steady state that features heterogenous returns* ($R_h > R_\ell$) *if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{1-\rho}{1+\rho\left(1-2\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\right)\right)}. \tag{62}$$

The main result out of the model is that the effects of wealth taxes on steady state productivity now depend on the persistence of productivity. We show that Z is increasing in the wealth tax rate (τ_a) as long as entrepreneurial productivity is persistent, $\rho > 0$. As in Section 3, an increase in wealth taxes increases the returns of high-productivity entrepreneurs and reduces those of low-productivity entrepreneurs (see Lemma 11, Appendix E). This translates into a higher wealth share of high-productivity entrepreneurs if and only if the entrepreneurs who currently have high-productivity are expected to be of high-productivity in the future.

Proposition 12. ***(Efficiency Gains from Wealth Taxation)*** *For all* $\tau_a < \overline{\tau}_a$*, an increase in the wealth tax* (τ_a) *increases productivity,* $\frac{dZ}{d\tau_a} > 0$*, if and only if entrepreneurial productivity is persistent,* $\rho > 0$*.*

The remainder of our results also have parallels in these model that we omit for space. The behavior of returns implies that high-productivity entrepreneurs benefit from wealth taxes, while entrepreneurs as a group see their welfare go down because average returns decrease in response to an increase in wealth taxes. Workers benefit as before through the increase in wages following the increase in productivity. The choice of optimal taxes

[20]Equation (61) also implies that steady state productivity is independent of μ. That is because only the wealth shares are relevant for determining productivity and returns in equilibrium and not the underlying mass of individuals. This is a consequence of the linearity of the problem that allows aggregation at the type level. The share of high-productivity entrepreneurs, μ, plays a role in (23) because it affects the distribution of assets through the birth-and-death process.

depends, of course, on the choice of government's objective function. However, the condition characterizing optimal taxes takes a familiar form, balancing the increase in wages (captured by the elasticity of wages to productivity), with the decrease in returns (captured by the average elasticity of returns). See Appendix E.3 for examples of this.

6 Conclusion (preliminary)

We studied the taxation of capital through capital income and wealth taxes. In the heterogeneous-returns equilibrium that emerges under a broad set of parameter choices, an increase in wealth taxes leads to higher aggregate productivity and output. Higher wealth taxes benefit workers through the increase in wages that follows the increases in productivity and output. High-productivity entrepreneurs also benefit and low-productivity entrepreneurs generally lose, reflecting the shift in the tax burden from high-return individuals under capital income taxes to low-return (but wealthy) individuals under wealth taxes.

Turning to optimal taxation, when the government can use both tools simultaneously, the optimal policy depends on the model parameters in the form of thresholds in the capital intensity of production. For sufficiently high capital share parameters, the optimal policy is a positive wealth tax combined with a capital income subsidy. Endogenizing entrepreneurial productivity through costly innovation effort strengthens these channels.

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A Entrepreneurial problem

Profits and input choice: We start with an entrepreneur's labor choice given their capital:

$$\pi\left(z,k\right)=\max_{n}\left(zk\right)^{\alpha}n^{1-\alpha}-wn, \tag{63}$$

which gives the following labor demand

$$n^{\star}\left(z,k\right) = \left(\frac{1-\alpha}{w}\right)^{1/\alpha} zk. \tag{64}$$

Substituting the optimal labor demand into the profit, the entrepreneur's capital choice is given by

$$k^{\star}\left(z,a\right) = \underset{0\leq k\leq\lambda a}{\operatorname{argmax}} \left[\alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha} z-r\right]k. \tag{65}$$

The optimal capital decision of the entrepreneur is therefore characterized by the following function:

$$k^{\star}\left(z,a\right) = \begin{cases} \lambda a & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z>r \\ [0,\lambda a] & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z=r \\ 0 & \text{if } \alpha\left(\frac{1-\alpha}{w}\right)^{(1-\alpha)/\alpha}z<r. \end{cases} \tag{66}$$

Entrepreneurs whose marginal return to capital is greater than the interest rate borrow up to the limit and sets λa and those whose return is below the interest rate does not produce zero output and earns the return r in the bond market on wealth a.

This leads to the profit function in equation (7).

Savings: Given taxes τ_a and τ_k and constant prices, an entrepreneur's optimal savings problem can be written as

$$V_i\left(a\right) = \max_{a'}\log\left(c\right)+\beta\delta V\left(a'\right)$$

subject to

$$c+a' = R_i a,$$

where $R_i = 1 - \tau_a + (1 - \tau_k)(r + \pi^{\star}(z_i))$ as in the main text and $i \in \{\ell, h\}$ gives the productivity type of the entrepreneur.

We solve the entrepreneur's saving problem via guess and verify. To this end, we guess that the value function of an entrepreneur of type $i \in \{\ell, h\}$ has the form

$$V_i(a) = m_i + n \log(a),$$

where $m_\ell, m_h, n \in \mathbb{R}$ are coefficients. Under this guess the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{R_i a - a_i^{'}} = \frac{\beta\delta n}{a_i^{'}}. \tag{67}$$

Solving for savings gives:

$$a_i^{'} = \frac{\beta\delta n}{1 + \beta\delta n} R_i a. \tag{68}$$

Replacing the savings rule into the value function gives:

$$V_i(a) = \log\left(R_i a - a_i^{'}\right) + \beta\delta V_i\left(a_i^{'}\right) \tag{69}$$

$$m_i + n\log(a) = \log\left(R_i a - a_i^{'}\right) + \beta\delta m_i + \beta\delta n \log\left(a_i^{'}\right) \tag{70}$$

$$m_i + n\log(a) = \beta\delta n \log(\beta\delta n) + (1 + \beta\delta n)\log\left(\frac{R_i}{1 + \beta\delta n}\right) + \beta\delta m_i + (1 + \beta\delta n)\log(a) \tag{71}$$

Matching coefficients:

$$n = 1 + \beta\delta n; \tag{72}$$

$$m_i = \beta\delta n \log(\beta\delta n) + (1 + \beta\delta n)\log\left(\frac{R_i}{1 + \beta\delta n}\right) + \beta\delta m_i. \tag{73}$$

The solution to the first equation implies:

$$n = \frac{1}{1 - \beta\delta}, \tag{74}$$

which in turn delivers the optimal saving decision of the entrepreneur in equation (9) with constant saving rate $\beta\delta$.

Finally, we solve for the remaining coefficients from the system of linear equations:

$$m_i = \frac{1}{(1-\beta\delta)^2}\left[\log (\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta} + \log R_i\right] \tag{75}$$

Together this implies that the value of an entrepreneur is

$$V_i(a) = \frac{\log (\beta\delta)^{\beta\delta}(1-\beta\delta)^{1-\beta\delta}}{(1-\beta\delta)^2} + \frac{1}{(1-\beta\delta)^2}\log R_i + \frac{1}{1-\beta\delta}\log(a). \tag{76}$$

B Proofs for benchmark model

This appendix presents the proofs for the results listed in the paper. We reproduce the statement of all results for the reader's convenience.

Lemma 1. ***(Aggregate Variables in Equilibrium)*** *In the heterogenous return equilibrium* $((\lambda-1)\mu A_h < (1-\mu)A_\ell)$*, output, wages, interest rate, and gross returns are:*

$$Y = (ZK)^\alpha L^{1-\alpha} \tag{77}$$

$$w = (1-\alpha)\left({}^{ZK}/_{L}\right)^\alpha \tag{78}$$

$$r = \alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_\ell \tag{79}$$

$$R_\ell = (1-\tau_a) + (1-\tau_k)\,\alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_\ell \tag{80}$$

$$R_h = (1-\tau_a) + (1-\tau_k)\,\alpha\left({}^{ZK}/_{L}\right)^{\alpha-1} z_\lambda. \tag{81}$$

Proof. We start by considering the labor market clearing condition

$$\mu n^\star(z_h, K_h) + (1-\mu)\,n^\star(z_\ell, K_\ell) = L. \tag{82}$$

Replacing for the optimal labor demand (64) we get

$$\left(\frac{1-\alpha}{w}\right)^{1/\alpha}(z_h\mu K_h + z_\ell(1-\mu)K_\ell) \quad = \quad L; \tag{83}$$

$$\left(\frac{1-\alpha}{w}\right)^{1/\alpha} ZK \quad = \quad L. \tag{84}$$

Manipulating this expression we get wages as:

$$w = (1-\alpha)\left({}^{ZK}/_{L}\right)^\alpha. \tag{85}$$

Replacing into the equilibrium interest rate we get:

$$r = \alpha \left(\frac{1-\alpha}{w} \right)^{\frac{1-\alpha}{\alpha}} z_\ell = \alpha \left({}^{ZK}\!/\!{}_{L} \right)^{\alpha - 1} z_\ell \tag{86}$$

These two expressions also let us rewrite the profit rate of the high-productivity entrepreneurs (from 7):

$$\pi^\star (z_h) = \left(\alpha \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} z_h - r \right) \lambda = \alpha \left({}^{ZK}\!/\!{}_{L} \right)^{\alpha - 1} (z_h - z_\ell) \lambda \tag{87}$$

We can then use the equilibrium profit rates of entrepreneurs to rewrite the gross returns of entrepreneurs:

$$R_\ell = (1-\tau_a) + (1-\tau_k)\, r = (1-\tau_a) + (1-\tau_k)\, \alpha \left({}^{ZK}\!/\!{}_{L} \right)^{\alpha - 1} z_\ell \tag{88}$$

and

$$R_h = (1-\tau_a) + (1-\tau_k) \left(r + \pi^\star (z_h) \right) \tag{89}$$

$$= (1-\tau_a) + (1-\tau_k)\, \alpha \left({}^{ZK}\!/\!{}_{L} \right)^{\alpha - 1} \left(z_\ell + \lambda (z_h - z_\ell) \right) \tag{90}$$

$$= (1-\tau_a) + (1-\tau_k)\, \alpha \left({}^{ZK}\!/\!{}_{L} \right)^{\alpha - 1} z_\lambda. \tag{91}$$

Finally we consider aggregate output, for this note that the ratio of labor to capital is constant across entrepreneurs which allows us to aggregate in terms of the total capital of each type. From (64) we can express the output of an individual entrepreneur with productivity z and capital k as:

$$y(z,k) = \left(\frac{1-\alpha}{w} \right)^{(1-\alpha)/\alpha} zk = \left({}^{ZK}\!/\!{}_{L} \right)^{\alpha - 1} zk,$$

where the second equality comes after replacing the wage from (85). Aggregate output is the sum of the total output produced by each type of entrepreneur:

$$Y = \left({}^{ZK}\!/\!{}_{L} \right)^{\alpha - 1} \left(z_h \mu K_h + z_\ell (1-\mu) K_\ell \right) = (ZK)^\alpha L^{1-\alpha}. \tag{92}$$

This completes the derivation of the results.

□

Lemma 2. *The steady state after-tax returns are independent of the steady state level of capital and of capital income taxes and satisfy*

$$R_\ell = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a) \right) \frac{z_\ell}{Z} \quad \textit{and} \quad R_h = 1 - \tau_a + \left(\frac{1}{\beta\delta} - (1-\tau_a) \right) \frac{z_\lambda}{Z}. \tag{93}$$

Moreover, the wealth-weighted returns depend only on the entrepreneurial saving rate,

$s_h R_h + (1 - s_h) R_\ell = {}^1\!/\!{\beta\delta}.$

Proof. The proof is immediate by replacing (19) into the expression for returns in terms of aggregate variables.

$$R_\ell = (1 - \tau_a) + (1 - \tau_k)\,\alpha\,({}^{ZK}\!/\!{}_{L})^{\alpha-1} z_\ell; \tag{94}$$

$$R_h = (1 - \tau_a) + (1 - \tau_k)\,\alpha\,({}^{ZK}\!/\!{}_{L})^{\alpha-1} z_\lambda. \tag{95}$$

The second part follows by replacing again and recalling the definition of aggregate productivity,

$$s_h R_h + (1 - s_h) R_\ell = (1 - \tau_a) + \left(\frac{1}{\beta\delta} - (1 - \tau_a)\right) \frac{s_h z_\lambda + (1 - s_h) z_\ell}{Z} = \frac{1}{\beta\delta}. \tag{96}$$

□

Proposition 1. ***(Existence and Uniqueness of Steady State)*** *There exists a unique steady state productivity, $Z \in (z_\ell, z_h)$, given a share of high-productivity entrepreneurs, μ. The steady state equilibrium features heterogenous returns ($R_h > R_\ell$) if and only if $\lambda < \overline{\lambda}_p \equiv 1 + \frac{(1-\delta)(1-\mu)}{(1-\delta)\mu + \delta(1-\delta\beta(1-\tau_a))\left(1-\frac{z_\ell}{z_h}\right)}$.*

Proof. We study the behavior of the quadratic equation in (23). We show that it has a single admissible root as shown in Figure 8. More precisely, we show there exists a unique solution to the quadratic equation in the interval

$$Z \in \left(\max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right\}, z_\lambda\right).$$

This interval is relevant for the proof of 3.

We start by defining the function H,

$$H(x) = (1 - \delta\eta) - \frac{(1-\delta)(\mu z_\lambda + (1-\mu) z_\ell) + \delta(1-\eta)(z_\lambda + z_\ell)}{x} + \delta(1-\eta)\frac{z_\ell z_\lambda}{x^2}, \tag{97}$$

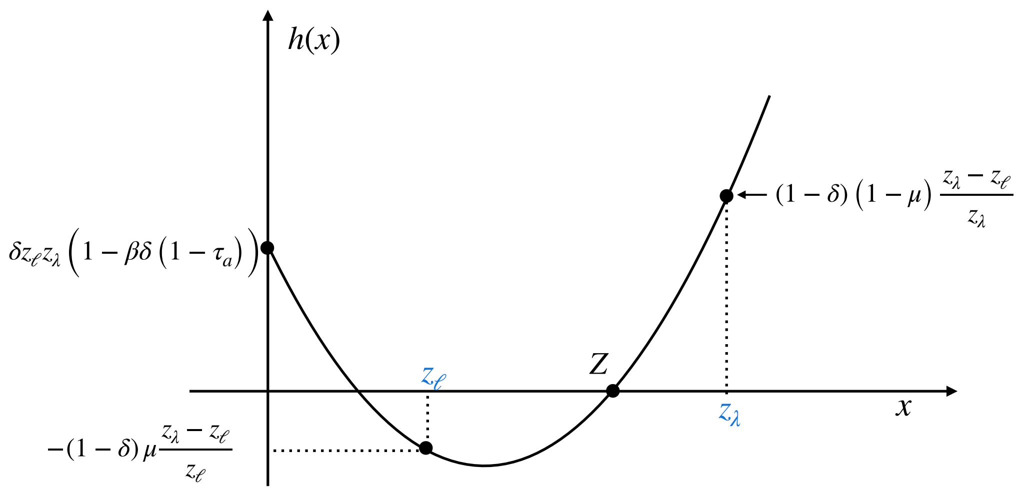
as the residual of the equation at x, where $\eta \equiv \beta\delta(1 - \tau_a)$. We verify directly that H has a root in the interval $\left(\max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right\}, z_\lambda\right)$:

$$H(z_\ell) = -\frac{(1-\delta)\mu}{z_\ell}(z_\lambda - z_\ell) < 0$$

$$H\left(\frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda\right) = -\frac{(1-\delta\eta)(1-\delta)\mu}{\delta(1-\eta)}\frac{1}{z_\lambda}(z_\lambda - z_\ell) < 0$$

$$H(z_\lambda) = \frac{(1-\delta)(1-\mu)}{z_\lambda}(z_\lambda - z_\ell) > 0$$

Figure 8: Steady State Productivity (Z)



Note: *The figure plots* $h(x) = \left(1-\delta^2\beta(1-\tau_a)\right)x^2 - \left[(1-\delta)(\mu z_\lambda + (1-\mu)z_\ell) + \delta(1-\delta\beta(1-\tau_a))(z_\lambda + z_\ell)\right]x + \delta(1-\delta\beta(1-\tau_a))z_\ell z_\lambda = 0$. *The steady state productivity corresponds to the larger root of h, marked with a circle on the horizontal axis.*

The existence of the unique root is guaranteed by the intermediate value theorem and the fact that the function is quadratic.

Now we derive sufficient conditions for the economy to be in the equilibrium with with excess supply of funds: $((1-\mu)A_\ell > (\lambda-1)\mu A_h)$. This happens if and only if $Z \leq z_h$. So now we find conditions that guarantee that $H(z_h) > 0$ which implies that $Z \leq z_h$ since $H(Z) = 0$ and $H(z)$ is increasing in $z \geq Z$.

$$H(z_h) = (1-\delta\eta) - \frac{(1-\delta)(\mu z_\lambda + (1-\mu)z_\ell) + \delta(1-\eta)(z_\lambda + z_\ell)}{z_h} + \delta(1-\eta)\frac{z_\ell z_\lambda}{z_h^2} > 0,$$

which after some manipulation gives:

$$\lambda < \overline{\lambda} \equiv 1 + \frac{(1-\delta)(1-\mu)}{(1-\delta)\mu + \delta(1-\delta\beta(1-\tau_a))\left(1-\frac{z_\ell}{z_h}\right)}.$$

Note that $\overline{\lambda} < 1/\mu$ always and so it is the relevant bound for λ.

A final detail is left to be verified, that is that $z_h \geq \max\left\{z_\ell, \frac{\eta-\delta}{\eta}z_\lambda\right\}$, the first case is verified immediately, the second case applies if $\frac{\delta(1-\eta)}{1-\delta\eta} > \frac{z_\ell}{z_\lambda}$. A sufficient condition for $z_h \geq \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda$ is:

$$\begin{aligned}
z_h &\geq \frac{\delta(1-\eta)}{1-\delta\eta} z_\lambda \\
(1-\delta) &\geq \delta(1-\eta)(\lambda-1)\left(1-\frac{z_\ell}{z_h}\right) \\
(1-\delta) &\geq \delta(1-\eta)(\lambda-1)\left(1-\frac{\delta(1-\eta)}{1-\delta\eta}\right) \\
\frac{1-\delta\eta}{\delta(1-\eta)} &\geq \lambda-1
\end{aligned}$$

For this bound not to bind we need that it is above $\overline{\lambda}$:

$$\begin{aligned}
\frac{1-\delta\eta}{\delta(1-\eta)} &\geq \overline{\lambda}-1 \\
(1-\delta\eta)\left((1-\delta)\mu+\delta(1-\eta)\left(1-\frac{z_\ell}{z_h}\right)\right) &\geq (1-\delta)\delta(1-\eta)(1-\mu) \\
(1-\delta\eta)\delta(1-\eta)\left(1-\frac{z_\ell}{z_h}\right) &\geq (1-\delta)\left[\delta(1-\eta)-(\delta+1-2\delta\eta)\mu\right]
\end{aligned}$$

The condition is most stringent when $\mu = 0$ (counterfactually). This leads to the sufficient condition being

$$\begin{aligned}
(1-\delta\eta)\left(1-\frac{z_\ell}{z_h}\right) &\geq (1-\delta) \\
\delta\frac{1-\eta}{1-\delta\eta} &\geq \frac{z_\ell}{z_h}
\end{aligned}$$

which is verified by assumption. So the upper bound $\overline{\lambda}$ is sufficient for guaranteeing that $z_h \geq \max\left\{z_\ell, \frac{\eta-\delta}{\eta} z_\lambda\right\}$.

$\square$

Corollary 2. *The condition for the steady state to feature heterogeneous returns can be restated as an upper bound on wealth taxes given a value for λ:*

$$\lambda < \overline{\lambda} \longleftrightarrow \tau_a < \overline{\tau} = 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu}{(\lambda-1)\left(1-\frac{z_\ell}{z_h}\right)}\right) \tag{98}$$

Proof. For a given λ there is also a bound on τ_a. The condition for the bound is

$$0 < (1-\delta)(1-\mu) - \left((1-\delta)\mu + \delta(1-\beta\delta(1-\tau_a))\left(1-\frac{z_\ell}{z_h}\right)\right)(\lambda-1). \tag{99}$$

We manipulate this equation to obtain the bound on τ_a:

$$1-\beta\delta(1-\tau_a) < \frac{\frac{(1-\delta)(1-\mu)}{(\lambda-1)} - (1-\delta)\mu}{\delta\left(1-\frac{z_\ell}{z_h}\right)}$$

$$\tau_a < 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu}{(\lambda-1)\left(1-\frac{z_\ell}{z_h}\right)}\right).$$

□

Lemma 3. ***(Saving and Dissaving in Steady State)*** *In steady state it holds that $\beta\delta R_\ell < 1 < \beta\delta R_h < 1/\delta$. Moreover, the wealth share of high-productivity entrepreneurs is higher than their population share, $s_h > \mu$.*

Proof. We start by showing that $R_\ell < 1/\beta\delta < R_h$. We verify this directly using the expression for the returns of high- and low-productivity entrepreneurs, the fact that $z_\ell < Z < z_\lambda$ and the steady state condition for the return on capital:

$$R_\ell = (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_\ell}{Z} < (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1} = \frac{1}{\beta\delta},$$

and

$$\frac{1}{\beta\delta} = (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1} < (1-\tau_a) + (1-\tau_k)\alpha Z^\alpha\left(\frac{K}{L}\right)^{\alpha-1}\frac{z_\lambda}{Z} = R_h,$$

Letting $\eta \equiv \delta\beta(1-\tau_a)$, we can also show that $\beta\delta R_h < 1/\delta$ if $\frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda < Z$. Thus,

$$\beta\delta R_\ell < 1 < \beta\delta R_h < 1/\delta \longleftrightarrow Z \in \left(\max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda\right\}, z_\lambda\right). \tag{100}$$

The interval for Z is non-empty. This is immediate because:

$$z_\ell < z_\lambda \qquad \text{and} \qquad \frac{\delta(1-\eta)}{1-\delta\eta} < 1.$$

Moreover, the lower bound depends on the ratio of productivities. We have $\max\left\{z_\ell, \frac{\delta(1-\eta)}{1-\delta\eta}z_\lambda\right\} = z_\ell$ if and only if $\frac{\delta(1-\eta)}{1-\delta\eta} \leq \frac{z_\ell}{z_\lambda}$. In the proof of Proposition 1 we establish that Z lies in the desired interval.

Finally, we prove that $s_h > \mu$. We know that $s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell}$, so $s_h > \mu$ is equivalent to $Z > \mu z_\lambda + (1-\mu) z_\ell$. We can verify if this is the case by evaluating at $\mu z_\lambda + (1-\mu) z_\ell$ the residual of the quadratic equation H defined in (23):

$$H\left(\frac{z_\lambda + z_\ell}{2}\right) = -\delta(1-\eta)(1-\mu)\mu\left(\frac{z_\lambda - z_\ell}{\mu z_\lambda + (1-\mu) z_\ell}\right)^2 < 0$$

The residual is always negative. So it must be that $Z > \mu z_\lambda + (1-\mu) z_\ell$ and thus $s_h > \mu$.

□

Proposition 2. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a < \overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases productivity, $\frac{dZ}{d\tau_a} > 0$.*

Proof. We first define the auxiliary function

$$H(x;\tau_a) = \left(1 - \delta^2\beta(1-\tau_a)\right) - \frac{(1-\delta)(\mu z_\lambda + (1-\mu) z_\ell) + \delta(1-\delta\beta(1-\tau_a))(z_\lambda + z_\ell)}{x} + \delta(1-\delta\beta(1-\tau_a))\frac{z_\ell z_\lambda}{x^2},$$

which characterizes the steady state if and only if $\lambda < \overline{\lambda}_p$. Simple manipulation of the function gives:

$$H(x;\tau_a) = F(x) - \left(1 - \frac{z_\ell}{x}\right)\left(1 - \frac{z_\lambda}{x}\right)\delta^2\beta(1-\tau_a),$$

where $F(x)$ is a function of only x that does not depend on taxes. We now establish that H is decreasing in τ_a for $x \in (z_\ell, z_\lambda)$, which is the interval of the steady state value of Z:

$$\frac{d\tilde{H}(x,\tau_a)}{d\tau_a} = \underbrace{\left(1 - \frac{z_\ell}{x}\right)}_{(+)}\underbrace{\left(1 - \frac{z_\lambda}{x}\right)}_{(-)}\delta^2\beta < 0.$$

This gives the desired result: $\frac{dZ}{d\tau_a} > 0$ because as we proved in the previous proposition H is increasing in x in the relevant interval.

□

Lemma 4. ***(Wealth Shares and Return Dispersion in Steady State)*** *The steady* *state aggregate variables satisfy, holding μ constant,*

$$s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell} > \mu \qquad \frac{ds_h}{dZ} > 0 \tag{101}$$

$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\,\mu}{s_h}\right) \qquad \xi_{R_h} \equiv \frac{d\log R_h}{d\log Z} > 0 \tag{102}$$

$$R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\,(1-\mu)}{(1-s_h)}\right) \qquad \xi_{R_\ell} \equiv \frac{d\log R_\ell}{d\log Z} < 0. \tag{103}$$

Moreover, the average returns decrease with wealth taxes and the average elasticity of returns with respect to Z (and hence the wealth tax) is negative,

$$\frac{d\left(\mu R_\ell + (1-\mu)\,R_h\right)}{d\tau_a} < 0 \qquad \text{and} \qquad \mu\xi_{R_h} + (1-\mu)\,\xi_{R_\ell} < 0, \tag{104}$$

and the top wealth shares $s_{h,t}$, defined in (27), also increase.

Proof. The result for the wealth share of the high-types is immediate from the definition of Z.

From the steady state level of wealth of high-productivity entrepreneurs we know that:

$$R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\,\mu}{s_h}\right)$$

which implies:

$$\frac{dR_h}{dZ} = \frac{(1-\delta)\,\mu}{\beta\delta^2}\frac{1}{s_h^2}\frac{ds_h}{dZ} > 0$$

A similar calculation delivers:

$$R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\,(1-\mu)}{(1-s_h)}\right) \qquad \frac{dR_\ell}{dZ} = -\frac{(1-\delta)\,(1-\mu)}{\beta\delta^2}\frac{1}{(1-s_h)^2}\frac{ds_h}{dZ} < 0.$$

We know the share of wealth of the high-types is increasing along with the overall wealth in the economy, so A_h must increase as well, this will imply that R_h must have risen. From Lemma 5 and the steady state condition for the assets of high-productivity entrepreneurs:

$$\frac{dR_h}{d\tau_a} = \frac{d\left(\frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\mu}{s_h}\right)\right)}{d\tau_a} = \frac{(1-\delta)\,\mu}{\beta\delta^2}\frac{1}{s_h^2}\frac{ds_h}{d\tau_a} = \frac{(1-\delta)\,\mu}{\beta\delta^2}\frac{z_\lambda - z_\ell}{(Z - z_\ell)^2}\frac{dZ}{d\tau_a} > 0,$$

and:

$$\frac{dR_\ell}{d\tau_a} = -\frac{(1-\delta)\,(1-\mu)}{\beta\delta^2}\frac{1}{(1-s_h)^2}\frac{ds_h}{d\tau_a} = \frac{-(1-\delta)\,(1-\mu)}{\beta\delta^2}\frac{z_\lambda - z_\ell}{(z_\lambda - Z)^2}\frac{dZ}{d\tau_a} < 0.$$

With this we get:

$$\frac{d\left(\mu R_h+(1-\mu)R_\ell\right)}{d\tau_a}=\frac{1-\delta}{\beta\delta^2}\left(\frac{\left(\mu\left(1-s_h\right)+\left(1-\mu\right)s_h\right)\left(\mu-s_h\right)}{s_h^2\left(1-s_h\right)^2}\right)\frac{ds_h}{d\tau_a}$$

so that $\frac{d(\mu R_h+(1-\mu)R_\ell)}{d\tau_a}\geq 0$ if and only if $s_h\leq\mu$. We have already established that $s_h>\mu$, then $\frac{d(\mu R_h+(1-\mu)R_\ell)}{d\tau_a}<0$.

Finally, we consider the weighted product of returns, which is also decreasing in taxes.

$$\begin{aligned}\frac{dR_h^{\mu}R_\ell^{1-\mu}}{d\tau_a}&=\left(1-\mu\right)R_h^{\mu}R_\ell^{-\mu}\frac{dR_\ell}{d\tau_a}+\mu R_h^{\mu-1}R_\ell^{1-\mu}\frac{dR_h}{d\tau_a}\\&<R_h^{\mu}R_\ell^{1-\mu}\frac{(1-\delta)}{\beta\delta^2}R_\ell\left[\frac{\left(\mu\left(1-s_h\right)+\left(1-\mu\right)s_h\right)\left(\mu-s_h\right)}{s_h^2\left(1-s_h\right)^2}\right]\frac{ds_h}{d\tau_a}\end{aligned}$$

which is negative because, as before, $s_h<\mu$. This completes this part of the proof.

The result for top wealth shares is immediate from the their definition as a function of after-tax returns (equation 27) and the fact that R_h increases with wealth taxes (see Lemma 4). An increase in wealth taxes increases the returns of high-productivity entrepreneurs (R_h), which in turn increases their savings rate and asset holdings. The effect is compounded with age because savings rate are constant, increasing more the wealth holdings of older/wealthier entrepreneurs.

□

Lemma 5. ***(Aggregate Variables in Steady State)*** *If $\tau<\overline{\tau}_a$ and under Assumption* **??***, the steady state level of aggregate capital is*

$$K=\left(\alpha\frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}}Z^{\frac{\alpha}{1-\alpha}}L \tag{105}$$

and the steady state elasticities of aggregate variables with respect to productivity are

$$\xi_K=\xi_Y=\xi_w=\xi\equiv\frac{\alpha}{1-\alpha}, \tag{106}$$

where $\xi_x\equiv\frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z.

Moreover, the wealth levels of each entrepreneurial type in steady state are

$$A_h=\frac{1}{\mu}\frac{Z-z_\ell}{z_\lambda-z_\ell}K \qquad \frac{dA_h}{dZ}\propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(Z-\alpha z_\ell\right)>0 \tag{107}$$

$$A_\ell=\frac{1}{1-\mu}\frac{z_\lambda-Z}{z_\lambda-z_\ell}K \qquad \frac{dA_\ell}{dZ}\propto Z^{\frac{2\alpha-1}{1-\alpha}}\left(\alpha z_\lambda-Z\right), \tag{108}$$

where $\frac{dA_\ell}{dZ} < 0$ *if and only if* $\alpha z_\lambda < Z$.

Proof. Using Assumption 2 we obtain:

$$K = \left(\alpha \frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}} L$$

which is increasing in Z. From this it is immediate that $Y = (ZK)^{\alpha} L^{1-\alpha}$ is also increasing in Z. For wages we get

$$w = (1-\alpha)\left({}^{ZK}/_{L}\right)^{\alpha} = (1-\alpha)\frac{Y}{L} = (1-\alpha)\left(\alpha \frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{\alpha}{1-\alpha}} Z^{\frac{\alpha}{1-\alpha}}.$$

The elasticities follow immediately.

Since K and s_h increase it must be the case that $A_h = \frac{s_h K}{\mu}$ increases as well. We are left with the response of A_ℓ. To get it we first write A_ℓ in terms of Z using the definition of the wealth share of the high-types:

$$\begin{aligned} A_\ell &= \frac{\left(1-s_h\right)K}{1-\mu} \\ &= \frac{1}{1-\mu}\left(1-\frac{Z-z_\ell}{z_\lambda - z_\ell}\right)K \\ &= \frac{1}{1-\mu}\left(\alpha \frac{\beta\delta\left(1-\theta\right)}{1-\beta\delta}\right)^{\frac{1}{1-\alpha}} L\frac{z_\lambda - Z}{z_\lambda - z_\ell} Z^{\frac{\alpha}{1-\alpha}} \end{aligned}$$

Taking derivatives shows that A_ℓ decreases with Z (and hence with τ_a):

$$\frac{dA_\ell}{dZ} \propto \frac{Z^{\frac{\alpha}{1-\alpha}-1}}{z_\lambda - z_\ell}\left[\alpha z_\lambda - Z\right]$$

which is negative if $\alpha z_\lambda < Z$.

□

Proposition 3. *For all* $\tau_a < \overline{\tau}_a$ *and* $\mu \in (0,1)$, *an increase in wealth taxes increases increases the welfare for workers,* ${}^{dV_w}/_{d\tau_a} > 0$, *and the welfare of high-productivity entrepreneurs,* ${}^{dV_h(\overline{a})}/_{d\tau_a} > 0$. *Moreover, it increases the welfare of low-productivity entrepreneurs,* ${}^{dV_\ell(\overline{a})}/_{d\tau_a} > 0$, *if* $\xi_K > \frac{-1}{1-\beta\delta}\xi_{R_\ell}$, *and the welfare of newborn entrepreneurs,* ${}^{d(\mu V_h(\overline{a}) + (1-\mu)V_\ell(\overline{a}))}/_{d\tau_a} > 0$, *if* $\xi_K > \frac{-1}{1-\beta\delta}\left(\mu\xi_{R_h} + (1-\mu)\,\xi_{R_\ell}\right)$, *where* $\xi_K = {}^{\alpha}/_{1-\alpha}$ *and* ξ_{R_i} *are the elasticities of capital and return with respect to productivity.*

Proof. We begin with worker welfare. Recall that $V_w = \frac{1}{1-\beta\delta}\log(w)$ and so $dV_w/d\tau_a = \frac{1}{1-\beta\delta}\xi_w d\log Z/d\tau_a$, where $\xi_w = \alpha/1-\alpha > 0$ is the elasticity of wages with respect to productivity. Recall that $d\log Z/d\tau_a > 0$ from proposition 2. This gives the result.

The value of entrepreneurs is $V_i(\overline{a}) = m_i + \frac{1}{1-\beta\delta}\log(\overline{a})$, where $m_i = \frac{1}{(1-\beta\delta)^2}[\beta\delta\log\beta\delta + (1-\beta\delta)\log(1-\beta\delta) + \log R_i]$. Hence, the change in the welfare of an entrepreneur with productivity z_i, $i \in \{\ell, h\}$, is

$$dV_i(\overline{a})/d\tau_a = \frac{1}{1-\beta\delta}\left(\xi_K + \frac{1}{1-\beta\delta}\xi_{R_i}\right) d\log Z/d\tau_a.$$

It is immediate that $dV_h(\overline{a})/d\tau_a > 0$ because $\xi_K, \xi_{R_h} > 0$ from Lemmas 5 and **??**.

For low-productivity entrepreneurs to benefit from an increase in wealth taxes it must be that

$$\xi_K > \frac{-1}{1-\beta\delta}\xi_{R_\ell}.$$

Finally, for entrepreneurs to benefit from an increase in wealth taxes it must be that

$$\xi_K > \frac{-1}{1-\beta\delta}\left(\mu\xi_{R_h} + (1-\mu)\xi_{R_\ell}\right).$$

Recall that $\xi_K = \alpha/1-\alpha$ and so there are cutoffs on wealth taxes above which low-productivity entrepreneurs and entrepreneurs as a whole stop benefiting from an increase in wealth taxes. Alternatively we can interpret the conditions evaluating the elasticity of returns at a fixed value of τ_a, say $\tau_a = 0$, so as to know the lowest α for which entrepreneurs benefit from a marginal increase in taxes at that point.

$\square$

Proposition 4. ***(Optimal Taxes)*** *Under Assumption 2, there exist a unique tax combination $(\tau_a^\star, \tau_k^\star)$ that maximizes the utilitarian welfare, an interior solution $\tau_a^\star < \overline{\tau}_a$ is the solution to:*

$$n_w\xi_w + (1-n_w)\xi_K = -\frac{1-n_w}{1-\beta\delta}\left(\mu\xi_Z^{R_h} + (1-\mu)\xi_Z^{R_\ell}\right) \tag{109}$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ is the elasticity of variable x with respect to Z. Furthermore, there exist *two cutoff values for* α, $\underline{\alpha}$ *and* $\overline{\alpha}$, *such that* $(\tau_a^\star, \tau_k^\star)$ *satisfies the following properties:*

$$\begin{aligned} \tau_a^\star &\in \left[1 - \frac{1}{\beta\delta}, 0\right) \text{ and } \tau_k^\star > \theta & & \text{if } \alpha < \underline{\alpha} \\ \tau_a^\star &\in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0, \theta] & & \text{if } \underline{\alpha} \leq \alpha \leq \bar{\alpha} \\ \tau_a^\star &\in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 & & \text{if } \alpha > \bar{\alpha} \end{aligned}$$

where $\tau_a^{\max} \geq 1$, $\underline{\alpha}$ *and* $\overline{\alpha}$ *are the solutions to equation (**??**) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$, *respectively.*

Proof. For aggregate welfare:

$$\mathcal{W} = \frac{1}{1-\beta\delta}\left(n_w \log w + (1-n_w)\log \overline{a}\right) + \frac{1-n_w}{(1-\beta\delta)^2}\left(\mu \log R_h + (1-\mu)\log R_\ell\right) + v \tag{110}$$

Then, the first order condition is

$$0 = n_w \frac{d\log w}{d\tau_a} + (1-n_w)\frac{d\log \overline{a}}{d\tau_a} + \frac{1-n_w}{1-\beta\delta}\left(\mu \frac{d\log R_h}{d\tau_a} + (1-\mu)\frac{d\log R_\ell}{d\tau_a}\right), \tag{111}$$

$$0 = \left[n_w \frac{d\log w}{d\log Z} + (1-n_w)\frac{d\log K}{d\log Z} + \frac{1-n_w}{1-\beta\delta}\left(\mu \frac{d\log R_h}{d\log Z} + (1-\mu)\frac{d\log R_\ell}{d\log Z}\right)\right]\frac{d\log Z}{d\tau_a}, \tag{112}$$

$$0 = \left[n_w \xi_w + (1-n_w)\xi_K + \frac{1-n_w}{1-\beta\delta}\left(\mu \xi_{R_h} + (1-\mu)\xi_{R_\ell}\right)\right]\frac{d\log Z}{d\tau_a}. \tag{113}$$

From proposition 2 we know that ${d\log Z}/{d\tau_a} > 0$ so that an interior solution must equate the first term to zero,

$$n_w \xi_w + (1-n_w)\xi_K = -\frac{1-n_w}{1-\beta\delta}\left(\mu \xi_{R_h} + (1-\mu)\xi_{R_\ell}\right).$$

We further know that $\xi_w = \xi_K = {\alpha}/{1-\alpha}$ from Lemma 5 and that $\mu \xi_{R_h} + (1-\mu)\xi_{R_\ell} < 0$ from Lemma 4. Further, the elasticities of returns are independent of α. Because of this we can define cutoffs for α by evaluating the right hand side of the equation at $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$. If α is exactly equal to the cutoff then the optimal τ_a is either 0 or $\frac{\theta(1-\beta)}{\beta(1-\theta)}$. The monotonicity of the right hand side lets us define the intervals shown in the proposition and the uniqueness of the solution. To see the monotonicity consider the explicit formulas obtained from Lemma 4.

□

C Proofs for endogenous innovation model

Proposition 5. ***(Existence of Unique Steady State Equilibrium in Innovation Effort)*** *There exists an upper bound for the wealth tax $\overline{\tau}_a^{\mu}$ such that for $\tau_a < \overline{\tau}_a^{\mu}$, there exists a unique steady state equilibrium that features heterogeneous returns. That is, there is a unique level of the share of high-productivity entrepreneurs, $\mu^{\star}$, such that the optimal level of effort exerted by innovators satisfies equation (46), $\mu^{\star} = e\left(Z\left(\mu^{\star}\right)\right)$, and $Z\left(\mu^{\star}\right) \in \left(z_{\ell}, z_h\right)$ satisfies equation (23). The upper bound for the wealth tax satisfies*

$$\overline{\tau}_a^{\mu} = 1 - \frac{1}{\beta\delta}\left(1 - \frac{1-\delta}{\delta}\frac{1-\lambda\mu^{\star}\left(\overline{\tau}_a^{\mu}\right)}{\left(\lambda-1\right)\left(1-\frac{z_{\ell}}{z_h}\right)}\right), \tag{114}$$

where we make the dependence of μ on τ_a explicit.

Proof. We break the proof in two parts. First tackling existence and then the uniqueness of the equilibrium.

Existence We provide two proofs of this result. The first one is longer but it proves to be instructive of the workings of the model and relies on Cellina's fixed point theorem, as found in Border (1985). The second one is more direct and relies on Brouwer's fixed point theorem. The objective in both cases is to show that the mapping of the share of high-productivity entrepreneurs into itself defined by (46) (and the other equilibrium conditions) has a fixed point in the space $\mathcal{M} \equiv [0,1]$.

We start by stating Cellina's fixed point theorem. The theorem breaks the construction of a mapping φ of $\mathcal{M}$ into itself in two steps that represent the way in which the share of high-productivity entrepreneurs implies a level of productivity that in turn implies a share. The theorem is as follows:

Theorem. ***[Cellina 1969; Border 1985, Thm. 15.1]*** *Let $\mathcal{M} \subseteq \mathbb{R}^m$ be nonempty, compact, and convex. Let $\varphi : \mathcal{M} \rightrightarrows \mathcal{M}$ be a correspondence defined on K. Suppose there is a nonempty-, compact-, and convex-valued correspondence $\gamma : \mathcal{M} \rightrightarrows \mathcal{K}$ defined on $\mathcal{M}$ with values in $\mathcal{K} \subseteq \mathbb{R}^n$, a compact and convex set, and also a continuous function $f : \mathcal{M} \times \mathcal{K} \to \mathcal{M}$ such that, for every $\mu \in \mathcal{M}$, $\varphi\left(\mu\right) = \{f\left(\mu, Z\right) | Z \in \gamma\left(\mu\right)\}$. Then, φ has a fixed point.*

To apply Cellina's theorem we set $\mathcal{M} \equiv [0,1]$ as the space of shares, with typical element μ, and $\mathcal{K} = [z_{\ell}, z_h]$ as the space of productivities, with typical element Z. Both sets are nonempty, compact and convex, satisfying the theorem's requirements.

We then define the correspondence $\gamma(\mu) \equiv \min\left\{z_h, \text{Roots}^+(H,\mu)\right\}$ as the largest admissible root of the quadratic function H, defined in (23), determines equilibrium productivity:

$$
\begin{aligned}
H(Z;\mu) = \left(1-\delta^2\beta(1-\tau_a)\right)Z^2 - \left[(1-\delta)(\mu z_\lambda + (1-\mu) z_\ell) + \delta(1-\delta\beta(1-\tau_a))(z_\lambda + z_\ell)\right] Z \\
+ \delta(1-\delta\beta(1-\tau_a)) z_\ell z_\lambda.
\end{aligned}
$$

With this definition, γ is a function (a single-valued correspondence), and hence γ is nonempty-, compact-, and convex-valued.

Next, we define the function $f(\mu, Z) \equiv \max\left\{\min\left\{\left(\Lambda'\right)^{-1}(\log R_h - \log R_\ell), 1\right\}, 0\right\}$ as the solution to (46), where $\left(\Lambda'\right)^{-1}$ is the inverse of the derivative of Λ. This inverse exists and is continuous because Λ is convex and twice-continuously-differentiable functions. f takes as given μ and Z and provides a value of optimal effort, that gives a new value of μ. Notice that μ does not enter directly into f because returns are entirely determined given Z and τ_a, as seen in equation (20). So, the function is immediately (and vacuously) continuous on μ. The returns are themselves continuous in Z, see (20), so that f is itself continuous in Z.

Finally, we define the correspondence as $\varphi(\mu) \equiv \{f(\mu, Z) \,|\, Z = \gamma(\mu)\}$. All the conditions are satisfied and therefore a fixed point of φ exists. Any such fixed point is an equilibrium of the economy.

We now provide an alternative, and more direct proof based on Brouwer's fixed point theorem.

Theorem. ***[Brouwer 1912; Border 1985, Thm. 6.1]*** *Let $\mathcal{M} \subseteq \mathbb{R}^m$ be nonempty, compact, and convex. Let $\varphi : \mathcal{M} \to \mathcal{M}$ be a continuous function defined on K. Then, φ has a fixed point.*

To apply Brouwer's theorem we just need to show that the function

$$\varphi(\mu) \equiv f(\mu, \gamma(\mu)) = f\left(\mu, \min\left\{z_h, \text{Roots}^+(H,\mu)\right\}\right)$$

is continuous, with f and γ defined as above. This in fact the case because because the roots of the quadratic equation H, the minimum, the maximum, and f are continuous all continuous.

□

Proposition 6. *The equilibrium share of high-productivity entrepreneurs, $\mu^\star$, is increasing in wealth taxes, τ_a. That is, let $\tau_a^1 > \tau_a^2$, the equilibrium share of high-productivity entrepreneurs under τ_a^1, $\mu_1^\star$, is higher than the equilibrium shares of high-productivity entrepreneurs under τ_a^2, $\mu_1^\star > \mu_2^\star$.*

Proof. The proof levers on Theorem 3 in Villas-Boas (1997). We state here for completeness.

Theorem. ***[Villas-Boas 1997, Thm. 3]*** *Consider the mapping $\varphi_1 : \mathcal{M} \to \mathcal{M}$, the mapping $\varphi_2 : \mathcal{M} \to \mathcal{M}$, and a transitive, and reflexive order order $\geq$ on the set $\mathcal{M}$, such that both φ_1 and φ_2 have at least one fixed point in $\mathcal{M}$. If* *i.* φ_1 *is a weakly decreasing mapping, i.e.,* $\forall_{\mu',\mu\in\mathcal{M}}\, \mu' \geq \mu \longrightarrow \varphi_1\left(\mu'\right) \leq \varphi_1(\mu)$*;*

ii. φ_1 *is higher than* φ_2*, that is* $\varphi_1(\mu) > \varphi_2(\mu)$ *for all* $\mu \in \mathcal{M}$*,*

then, there is no fixed point $\mu_2^\star$ *of* φ_2 *which is* $>$ *than a fixed point* $\mu_1^\star$ *of* φ_1*.*

Remark. The theorem can be strengthened as it implies that the two mappings cannot have the same interior fixed point, so that we can conclude that for any (interior) fixed point $\mu_2^\star$ of φ_2 and any (interior) fixed point $\mu_1^\star$ of φ_1, it holds that $\mu_1^\star > \mu_2^\star$, that is that the order is strict. To see this, consider a fixed point $\mu_2^\star$ of φ_2 a fixed point $\mu_1^\star$ of φ_1. We already know that $\mu_1^\star \geq \mu_2^\star$ from the Theorem. Now, suppose that $\mu_2^\star = \mu_1^\star = \mu^\star$ and that $\mu^\star$ is interior. Because φ_1 is higher than φ_2 and $\mu^\star$ is a common fixed point we have $\mu^\star = \varphi_1(\mu^\star) > \varphi_2(\mu^\star) = \mu^\star$, which is a contradiction.

We now turn to verify the conditions of the Theorem to establish our desired result. Our space of interest is $\mathcal{M} \equiv [0,1]$, and so we take the order $\geq$ to be the natural order on $\mathbb{R}$, which satisfies the conditions of being a transitive, and reflexive order. We define the mappings as $\varphi_1(\mu) \equiv \varphi\left(\mu, \tau_a^1\right)$ and $\varphi_2(\mu) \equiv \varphi\left(\mu, \tau_a^2\right)$ with $\tau_a^1 > \tau_a^2$ and φ as in Proposition 5. These mappings have a unique fixed point in $K \equiv [0,1]$.

We first establish that φ is decreasing and that φ_1 is higher than φ_2. This result is proven in Lemma 8 as part of the proof of Proposition 5.

Then, we establish that an increase in wealth taxes increases effort for any given level of the share of high-productivity entrepreneurs, giving us the second condition of the Theorem. Crucially, this condition speaks to the behavior of φ for any fixed level of μ as τ_a changes. Thus, the setup of Section 3 applies. In particular, Proposition 2, which shows that $\frac{dZ}{d\tau_a} > 0$, together with Lemma 4, which shows that $\frac{dR_h}{dZ} > 0$ and $\frac{dR_\ell}{dZ} < 0$, imply that the dispersion of returns increases with τ_a, holding μ fixed, that is, $d(\log R_h - \log R_\ell)/d\tau_a > 0$. This increase leads to a higher level of effort, see (46). In other words, all the conditions for the theorem are verified and so it must be that all the (interior) equilibrium shares of high-productivity entrepreneurs under the higher wealth tax, τ_a^1, are higher as the equilibrium shares under the low wealth tax, τ_a^2.

Remark. In establishing the second condition for the theorem we make use of Lemma 4 instead of Lemma 7, provided as part of the proof of 5. The difference lies in the nature of the mapping being constructed. The mapping required for the construction of φ in this proof takes into account the equilibrium response of Z to μ and to τ_a, while the one constructed in Lemma 7 does not. Instead that mapping is interested on how arbitrary levels of productivity affect returns, and, through them, the innovation effort.

$\square$

Proposition 7. ***(Productivity Gains from Wealth Taxation with Innovation)*** *The equilibrium level of productivity,* $Z^\star$*, is increasing in wealth taxes,* τ_a*.*

Proof. This result follows from Propositions 2 and 6. To see it, consider equation (23) that defines the equilibrium level of productivity given μ and τ_a and differentiate to obtain $dZ/d\tau_a$. As in the proof of propositions 2 we re-write the function as

gives:

$$0 = \underbrace{\left[Z^2 - \left(\left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right) z_\ell\right) + \delta\left(z_\lambda + z_\ell\right)\right) Z + \delta z_\ell z_\lambda\right]}_{F(x;\mu)} + \underbrace{\left(Z - z_\ell\right)\left(z_\lambda - Z\right)\delta^2\beta\left(1-\tau_a\right)}_{G(x;\tau_a)}$$

We now look at the derivative with respect to τ_a

$$\begin{aligned}
0 &= 2Z\frac{dZ}{d\tau_a} - \left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right) z_\ell\right)\frac{dZ}{d\tau_a} - \left(1-\delta\right)\left(z_\lambda - z_\ell\right) Z\frac{d\mu}{d\tau_a} - \delta\left(z_\lambda + z_\ell\right)\frac{dZ}{d\tau_a} \\
&\quad + \left[\left(z_\lambda - Z\right) - \left(Z - z_\ell\right)\right]\delta^2\beta\left(1-\tau_a\right)\frac{dZ}{d\tau_a} - \left(Z - z_\ell\right)\left(z_\lambda - Z\right)\delta^2\beta \\
0 &= \left[2Z - \left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right) z_\ell\right) - \delta\left(z_\lambda + z_\ell\right) + \left[\left(z_\lambda - Z\right) - \left(Z - z_\ell\right)\right]\delta^2\beta\left(1-\tau_a\right)\right]\frac{dZ}{d\tau_a} \\
&\quad - \left(1-\delta\right)\left(z_\lambda - z_\ell\right) Z\frac{d\mu}{d\tau_a} - \left(Z - z_\ell\right)\left(z_\lambda - Z\right)\delta^2\beta
\end{aligned}$$

This gives

$$\begin{aligned}
\frac{dZ}{d\tau_a} &= \frac{\left(Z - z_\ell\right)\left(z_\lambda - Z\right)\delta^2\beta}{2Z - \left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right) z_\ell\right) - \delta\left(z_\lambda + z_\ell\right) + \left[\left(z_\lambda - Z\right) - \left(Z - z_\ell\right)\right]\delta^2\beta\left(1-\tau_a\right)} \\
&\quad + \frac{\left(1-\delta\right)\left(z_\lambda - z_\ell\right) Z}{2Z - \left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right) z_\ell\right) - \delta\left(z_\lambda + z_\ell\right) + \left[\left(z_\lambda - Z\right) - \left(Z - z_\ell\right)\right]\delta^2\beta\left(1-\tau_a\right)}\frac{d\mu}{d\tau_a}
\end{aligned}$$

The first term is positive as implied by proposition 2. The second term is the product of two factors. The first factor is positive (has the same denominator as the first term in the sum) and the second factor is positive following proposition 6. This completes the proof.

Alternatively, we can write the quadratic equation as the sum of two functions, one that depends on μ, F, and one on τ_a, G:

$$H\left(x;\mu,\tau_a\right) = \underbrace{\left[1 - \left(\frac{\left(1-\delta\right)\left(\mu z_\lambda + \left(1-\mu\right) z_\ell\right) + \delta\left(z_\lambda + z_\ell\right)}{x}\right) + \frac{\delta z_\ell z_\lambda}{x^2}\right]}_{F(x;\mu)} + \underbrace{\left(1 - \frac{z_\ell}{x}\right)\left(\frac{z_\lambda}{x} - 1\right)\delta^2\beta\left(1-\tau_a\right)}_{G(x;\tau_a)}$$

We had already established that G is decreasing in τ_a for $x \in \left(z_\ell, z_\lambda\right)$, which is the interval of the steady state value of Z:

$$\frac{dG\left(x,\tau_a\right)}{d\tau_a} = \underbrace{\left(1 - \frac{z_\ell}{x}\right)}_{(+)}\underbrace{\left(1 - \frac{z_\lambda}{x}\right)}_{(-)}\delta^2\beta < 0.$$

We can now establish that F is decreasing in μ for $x \in (z_\ell, z_\lambda)$:

$$\frac{dF(x;\mu)}{d\mu} = -\frac{(1-\delta)\overbrace{(z_\lambda - z_\ell)}^{(+)}}{x} < 0$$

Further, proposition 6 implies that ${d\mu}/{\tau_a} > 0$, so that $\frac{dF(x;\mu)}{d\tau_a} = \frac{dF(x;\mu)}{d\mu}\frac{d\mu}{\tau_a} < 0$. Together, these results imply that $\frac{dH(x;\mu(\tau_a),\tau_a)}{d\tau_a} < 0$. This gives the desired result: $\frac{dZ}{d\tau_a} > 0$ because as we proved in the Proposition 1, H is increasing in x in the relevant interval.

$\square$

Proposition 8. *Under Assumption 2, an interior solution ($\tau^\star_{a,\mu} < \overline{\tau}^\mu_a$) to the optimal tax combination $\left(\tau^\star_{a,\mu}, \tau^\star_{k,\mu}\right)$ that maximizes the welfare of newborns, $\mathcal{W}$, is the solution to the following equation:*

$$0 = \Bigg(\underbrace{n_w \xi^Z_w + (1-n_w)\,\xi^Z_K}_{\text{Productity Effect on } K \text{ and } w\ (+)} + \underbrace{\frac{1-n_w}{1-\beta\delta}\left(\mu\xi^Z_{R_h} + (1-\mu)\,\xi^Z_{R_\ell}\right)}_{\text{Productivity Effect on Returns (-)}} \Bigg)\frac{d\log Z}{d\tau_a} \tag{115}$$
$$+ \underbrace{\frac{1-n_w}{1-\beta\delta}\left(\mu\xi^\mu_{R_h} + (1-\mu)\,\xi^\mu_{R_\ell}\right)\frac{d\mu}{d\tau_a}}_{\text{Innovation Effect on Returns } (+)}$$

where $\xi^Z_x \equiv \frac{\partial \log x}{\partial \log Z}$ is the elasticity of variable x with respect to Z and $\xi^\mu_x \equiv \frac{\partial \log x}{\partial \mu}$. Recall from Lemma 5 that $\xi^Z_w = \xi^Z_K = {\alpha}/{1-\alpha}$.

Furthermore, there are two cutoff values for α, $\underline{\alpha}_\mu$ and $\overline{\alpha}_\mu$, such that $\left(\tau^\star_{a,\mu}, \tau^\star_{k,\mu}\right)$ has the following properties:

$$\begin{aligned}
\tau^\star_{a,\mu} &\in \left[1-\frac{1}{\beta\delta}, 0\right) \text{ and } \tau^\star_{k,\mu} > \theta && \text{if } \alpha < \underline{\alpha}_\mu \\
\tau^\star_{a,\mu} &\in \left[0, \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}\right] \text{ and } \tau^\star_{k,\mu} \in [0,\theta] && \text{if } \underline{\alpha}_\mu \le \alpha \le \overline{\alpha}_\mu \\
\tau^\star_{a,\mu} &\in \left(\frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}, \tau^{\max}_a\right) \text{ and } \tau^\star_{k,\mu} < 0 && \text{if } \alpha > \overline{\alpha}_\mu
\end{aligned}$$

where $\tau^{\max}_a \geq 1$ is such that $R_\ell \geq 0$, and $\underline{\alpha}_\mu$ and $\overline{\alpha}_\mu$ are the solutions to equation ((115)) with $\tau_a = 0$ and $\tau_a = \frac{\theta(1-\beta\delta)}{\beta\delta(1-\theta)}$, respectively. When $\theta = 0$ and there are no revenue needs, so $\underline{\alpha}_{,\mu} = \overline{\alpha}_{,\mu}$.

Proof. The interior solution to optimal taxes satisfies the following first order condition

$$0 = \frac{d\mathcal{W}}{d\tau_a} = n_w \frac{dV_w(w)}{d\tau_a} + (1-n_w)\frac{d}{d\tau_a}\left(\mu V_h(\overline{a}) + (1-\mu)V_\ell(\overline{a}) - \frac{\Lambda(\mu)}{(1-\beta\delta)^2}\right).$$

Replacing for the value of workers and entrepreneurs we get

$$\begin{aligned}\mathcal{W} = &\frac{n_w}{1-\beta\delta}\log w + \frac{1-n_w}{1-\beta\delta}\log(\overline{a}) \\ &+ \frac{1-n_w}{1-\beta\delta}\left(\frac{\beta\delta\log\beta\delta + (1-\beta\delta)\log(1-\beta\delta)}{1-\beta\delta} + \frac{\mu\log R_h + (1-\mu)\log R_\ell}{1-\beta\delta} - \frac{\Lambda(\mu)}{(1-\beta\delta)}\right).\end{aligned}$$

Recall that $V_i(a) = m_i + n\log(a)$, where $n = \frac{1}{1-\beta\delta}$ and $m_i = \frac{1}{(1-\beta\delta)^2}[\beta\delta\log\beta\delta + (1-\beta\delta)\log(1-\beta\delta) + \log R_i]$.

Then, the first order condition is

$$\begin{aligned}0 &= n_w\frac{d\log w}{d\tau_a} + (1-n_w)\frac{d\log(\overline{a})}{d\tau_a} + \frac{1-n_w}{1-\beta\delta}\left(\frac{d(\mu\log R_h + (1-\mu)\log R_\ell)}{d\tau_a} - \frac{d\Lambda(\mu)}{d\tau_a}\right) \\ 0 &= n_w\frac{d\log w}{d\tau_a} + (1-n_w)\frac{d\log(\overline{a})}{d\tau_a} \\ &\quad + \frac{1-n_w}{1-\beta\delta}\left(\left(\mu\frac{d\log R_h}{d\tau_a} + (1-\mu)\frac{d\log R_\ell}{d\tau_a}\right) + \left[\underbrace{(\log R_h - \log R_\ell) - \Lambda'(\mu)}_{=0}\right]\frac{d\mu}{d\tau_a}\right)\end{aligned}$$

The last term is equal to zero because individuals already optimize over their innovation effort, so that, by the Pareto principle, taxes cannot improve on their choice. This leaves the local effects of wealth taxes taking μ as given (at its equilibrium level). We can further simplify these effects by noticing that, given μ, the effect of wealth taxes is only felt through the change in equilibrium productivity. This gives,

$$0 = n_w\frac{d\log w}{d\tau_a} + (1-n_w)\frac{d\log(\overline{a})}{d\tau_a} + \frac{1-n_w}{1-\beta\delta}\left(\mu\frac{d\log R_h}{d\tau_a} + (1-\mu)\frac{d\log R_\ell}{d\tau_a}\right),$$

where the issue is how to get the derivative of returns taking into account the effect of taxes in equilibrium.

We start from the representation of steady state returns in terms of μ and Z:

$$R_\ell = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)(1-\mu)(z_\lambda - z_\ell)}{z_\lambda - Z}\right) \quad \text{and} \quad R_h = \frac{1}{\beta\delta^2}\left(1 - \frac{(1-\delta)\mu(z_\lambda - z_\ell)}{Z - z_\ell}\right),$$

so that we can express the change in returns as

$$\frac{d\log R_i}{d\tau_a} = \frac{d\log R_i}{d\log Z}\frac{d\log Z}{d\tau_a} + \frac{d\log R_i}{d\mu}\frac{d\mu}{d\tau_a}$$

and the optimal tax formula as

$$
\begin{aligned}
0 = n_w \frac{d\log w}{d\log Z} + (1-n_w)\frac{d\log(\overline{a})}{d\log Z} + \frac{1-n_w}{1-\beta\delta}\left(\mu\frac{\partial\log R_h}{\partial\log Z} + (1-\mu)\frac{\partial\log R_\ell}{\partial\log Z}\right) \\
+ \frac{1-n_w}{1-\beta\delta}\left(\mu\frac{\partial\log R_h}{\partial\mu} + (1-\mu)\frac{\partial\log R_\ell}{\partial\mu}\right)\frac{\frac{d\mu}{d\tau_a}}{\frac{d\log Z}{d\tau_a}}
\end{aligned}
$$

We can write it more succinctly by defining the elasticities with respect as $\xi_x^Z \equiv {}^{\partial\log x}/_{\partial\log Z}$ and with respect to innovation as $\xi_x^\mu \equiv {}^{\partial\log x}/_{\partial\mu}$:

$$
0 = n_w\xi_w^Z + (1-n_w)\xi_K^Z + \frac{1-n_w}{1-\beta\delta}\left(\mu\xi_{R_h}^Z + (1-\mu)\xi_{R_\ell}^Z\right) + \frac{1-n_w}{1-\beta\delta}\left(\mu\xi_{R_h}^\mu + (1-\mu)\xi_{R_\ell}^\mu\right)\frac{\frac{d\mu}{d\tau_a}}{\frac{d\log Z}{d\tau_a}}
$$

We now turn to show that the sign of the last two effects. We first obtaining a useful expression for the equilibrium level of μ. We know that (23) must be satisfied in equilibrium so we can write μ as follows,

$$
\mu = \frac{Z-z_\ell}{z_\lambda - z_\ell}\left(1 - \frac{\delta(1-\eta)}{1-\delta}\frac{(z_\lambda - Z)}{Z}\right) \tag{116}
$$

$$
1-\mu = \frac{z_\lambda - Z}{z_\lambda - z_\ell}\left(1 + \frac{\delta(1-\eta)}{1-\delta}\frac{(Z-z_\ell)}{Z}\right) \tag{117}
$$

With this expression for μ we can now establish the effect of productivity and innovation on returns.

Effect on welfare of change in returns from productivity:

From the expression for steady state returns we know that

$$
\frac{\partial R_h}{\partial Z} = \frac{(1-\delta)\mu}{\beta\delta^2}\frac{z_\lambda - z_\ell}{(Z-z_\ell)^2} \qquad \text{and} \qquad \frac{\partial R_\ell}{\partial Z} = -\frac{(1-\delta)(1-\mu)}{\beta\delta^2}\frac{z_\lambda - z_\ell}{(z_\lambda - Z)^2}.
$$

We use this directly for the effect of productivity.

$$
\begin{aligned}
\mu\frac{Z}{R_h}\frac{\partial R_h}{\partial Z}+(1-\mu)\frac{Z}{R_\ell}\frac{\partial R_\ell}{\partial Z} &= \frac{(1-\delta)(z_\lambda-z_\ell)}{\beta\delta^2}Z\left(\frac{1}{R_h}\left(\frac{\mu}{Z-z_\ell}\right)^2-\frac{1}{R_\ell}\left(\frac{1-\mu}{z_\lambda-Z}\right)^2\right)\\
&= (1-\delta)(z_\lambda-z_\ell)Z\left(\frac{\left(\frac{\mu}{Z-z_\ell}\right)^2}{\left(1-(1-\delta)(z_\lambda-z_\ell)\frac{\mu}{Z-z_\ell}\right)}\right.\\
&\qquad\left.-\frac{\left(\frac{1-\mu}{z_\lambda-Z}\right)^2}{\left(1-(1-\delta)(z_\lambda-z_\ell)\left(\frac{1-\mu}{z_\lambda-Z}\right)\right)}\right)\\
&= \frac{1}{(1-\delta)\delta(z_\lambda-z_\ell)}\left(\frac{((1-\delta\eta)Z-\delta(1-\eta)z_\lambda)^2}{\eta Z+(1-\eta)z_\lambda}\right.\\
&\qquad\left.-\frac{((1-\delta\eta)Z-\delta(1-\eta)z_\ell)^2}{\eta Z+(1-\eta)z_\ell}\right)\\
&< 0
\end{aligned}
$$

Effect on welfare of change in returns from innovation:

We first get the derivative of returns with respect to μ:

$$
\frac{\partial R_h}{\partial\mu}=-\frac{1}{\beta\delta^2}\frac{(1-\delta)(z_\lambda-z_\ell)}{Z-z_\ell}\qquad\text{and}\qquad\frac{\partial R_\ell}{\partial\mu}=\frac{1}{\beta\delta^2}\frac{(1-\delta)(z_\lambda-z_\ell)}{z_\lambda-Z}.
$$

Now we use this to establish the effect of innovation on returns:

$$
\begin{aligned}
\mu\frac{1}{R_h}\frac{\partial R_h}{\partial\mu}+(1-\mu)\frac{1}{R_\ell}\frac{\partial R_\ell}{\partial\mu} &= \frac{(1-\delta)}{\beta\delta^2}\left[-\frac{1}{R_h}\mu\frac{z_\lambda-z_\ell}{Z-z_\ell}+\frac{1}{R_\ell}(1-\mu)\frac{(z_\lambda-z_\ell)}{z_\lambda-Z}\right]\\
&= (1-\delta)\left[\frac{(1-\mu)\frac{(z_\lambda-z_\ell)}{z_\lambda-Z}}{1-(1-\delta)\frac{(1-\mu)(z_\lambda-z_\ell)}{z_\lambda-Z}}-\frac{\mu\frac{z_\lambda-z_\ell}{Z-z_\ell}}{1-(1-\delta)\frac{\mu(z_\lambda-z_\ell)}{Z-z_\ell}}\right]\\
&= \frac{1}{\delta}\left(\frac{(1-\delta\eta)Z-\delta(1-\eta)z_\ell}{\eta Z+(1-\eta)z_\ell}-\frac{(1-\delta\eta)Z-\delta(1-\eta)z_\lambda}{\eta Z+(1-\eta)z_\lambda}\right)\\
&> 0
\end{aligned}
$$

□

D Intensive margin entrepreneurial effort

Consider a model like that in Section 2 where entrepreneurs can exert effort every period to increase their output. We capture the effect of effort as modifying the production function of entrepreneurs to:

$$y = (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma}. \tag{118}$$

where $\gamma \in [0,1)$. Exerting effort has a utility cost of $h(e)$, where $h'(e) > 0$ and $h''(e) \geq 0$ but no dynamic effects. The utility function is now

$$u(c,e) = \log(c - h(e)). \tag{119}$$

D.1 Entrepreneurial problem with effort in production

We solve for the entrepreneur's static effort and labor demand choices. The solution is characterized by the following first order conditions:

$$u_e h'(e) = (1-\tau_k) u_c \cdot \gamma (zk)^{\alpha} g(e)^{\gamma-1} n^{1-\alpha-\gamma} g'(e) \qquad w = (1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma} n^{-\alpha-\gamma} \tag{120}$$

which imply:

$$n = \left[\frac{(1-\alpha-\gamma)(zk)^{\alpha} g(e)^{\gamma}}{w}\right]^{\frac{1}{\alpha+\gamma}} \tag{121}$$

replacing:

$$\frac{u_e}{u_c}\frac{h'(e)}{g'(e)} = (1-\tau_k)\gamma (zk)^{\frac{\alpha}{\alpha+\gamma}} g(e)^{\frac{-\alpha}{\alpha+\gamma}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha+\gamma}} \tag{122}$$

$$g(e) = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} zk \tag{123}$$

For tractability we impose that $\frac{h'(e)}{g'(e)} = \psi$ is constant, say with $h(e) = \psi e$ and $g(e) = e$. If that is the case we can write labor demand as:

$$n = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}} \left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}} zk \tag{124}$$

and profits as:

$$\pi(z,k) = (zk)^{\alpha} g(e)^{\gamma} n^{1-\alpha-\gamma} - wn - rk \tag{125}$$
$$= \left[\underbrace{(\alpha+\gamma)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z - r}_{\pi^{\star}(z)}\right] k$$

Both profits and effort are proportional to how much capital the entrepreneur uses. The entrepreneur will only demand capital and operate their firm if the (after-tax) profits net of the effort cost are positive, that is:

$$k \geq 0 \longleftrightarrow (1-\tau_k)\,\pi^{\star}(z) - \underbrace{\frac{u_e h'(e)}{u_c}}_{\text{Shadow Price}} \varepsilon(z) \geq 0, \tag{126}$$

where the shadow price of the effort cost is equal to ψ given our assumptions and

$$\varepsilon(z) \equiv \frac{e(z,k)}{k} = \left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\alpha+\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z. \tag{127}$$

In order to demand capital the entrepreneur must make profits to cover the cost of effort.

The optimal demand for capital is then:

$$k^{\star}(z,a) \;=\; \begin{cases} \lambda a & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z > r \\ [0,\lambda a] & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z = r \\ 0 & \text{if } \alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z < r \end{cases} \tag{128}$$

With this demand for capital we can replace back and get the level of profits, effort and labor demand.

Before proceeding to the optimal savings choice of the agent we need to determine the level of the capital demand for each type of entrepreneur. The relevant case has high-productivity entrepreneurs demanding $k^{\star}(z_h,a) = \lambda a$ for a total demand of $K_h = \lambda\mu A_h$. The remaining assets are used by the low-productivity entrepreneurs who will be indifferent between any production level. The total demand for capital required to clear the market is $K_L = (1-\mu)A_L - (\lambda-1)\mu A_h$.

Let $\lambda_{\ell,\iota} \equiv \frac{k_\iota}{a_\iota}$ be the ratio of capital to assets of low-productivity entrepreneur ι, for $\iota \in [0,1]$. We will show that the savings choice of the entrepreneur is independent of the value of $\lambda_{\ell,\iota}$.

Now we turn to the value function:

$$V_\iota(a,z) = \max_{\{c,a'\}} \ln\left(c - \psi e_\iota(z,a)\right) + \beta\delta E\left[V_\iota\left(a',z'\right)|z\right] \qquad \text{s.t. } c + a' = R_\iota(z)\, a \tag{129}$$

where $R(z) \equiv (1-\tau_a) + (1-\tau_k)\left(r + \pi^\star(z)\,\lambda_\iota(z)\right)$, and where $e_\iota(z,a) = \varepsilon(z)\,\lambda_\iota(z)\,a$. The value of $\lambda_\iota(z)$ satisfies:

$$\lambda_\iota(z) = \begin{cases} \lambda & \text{if } z = z_h \\ \lambda_{\iota,\ell} & \text{if } z = z_\ell. \end{cases} \tag{130}$$

We solve the dynamic programming problem of the entrepreneur via guess and verify. To this end, we guess that the value function of an entrepreneur of type $i \in \{\ell, h\}$ has the form

$$V_{i,\iota}(a) = m_{i,\iota} + n\log(a)\,, \tag{131}$$

where $\{m_{\ell,\iota}, m_{h,\iota}\}_{\iota\in\{0,1\}}, n \in \mathbb{R}$ are coefficients. Under this guess the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{\left(R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}\right)a - a_i'} = \frac{\beta\delta n}{a_i'} \longrightarrow a_i' = \frac{\beta\delta n}{1+\beta\delta n}\left(R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}\right)a. \tag{132}$$

Replacing the savings rule into the value function gives:

$$V_{i,\iota}(a) = \log\left(\left(R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}\right)a - a_i'\right) + \beta\delta V_{i,\iota}\left(a_i'\right) \tag{133}$$

$$m_{i,\iota} + n\log(a) = \log\left(\left(R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}\right)a - a_i'\right) + \beta\delta m_{i,\iota} + \beta\delta n\log\left(a_i'\right) \tag{134}$$

Matching coefficients:

$$n = 1 + \beta\delta n \tag{135}$$

$$m_{i,\iota} = \beta\delta n\log(\beta\delta n) + (1+\beta\delta n)\log\left(\frac{R_{i,\iota} - \psi\varepsilon_i\lambda_{i,\iota}}{1+\beta\delta n}\right) + \beta\delta m_{i,\iota}. \tag{136}$$

The solution to the first equation implies:

$$n = \frac{1}{1-\beta\delta}, \tag{137}$$

which in turn delivers the optimal saving decision of the entrepreneur:

$$a' \;=\; \beta\delta\left(R_\iota\left(z\right) - \psi\varepsilon\left(z\right)\lambda_\iota\left(z\right)\right)a. \tag{138}$$

Finally, we solve for the remaining coefficients for the relevant case in which high-productivity entrepreneurs are all constrained and low-productivity entrepreneurs are indifferent between any level of production. In that case, it holds that:

$$\begin{aligned} R_\iota\left(z_\ell\right) - \psi\varepsilon\left(z_\ell\right)\lambda_\iota\left(z_\ell\right) &= \left(1-\tau_a\right) + \left(1-\tau_k\right)r + \left[\left(1-\tau_k\right)\pi^\star\left(z\right) - \psi\varepsilon\left(z_\ell\right)\right]\lambda_\iota\left(z_\ell\right) \\ &= \left(1-\tau_a\right) + \left(1-\tau_k\right)r \end{aligned} \tag{139}$$

which is independent of the identity of the entrepreneur. It also holds that

$$\begin{aligned} R_\iota\left(z_h\right) - \psi\varepsilon\left(z_h\right)\lambda_\iota\left(z_h\right) &= \left(1-\tau_a\right) + \left(1-\tau_k\right)\left(r + \pi^\star\left(z_h\right)\lambda\right) - \psi\varepsilon\left(z_h\right)\lambda \\ &= \left(1-\tau_a\right) + \left(1-\tau_k\right)\left(\left(1-\lambda\right)r + \alpha\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z\lambda\right), \end{aligned} \tag{140}$$

which is also independent of the identity of the entrepreneur. Consequently, we can write without loss:

$$R_\iota\left(z\right) - \psi\varepsilon\left(z\right)\lambda_\iota\left(z\right) = R\left(z\right) - \psi\varepsilon\left(z\right)\lambda \equiv \hat{R}\left(z\right) \tag{141}$$

Having established these results, we can solve for m_ℓ and m_h from the system of linear equations:

$$m_i = \frac{1}{\left(1-\beta\delta\right)^2}\left(\log\left(\left(\beta\delta\right)^{\beta\delta}\left(1-\beta\delta\right)^{1-\beta\delta}\right)\right) + \frac{1}{\left(1-\beta\delta\right)^2}\log\hat{R}\left(z\right) \tag{142}$$

D.2 Equilibrium and aggregation

In equilibrium the interest rate is such that the low-productivity entrepreneurs are indifferent between lending their assets or using them in their own firm. Lending the assets gives them a (before-tax) return of r, using them gives them $\pi^\star\left(z_\ell\right)$ but it also entails a utility cost because of effort, which we know from the previous results is proportional to assets, same as returns and profits. The agents will be indifferent if the (after-tax) profits net of effort costs are zero:

$$0 = (1 - \tau_k)\,\pi^\star(z_\ell) - \underbrace{\frac{u_e h'(e)}{u_c}}_{\text{Shadow Price}}\,\varepsilon(z_\ell) \tag{143}$$

$$0 = (1 - \tau_k)\,\pi^\star(z_\ell) - \psi\varepsilon(z_\ell) \tag{144}$$

replacing for the optimal solution of the entrepreneur's problem:

$$r = \alpha\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}} z_\ell \tag{145}$$

We can then exploit the linearity of the savings function to aggregate results:

Lemma 6. *In the heterogenous return equilibrium* $((\lambda - 1)\,\mu A_h < (1-\mu)\,A_\ell)$, *output, wages, interest rate, and gross returns on savings are:*

$$Y = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}}\, L^{\frac{1-\alpha-\gamma}{1-\gamma}} \tag{146}$$

$$E = \left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{1}{1-\gamma}} (ZK)^{\frac{\alpha}{1-\gamma}}\, L^{\frac{1-\alpha-\gamma}{1-\gamma}} \tag{147}$$

$$w = (1-\alpha-\gamma)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}} \tag{148}$$

$$r = \alpha\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell \tag{149}$$

$$R_{\ell,\iota} = (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha + \gamma\lambda_\iota)\,z_\ell \tag{150}$$

$$R_h = (1-\tau_a) + (1-\tau_k)\left(\frac{(1-\tau_k)\,\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} (\alpha z_\lambda + \gamma\lambda z_h) \tag{151}$$

and

$$\hat{R}(z) \equiv R(z) - \psi\varepsilon(z)\,\lambda = \begin{cases} (1-\tau_a) + (1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell & \text{if } z = z_\ell \\ (1-\tau_a) + (1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\lambda & \text{if } z = z_h \end{cases} \tag{152}$$

Proof. We start by considering the labor market clearing condition, we get

$$
\begin{aligned}
n^{\star}\left(z_h, K_h\right)+n^{\star}\left(z_\ell, K_\ell\right) &= L \\
\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\gamma}{\alpha}}\left(z_h K_h+z_\ell K_\ell\right) &= L \\
(1-\alpha-\gamma)\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}} &= w \qquad (153)
\end{aligned}
$$

Turning to the total effort we get:

$$
\begin{aligned}
\left(\frac{E}{ZK}\right)^{\alpha} &= \left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\alpha+\gamma}\left(\frac{1-\alpha-\gamma}{w}\right)^{1-\alpha-\gamma} \\
E &= \left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{1}{1-\gamma}}(ZK)^{\frac{\alpha}{1-\gamma}} L^{\frac{1-\alpha-\gamma}{1-\gamma}} \qquad (154)
\end{aligned}
$$

replacing back and then applying the result to the interest rate we get the usual Cobb-Douglas expressions:

$$
w=(1-\alpha-\gamma)\frac{(ZK)^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{L} \qquad (155)
$$

$$
r=\alpha\frac{(ZK)^{\alpha}E^{\gamma}L^{1-\gamma-\alpha}}{ZK}z_\ell \qquad (156)
$$

We can go further by replacing E which itself depends on other aggregates:

$$
r=\alpha\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}} z_\ell \qquad (157)
$$

$$
w=(1-\alpha-\gamma)\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{ZK}{L}\right)^{\frac{\alpha}{1-\gamma}} \qquad (158)
$$

These two expressions also let us rewrite the profit rate (of capital) of entrepreneurs:

$$
\pi^{\star}(z)=\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\left(\alpha\left(z-z_\ell\right)+\gamma z\right)>0 \qquad (159)
$$

Notice that profits are always positive for both types of entrepreneurs.

We can then use the equilibrium profit rates of entrepreneurs to rewrite the gross returns of entrepreneurs:

$$
\begin{aligned}
R(z) &= \left(1-\tau_a\right)+\left(1-\tau_k\right)\left(r+\pi^{\star}(z)\lambda\right) \qquad (160) \\
&= \left(1-\tau_a\right)+\left(1-\tau_k\right)\left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}\left(\alpha\left(z_\ell+\lambda\left(z-z_\ell\right)\right)+\gamma\lambda z\right)
\end{aligned}
$$

we can express this as:

$$R(z)=\begin{cases}(1-\tau_a)+(1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha+\gamma\lambda)\,z_\ell & \text{if } z=z_\ell\\ (1-\tau_a)+(1-\tau_k)\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\alpha z_\lambda+\gamma\lambda z_h) & \text{if } z=z_h\end{cases} \tag{161}$$

We are loosely referring as λ to the ratio of capital to assets of the entrepreneur. This ratio can vary by entrepreneur for the low-productivity entrepreneurs.

The return net of effort cost is:

$$\begin{aligned}\hat{R}(z)&=R(z)-\psi\varepsilon(z)\lambda \\ &=(1-\tau_a)+(1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}(\lambda z+(1-\lambda)z_\ell)\end{aligned} \tag{162}$$

More explicitly:

$$\hat{R}(z)=\begin{cases}(1-\tau_a)+(1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}z_\ell & \text{if } z=z_\ell\\ (1-\tau_a)+(1-\tau_k)\,\alpha\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}\left(\frac{L}{ZK}\right)^{\frac{1-\alpha-\gamma}{1-\gamma}}z_\lambda & \text{if } z=z_h\end{cases} \tag{163}$$

Finally we consider aggregate output, for this note that the ratio of labor to capital is constant across entrepreneurs which allows us to aggregate in terms of the total capital of each type. We can express the output of an individual entrepreneur with productivity z and capital k as:

$$y(z,k)=\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}}zk \tag{164}$$

Aggregate output is the sum of the total output produced by each type of entrepreneur:

$$\begin{aligned}Y&=\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{\alpha}}\left(\frac{1-\alpha-\gamma}{w}\right)^{\frac{1-\alpha-\gamma}{\alpha}}(z_hK_h+z_\ell K_\ell)\\ &=\left(\frac{(1-\tau_k)\gamma}{\psi}\right)^{\frac{\gamma}{1-\gamma}}(ZK)^{\frac{\alpha}{1-\gamma}}L^{\frac{1-\alpha-\gamma}{1-\gamma}}\end{aligned} \tag{165}$$

For completeness we also consider the aggregate effort of high- and low-productivity entrepreneurs:

$$E_i\equiv\int e(z,k_{\iota,i})\,d\iota=\varepsilon(z_i)\int k_{\iota,i}d\iota=\left[\frac{(1-\tau_k)\gamma}{\psi}ZK^{-(1-\alpha-\gamma)}L^{1-\alpha-\gamma}\right]^{\frac{1}{1-\gamma}}z_iK_i \tag{166}$$

This completes the derivation of the results.

□

We now turn to the evolution of aggregates:Using the savings decision rules of each type, we can obtain the law of motions for aggregate wealth held by each type as

$$A_i^{'} = \delta^2 \beta \hat{R}_i A_i + (1-\delta)\,\overline{a}, \tag{167}$$

where $\overline{a} \equiv K = (1-\mu)\,A_\ell + \mu A_h$ is the wealth endowment of a newborn entrepreneur, equal to the total (average) wealth in the economy.

From the evolution of the assets of low- and high-productivity entrepreneurs, we obtain the low of motion for aggregate capital:

$$\frac{K^{'}}{K} = \delta^2 \beta \left(s_h \hat{R}_h + (1-s_h)\,\hat{R}_\ell \right) + (1-\delta)\,. \tag{168}$$

D.3 Steady state effects of wealth taxes with entrepreneurial effort

In steady state it must be that wealth weighted (effective) returns are constant in steady state:

$$s_h \hat{R}_h + (1-s_h)\,\hat{R}_\ell = \frac{1}{\beta\delta} \tag{169}$$

$$(1-\tau_a) + (1-\tau_k)\,\alpha \left(\frac{(1-\tau_k)\,\gamma}{\psi} \right)^{\frac{\gamma}{1-\gamma}} \left(\frac{L}{ZK} \right)^{\frac{1-\alpha-\gamma}{1-\gamma}} Z = \frac{1}{\beta\delta} \tag{170}$$

This is similar to the result in 19 but it includes the distortionary effect of capital income taxes on effort. As in the benchmark model this result implies that returns net of effort cost are:

$$\hat{R}\,(z) = \begin{cases} (1-\tau_a) + \left(\frac{1}{\beta\delta} - (1-\tau_a) \right) \frac{z_\ell}{Z} & \text{if } z = z_\ell \\ (1-\tau_a) + \left(\frac{1}{\beta\delta} - (1-\tau_a) \right) \frac{z_\lambda}{Z} & \text{if } z = z_h \end{cases} \tag{171}$$

which is the same as in Lemma 2 for the returns.

Equations (167) and (171) imply that equations 23 applies unchanged and determines the steady state level of productivity as in Section 2.5.

Consequently, Propositions 1 and 2 apply to this economy without modifications:

Proposition 9. *Propositions s 1 and 2 apply to this economy, so that a steady state equilibrium with heterogeneous returns exists if and only if $\lambda < \overline{\lambda}$, and an increase in wealth taxes in such an equilibrium increases productivity Z.*

The difference between the model in Section 2 and the model with effort is in the response of aggregate variables other than Z to changes in taxes. It turns out that all directions are maintained, but there are now two sources of changes on aggregates. The first source is, as in Section 2, a change in productivity. The second source is a direct effect of taxes on the effort of entrepreneurs. An increase in wealth taxes reduces capital income taxes to balance the government's budget, in turn reducing the distortions on the effort choice of entrepreneurs.

Before establishing the effects of a change in taxes on aggregate variables we revisit the role of government spending. The Government's constraint can be expressed just as before:

$$G = \tau_k \alpha Y + \tau_a K. \tag{172}$$

Assumption 2 still implies that:

$$\frac{1-\tau_k}{1-\beta\delta\left(1-\tau_a\right)} = \frac{1-\theta}{1-\beta\delta} \tag{173}$$

Then, steady state capital is, under Assumption 2:

$$K = \left(\alpha\beta\delta\frac{1-\theta}{1-\beta\delta}\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{\left(1-\tau_k\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L \tag{174}$$

Note that the level of capital depends directly on capital income taxes through their effect on effort. Alternatively, we can write the value of capital in terms of the level of wealth taxes:

$$K = \left(\alpha\beta\right)^{\frac{1-\gamma}{1-\alpha-\gamma}} \left(\frac{1-\theta}{1-\beta\delta}\right)^{\frac{1}{1-\alpha-\gamma}} \left(\frac{\left(1-\beta\delta\left(1-\tau_a\right)\right)\gamma}{\psi}\right)^{\frac{\gamma}{1-\alpha-\gamma}} Z^{\frac{\alpha}{1-\alpha-\gamma}} L \tag{175}$$

This makes it clear that aggregate capital increases with wealth taxes both through the efficiency gains (higher Z) and the decrease in distortions, lower τ_k.

Lemma 5. *If $\tau < \overline{\tau}_a$ and under Assumption 2, an increase in wealth taxes (τ_a) increases aggregate entrepreneurial effort, capital, output, and wages, $\frac{dE}{d\tau_a}, \frac{dK}{d\tau_a}, \frac{dY}{d\tau_a}, \frac{dw}{d\tau_a} > 0$. It also increases the wealth share of high-productivity entrepreneurs, $\frac{ds_h}{d\tau_a} > 0$, and the after-tax return net of effort costs of high-productivity entrepreneurs, $\frac{d\hat{R}_h}{d\tau_a} > 0$, while the after-tax returns net of effort costs of low-productivity entrepreneurs decreases, $\frac{d\hat{R}_\ell}{d\tau_a} < 0$.*

Proof. The wealth share of high-productivity entrepreneurs is tied to productivity by:

$$s_h = \frac{Z - z_\ell}{z_\lambda - z_\ell}$$

so that the wealth share changes in the same direction as productivity. Productivity increases following Proposition 9.

The results for after-tax returns net of effort costs follow from a straightforward modification of Lemma 4 which gives:

$$\frac{d\hat{R}_h}{d\tau_a} > 0 \qquad \text{and} \qquad \frac{d\hat{R}_\ell}{d\tau_a} < 0$$

Total capital increases with wealth taxes:

$$\frac{d \log K}{d \log \tau_a} = \frac{\gamma}{1-\alpha-\gamma} \frac{\beta\delta\tau_a}{1-\beta\delta\left(1-\tau_a\right)} + \frac{\alpha}{1-\alpha-\gamma} \frac{d \log Z}{d \log \tau_a} > 0$$

It follows immediately that output, wages, and total effort increase since they depend positively on ZK and negatively on capital income taxes τ_k.

□

D.4 Optimal taxes

Introducing an effort choice for entrepreneurs changes the choice of optimal taxes in two direct ways. First, the equilibrium level of wages and wealth depend on taxes directly through the effect of taxes on effort. Second, entrepreneurial welfare depends now on after-tax returns net of effort cost. However, only the first effect has an effect on the choice of optimal taxes. This is because in steady state the after-tax returns net of effort cost behave exactly like after-tax returns did in the model of Section 2. This leads to the following result:

Proposition 10. *Under Assumption 2, there exist a unique tax combination $(\tau_a^\star, \tau_k^\star)$ that maximizes the utilitarian welfare, an interior solution $\tau_a^\star < \overline{\tau}_a$ is the solution to:*

$$\frac{\gamma}{1-\alpha-\gamma} \frac{\beta\delta}{\frac{\log Z}{d\tau_a}} + \frac{\alpha}{1-\alpha-\gamma} = -\frac{1-n_w}{1-\beta\delta}\left(\mu\xi^Z_{\hat{R}_h} + \left(1-\mu\right)\xi^Z_{\hat{R}_\ell}\right) \tag{176}$$

where $\xi^Z_x \equiv \frac{d\log x}{d \log Z}$ is the elasticity of variable x with respect to Z. An interior solution is higher than a solution to taxes in Proposition 8 and is equal if and only if $\gamma = 0$.

Proof. Newborn welfare is

$$\mathcal{W} = \frac{1}{1-\beta\delta}\left(n_w \log w + (1-n_w)\log \overline{a}\right) + \frac{1-n_w}{(1-\beta\delta)^2}\left(\mu \log \hat{R}_h + (1-\mu)\log \hat{R}_\ell\right) + v \tag{177}$$

Then, the first order condition is

$$0 = n_w \frac{d\log w}{d\tau_a} + (1-n_w)\frac{d\log \overline{a}}{d\tau_a} + \frac{1-n_w}{1-\beta\delta}\left(\mu \frac{d\log \hat{R}_h}{d\tau_a} + (1-\mu)\frac{d\log \hat{R}_\ell}{d\tau_a}\right), \tag{178}$$

where we have

$$\frac{\log \overline{a}}{d\tau_a} = \frac{\log w}{d\tau_a} = \frac{\gamma}{1-\alpha-\gamma}\beta\delta + \frac{\alpha}{1-\alpha-\gamma}\frac{\log Z}{d\tau_a} \tag{179}$$

and

$$\mu \frac{d\log \hat{R}_h}{d\tau_a} + (1-\mu)\frac{d\log \hat{R}_\ell}{d\tau_a} = \left(\mu \xi^Z_{\hat{R}_h} + (1-\mu)\xi^Z_{\hat{R}_\ell}\right)\frac{\log Z}{d\tau_a}.$$

Joining gives

$$0 = \frac{\gamma}{1-\alpha-\gamma}\frac{\beta\delta}{\frac{\log Z}{d\tau_a}} + \frac{\alpha}{1-\alpha-\gamma} + \frac{1-n_w}{1-\beta\delta}\left(\mu \xi^Z_{\hat{R}_h} + (1-\mu)\xi^Z_{\hat{R}_\ell}\right). \tag{180}$$

Further, we can group terms as follows

$$0 = \left[\frac{\gamma}{1-\alpha-\gamma}\frac{\beta\delta}{\frac{\log Z}{d\tau_a}} + \frac{\alpha}{1-\alpha-\gamma} - \frac{\alpha}{1-\alpha}\right] + \left[\frac{\alpha}{1-\alpha} + \frac{1-n_w}{1-\beta\delta}\left(\mu \xi^Z_{\hat{R}_h} + (1-\mu)\xi^Z_{\hat{R}_\ell}\right)\right]$$

The second term is the same as the optimality condition in Proposition 8, were the elasticity of capital and wages was equal to $\alpha/1-\alpha$. Notice also that the average elasticity of returns net of effort cost is equal to the elasticity of returns in 8 as they do not depend on effort. The first term is positive as $\frac{\log Z}{d\tau_a} > 0$ and $\frac{\alpha}{1-\alpha-\gamma} \geq \frac{\alpha}{1-\alpha}$ and is equal to zero if and only if $\gamma = 0$ and effort does not affect production. This implies that:

1. Optimal wealth taxes are higher than in Proposition 8.
2. They are equal if and only if $\gamma = 0$.

□

E Persistence of Productivity

E.1 Entrepreneurial problem

Given taxes τ_a and τ_k and constant prices, an entrepreneur's optimal savings problem can be written as

$$V(a,z) = \max_{a'} \log(c) + \beta \sum_{z'} \Pi(z' \mid z) V(a',z') \qquad \text{s.t. } c + a' = R(z)a, \quad (181)$$

where $R(z) = 1 - \tau_a + (1-\tau_k)(r + \pi^\star(z))$ as in the main text.

We solve the dynamic programming problem of the entrepreneur via guess and verify. To this end, we guess that the value function of an entrepreneur of type $i \in \{\ell, h\}$ has the form

$$V_i(a) = m_i + n\log(a),$$

where $m_\ell, m_h, n \in \mathbb{R}$ are coefficients. Under this guess the optimal savings choice of the entrepreneur is characterized by

$$\frac{1}{R_i a - a_i^{'}} = \frac{\beta n}{a_i^{'}}.$$

Solving for savings gives:

$$a_i^{'} = \frac{\beta n}{1+\beta n} R_i a.$$

Replacing the savings rule into the value function gives:

$$\begin{aligned}
V_i(a) &= \log\left(R_i a - a_i^{'}\right) + \beta\left(pV_i\left(a_i^{'}\right) + (1-p)V_j\left(a_i^{'}\right)\right)\\
m_i + n\log(a) &= \log\left(R_i a - a_i^{'}\right) + \beta(pm_i + (1-p)m_j) + \beta n\log\left(a_i^{'}\right)\\
m_i + n\log(a) &= \beta n\log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta(pm_i + (1-p)m_j) + (1+\beta n)\log(a)
\end{aligned}$$

Matching coefficients:

$$\begin{aligned}
n &= 1+\beta n\\
m_i &= \beta n\log(\beta n) + (1+\beta n)\log\left(\frac{R_i}{1+\beta n}\right) + \beta(pm_i + (1-p)m_j),
\end{aligned}$$

where $j \neq i$. The solution to the first equation implies:

$$n = \frac{1}{1-\beta},$$

which in turn delivers the optimal saving decision of the entrepreneur:

$$a' \quad = \quad \beta R(z) a. \tag{182}$$

Finally, we solve for the remaining coefficients from the system of linear equations:

$$m_i = \frac{\beta}{1-\beta} \log\left(\frac{\beta}{1-\beta}\right) + \frac{1}{1-\beta} \log\left((1-\beta) R_i\right) + \beta\left(p m_i + (1-p) m_j\right)$$

The solution is given by:

$$m_i = \frac{\log(1-\beta)}{1-\beta} + \frac{\beta}{(1-\beta)^2} \log(\beta) + \frac{(1-\beta p)\log R_i + \beta(1-p)\log R_j}{(1-\beta)^2 (1-\beta(2p-1))}$$

E.2 Steady state equilibrium

We are interested in the equilibrium where the interest rate is determined by the return of low-productivity entrepreneurs.

Using the saving rules in equation (60), we derive the law of motion for the aggregate wealth of each group

$$\mu A_h^{'} = p\beta R_h \mu A_h + (1-p)\beta R_\ell (1-\mu) A_\ell, \tag{183}$$

$$(1-\mu) A_\ell^{'} = (1-p)\beta R_h \mu A_h + p\beta R_\ell (1-\mu) A_\ell, \tag{184}$$

and for the aggregate capital ($K \equiv (1-\mu) A_\ell + \mu A_h$), where $s_h = {}^{\mu A_h}/_{K}$

$$\frac{K^{'}}{K} = \beta\left(s_h R_h + (1-s_h) R_\ell\right). \tag{185}$$

Recall that the transition matrix for entrepreneurial productivity ensures that $\mu = {}^1/_2$.

As in Section (2.5) this condition and Lemma (1) imply that

$$\frac{K^{'}}{K} \quad = \quad \beta\left((1-\tau_a) + (1-\tau_k)\alpha Z^\alpha K^{\alpha-1} L^{1-\alpha}\right) \tag{186}$$

In steady state this implies that

$$s_h R_h + (1 - s_h) R_\ell = (1 - \tau_a) + (1 - \tau_k) \alpha Z^\alpha K^{\alpha - 1} L^{1-\alpha} = \frac{1}{\beta}, \tag{187}$$

moreover, a version of Lemma (2) applies:

$$R_\ell = 1 - \tau_a + \left(\frac{1}{\beta} - (1 - \tau_a)\right) \frac{z_\ell}{Z} \quad \text{and} \quad R_h = 1 - \tau_a + \left(\frac{1}{\beta} - (1 - \tau_a)\right) \frac{z_\lambda}{Z}. \tag{188}$$

Then we can use the law of motion of assets to obtain

$$\frac{(1-p) \beta R_\ell}{1 - p\beta R_h} = \frac{s_h}{1 - s_h} = \frac{1 - p\beta R_\ell}{(1-p) \beta R_h} \tag{189}$$

Replacing for the steady state of returns in terms of productivity we get a quadratic equation not unlike that in (23):

$$0 = (1 - \rho\beta (1 - \tau_a)) - (1 + \rho (1 - 2\beta (1 - \tau_a))) \left(\frac{\frac{z_h + z_\ell}{2}}{Z}\right) + \rho (1 - \beta (1 - \tau_a)) \frac{z_h z_\ell}{Z^2} = 0. \tag{190}$$

Studying this quadratic equation, we show that there is a unique steady state and obtain necessary and sufficient conditions for it to feature heterogeneous returns. Before providing the formal statement of our result, we discuss the logic behind the proof. Existence and uniqueness follow from analyzing the solution to equation (61). For $\rho \leq 0$, there is a unique solution. For $\rho > 0$, there are two positive roots. However, only the larger root satisfies $z_\ell < Z < z_\lambda$. Then, there is always a unique equilibrium.

Then, we turn to whether the equilibrium features return heterogeneity with $R_h > R_\ell$. This necessarily implies that there is misallocation, therefore Z is below its efficient level z_h. We obtain an upper bound on the collateral constraint parameter, $\overline{\lambda}$, that guarantees that $Z < z_h$. This upper bound turns out to be not only sufficient but also necessary for the result.

Proposition 11. *There exists a unique steady state that features heterogenous returns* $(R_h > R_\ell)$ *if and only if*

$$\lambda < \overline{\lambda} \equiv 1 + \frac{1 - \rho}{1 + \rho \left(1 - 2 \left(\beta (1 - \tau_a) + (1 - \beta (1 - \tau_a)) \frac{z_\ell}{z_h}\right)\right)}. \tag{191}$$

Corollary 3. *The condition for the steady state to feature heterogeneous returns can be restated as an upper bound on wealth taxes:*

$$\lambda < \overline{\lambda} \longleftrightarrow \tau_a \leq \overline{\tau}_a \equiv 1 - \frac{1}{\beta\left(1 - \frac{z_\ell}{z_h}\right)}\left[\frac{(\lambda - 1)(\rho + 1) - (1 - \rho)}{2(\lambda - 1)\rho} - \frac{z_\ell}{z_h}\right]. \tag{192}$$

Proof. First, we show that the steady state is unique when $(\lambda - 1) A_h < A_\ell$. In this case, the steady state Z is the solution to equation (61). We will show that the larger root of that equation is the steady state Z. For this, let $h(z)$ be a function defined as

$$\begin{aligned} h(z) = (1 - \beta(1 - \tau_a)(2p - 1)) z^2 - (z_\ell + z_\lambda)(p - \beta(1 - \tau_a)(2p - 1)) z \\ + (2p - 1) z_\ell z_\lambda (1 - \beta(1 - \tau_a)) \quad = \quad 0. \end{aligned}$$

It is easy to show that $h(z_\ell) = (1 - p) z_\ell (z_\ell - z_\lambda) < 0$ and $h(z_\lambda) = (1 - p) z_\lambda (z_\lambda - z_\ell) > 0$. Since $h(z)$ is a quadratic function and $z_\ell < Z < z_\lambda$.

Next, we prove that $(\lambda - 1)\mu A_h < (1 - \mu) A_\ell$ (excess supply of funds) iff $\lambda < \overline{\lambda}$ where

$$\overline{\lambda} \equiv 1 + \frac{(1 - p)}{p - (2p - 1)\left(\beta(1 - \tau_a) + (1 - \beta(1 - \tau_a))\frac{z_\ell}{z_h}\right)}.$$

The proof involves two steps. First, we show that $(\lambda - 1)\mu A_h < (1 - \mu) A_\ell$ iff $Z < z_h$. Second, we find the condition on λ so that $Z < z_h$. For the first step, substituting the definition of $Z = \frac{(z_h + (\lambda - 1)(z_h - z_\ell))\mu A_h + z_\ell (1 - \mu) A_\ell}{\mu A_h + (1 - \mu) A_\ell}$ into $Z < z_h$ and some algebra gives $(\lambda - 1)\mu A_h < (1 - \mu) A_\ell$. For the second step, we derive the condition on λ so that $h(z_h) > 0$ in equation (61). Thus to complete the proof, we evaluate $h(z_h)$:

$$\begin{aligned} h(z_h)/z_h^2 &= 1 - (2p - 1)\beta(1 - \tau_a) - \frac{(z_\ell + z_\lambda)}{z_h}(p - (2p - 1)\beta(1 - \tau_a)) \\ &+ (2p - 1)\frac{z_\ell z_\lambda}{z_h^2}(1 - \beta(1 - \tau_a)). \end{aligned}$$

Inserting $z_\lambda = z_h + (\lambda - 1)(z_h - z_\ell)$ gives

$$\begin{aligned} h(z_h)/z_h^2 &= 1 - (2p - 1)\beta(1 - \tau_a) - \frac{(z_\ell + z_h + (\lambda - 1)(z_h - z_\ell))}{z_h}(p - (2p - 1)\beta(1 - \tau_a)) \\ &+ (2p - 1)\frac{z_\ell(z_h + (\lambda - 1)(z_h - z_\ell))}{z_h^2}(1 - \beta(1 - \tau_a)). \end{aligned}$$

Next we combine the terms that include $\lambda-1$:

$$\begin{aligned} h\left(z_h\right)/z_h^2 &= \frac{\left(1-p\right)\left(z_h-z_\ell\right)}{z_h} \\ &- \frac{\left(\lambda-1\right)\left(z_h-z_\ell\right)}{z_h}\left(p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\right)\right). \end{aligned}$$

Since $p-\left(2p-1\right)\left(\beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\frac{z_\ell}{z_h}\right)>0$ for all p, then, $h\left(z_h\right)>0$ iff $\lambda-1<\frac{1-p}{p-(2p-1)\left(\beta(1-\tau_a)+(1-\beta(1-\tau_a))\frac{z_\ell}{z_h}\right)}$. Finally, recall that this equilibrium can only exist if $\lambda\leq 2$ (this gives $K_\ell\geq 0$). Inspecting the previous result it is immediate that $\overline{\lambda}\leq 2$ iff $p\geq 1/2$.

□

Proposition 12. ***(Efficiency Gains from Wealth Taxation)*** *For all $\tau_a<\overline{\tau}_a$, a marginal increase in wealth taxes (τ_a) increases aggregate productivity (Z), $\frac{dZ}{d\tau_a}>0$, if and only if entrepreneurial productivity is persistent, $\rho>0$.*

Proof. The steady state Z is given by the solution of $h\left(Z\right)=0$ where $h\left(z\right)$ is defined in equation (61). Differentiating $h\left(z\right)$ with respect to τ_a gives

$$\begin{aligned} \frac{d}{d\tau_a}h\left(z\right) &= \left(2p-1\right)\beta z^2-\left(2p-1\right)\beta\left(z_\ell+z_\lambda\right)z+\left(2p-1\right)\beta z_\ell z_\lambda \\ &= \left(2p-1\right)\beta z_\ell z_\lambda\left(z-z_\ell\right)\left(z-z_\lambda\right). \end{aligned}$$

We know that the steady state Z satisfies $z_\ell<Z<z_\lambda$, so we have $\left(z-z_\ell\right)\left(z-z_\lambda\right)<0$. Thus, $\frac{d}{d\tau_a}h\left(z\right)<0$ if and only if $p>1/2$. Notice also that $\frac{d}{d\tau_a}h\left(z\right)<0$ for all τ_a if $z_\ell<Z<z_\lambda$. Thus, $\frac{dZ}{d\tau_a}>0$ for all τ_a as long as the economy is in the first equilibrium which happens if and only if $\lambda\leq\overline{\lambda}$.

□

We also state and prove a couple of additional results here that aid in the explanation of the result in Proposition (12).

Lemma 10. ***(Savings Rates and Wealth Shares)*** *For all $\tau_a<\overline{\tau}_a$, the steady state saving rate of high-productivity entrepreneurs is positive and the saving rate of low-productivity entrepreneurs is negative: $\beta R_h>1>\beta R_\ell$. Furthermore, $s_h>1/2$ if and only if $\rho>0$.*

Proof. The gross saving rate of the entrepreneurs is βR_i. We first show that $\beta R_i>1$ if and only if $\overline{z}_i>Z$, where we slightly abuse notation by letting $\overline{z}_\ell=z_\ell$. The result follows immediately from expressing the savings rate in terms of Z by substituting R_i's from equation (188):

$$\begin{aligned}\beta R_i &> 1\\ \beta\left(1-\tau_a\right)+\left(1-\beta\left(1-\tau_a\right)\right)\overline{z}_i/Z &> 1\\ \overline{z}_i &> Z\end{aligned}$$

To finalize the proof recall from Proposition 11 that the steady state Z satisfies $z_\ell < Z < z_\lambda$, this gives the desired result.

Now, consider $s_h \geq 1/2$. We know that $s_h = \frac{Z-z_\ell}{z_\lambda - z_\ell}$, so $s_h > 1/2$ is equivalent to $Z > \frac{z_\lambda + z_\ell}{2}$. We can verify if this is the case by evaluating the residual of (61) at $\frac{z_\lambda+z_\ell}{2}$:

$$\begin{aligned}h\left(\frac{z_\lambda+z_\ell}{2}\right) &= -\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left(\frac{z_\lambda+z_\ell}{2}\right)^2+\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)z_\ell z_\lambda\\ &= -\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left[\left(\frac{z_\lambda+z_\ell}{2}\right)^2 - z_\ell z_\lambda\right]\\ &= -\left(2p-1\right)\left(1-\beta\left(1-\tau_a\right)\right)\left(\frac{z_\lambda-z_\ell}{2}\right)^2 < 0\end{aligned}$$

The residual is negative if and only if $p \geq 1/2$, $\rho > 0$. So it must be that $Z > \frac{z_\lambda+z_\ell}{2}$ and thus $s_h > 1/2$ for $p \geq 1/2$.

□

Lemma 11. ***(Wealth Shares and Returns in Steady State)*** *For all $\tau_a < \overline{\tau}_a$, the following equations and inequalities hold in steady state:*

$$s_h = \frac{1-\beta R_\ell}{\beta\left(R_h - R_\ell\right)} = \frac{Z-z_\ell}{z_\lambda - z_\ell} \qquad \frac{ds_h}{dZ} = \frac{1}{z_\lambda - z_\ell} > 0 \tag{193}$$

$$R_h = \frac{1}{\beta\left(2p-1\right)}\left(1-\frac{1-p}{s_h}\right) \qquad \frac{dR_h}{dZ} > 0 \tag{194}$$

$$R_\ell = \frac{1}{\beta\left(2p-1\right)}\left(1-\frac{1-p}{1-s_h}\right) \qquad \frac{dR_h}{dZ} < 0. \tag{195}$$

Moreover, the average returns are always decreasing with productivity, $\frac{d(R_\ell+R_h)}{dZ} < 0$, and the geometric average of returns decreases, $\frac{d(R_h R_\ell)}{dZ} < 0$, if and only if $\rho > 0$.

Proof. The wealth share is

$$s_h = \frac{Z-z_\ell}{z_\lambda - z_\ell} \tag{196}$$

so it is increasing in Z, $\frac{ds_h}{dZ} = \frac{1}{z_\lambda - z_\ell} > 0$.

Now we consider what happens to R_h as Z increases. We start by considering the evolution equation for A_h in steady state

$$(1 - p\beta R_h)\,\mu A_h \quad = \quad (1-p)\,\beta R_\ell\,(1-\mu)\,A_\ell.$$

Manipulating this expression gives

$$R_h = \frac{1}{p\beta} - \left(\frac{1-p}{p}\right)\left(\frac{1-s_h}{s_h}\right)R_\ell.$$

We can also use the law of motion for A_ℓ in steady state to obtain an expression for R_ℓ in terms of R_h and s_h:

$$R_\ell = \frac{1}{p\beta} - \left(\frac{1-p}{p}\right)\left(\frac{s_h}{1-s_h}\right)R_h$$

Replacing we can solve for R_h as a function of s_h:

$$R_h = \frac{1}{\beta\,(2p-1)}\left(1 - \frac{1-p}{s_h}\right) \tag{197}$$

We can now obtain the derivative of the high-type returns with respect to Z:

$$\frac{dR_h}{dZ} = \frac{1-p}{\beta\,(2p-1)}\frac{1}{s_h^2}\frac{ds_h}{dZ} > 0 \tag{198}$$

The sign follows from Proposition 12.

We can also obtain an expression for R_ℓ in terms of s_h:

$$R_\ell = \frac{1}{\beta\,(2p-1)}\left(1 - \frac{1-p}{1-s_h}\right) \tag{199}$$

This expression allows to obtain an alternative expression for the derivative of the low-type returns with respect to Z:

$$\frac{dR_\ell}{dZ} = -\frac{(1-p)}{\beta\,(2p-1)}\frac{1}{(1-s_h)^2}\frac{ds_h}{dZ} < 0 \tag{200}$$

Using the results in (197), (198), (199), and (200) we can obtain expressions for the derivative of the sum and product of returns with respect to wealth taxes:

$$\frac{d\,(R_h + R_\ell)}{dZ} = \frac{-\,(2s_h - 1)\,(1-p)}{\beta\,(2p-1)\left((1-s_h)^2\,s_h^2\right)}\frac{ds_h}{dZ} \tag{201}$$

$$\frac{d\,(R_h R_\ell)}{dZ} = \frac{-\,(2s_h - 1)\,p\,(1-p)}{[(1-s_h)\,s_h\beta\,(2p-1)]^2}\frac{ds_h}{dZ} \tag{202}$$

$\frac{d(R_h+R_\ell)}{dZ}$ is always negative because $s_h \geq 1/2$ if and only if $p \geq 1/2$, as we proved in the previous Lemma. $\frac{d(R_h R_\ell)}{dZ}$ is negative if and only if $s_h \geq 1/2$, again, this happens if and only if $p \geq 1/2$.

$\square$

E.3 Optimal taxes

We first discuss the welfare measure we use for the government objective. Because there is no stationary wealth distribution in the steady state of the model it is not possible to compute aggregate welfare directly. However, it is possible to define policy so as to maximize the welfare change with respect to some initial steady state. Let B denote the initial benchmark economy with a given capital income and wealth taxes and C denote a counterfactual economy with a higher wealth tax and a lower capital income tax. Define $\left\{c_t^{\mathrm{j}}(a,i)\right\}$ as the consumption path and $V^{\mathrm{j}}(a,i)$ as the value function of an individual of type $i \in \{w,h,\ell\}$ under economy $\mathrm{j} \in \{\mathrm{B},\mathrm{C}\}$. We ask each individual how much they value being dropped from B to C in terms of a consumption-equivalent welfare measure $\mathrm{CE}_1(a,i)$, which is defined by

$$E\sum_t \beta^{t-1}\log\left(\left(1+\mathrm{CE}_1(a,i)\right)c_t^{\mathrm{B}}(a,i)\right) = E\sum_t \beta^{t-1}\log\left(c_t^{\mathrm{C}}(a,i)\right). \tag{203}$$

We solve for $\mathrm{CE}_1(a,i)$. All terms containing wealth cancel, so drop wealth from the arguments and write

$$\log\left(1+\mathrm{CE}_{1,i}\right) = \begin{cases} \log\left(\frac{w^{\mathrm{C}}}{w^{\mathrm{B}}}\right) & \text{if } i = w \\ \frac{(1-\beta)\log\left(\frac{R_i^{\mathrm{C}}}{R_i^{\mathrm{B}}}\right)+\beta(1-p)\left(\log\left(\frac{R_\ell^{\mathrm{C}}}{R_\ell^{\mathrm{B}}}\right)+\log\left(\frac{R_h^{\mathrm{C}}}{R_h^{\mathrm{B}}}\right)\right)}{(1-\beta)(1-\beta(2p-1))} & \text{if } i \in \{\ell,h\}. \end{cases} \tag{204}$$

We compute the aggregate welfare gain as the population-weighted average of welfare gains. Letting $n_w \equiv L/(L+1)$ be the population share of workers and $n_h = n_\ell \equiv 1/(L+1)$ the share of entrepreneurs, we write

$$\log\left(1+\mathrm{CE}_1\right) = \sum_{i\in\{w,h,\ell\}} n_i \log\left(1+\mathrm{CE}_{1,i}\right). \tag{205}$$

We also define the average entrepreneurial welfare gain ($\mathrm{CE}_1^{\mathrm{e}}$) as

$$\begin{aligned}\log\left(1+\mathrm{CE}_1^{\mathrm{e}}\right) &= \mu\log\left(1+\mathrm{CE}_{1,h}\right)+(1-\mu)\log\left(1+\mathrm{CE}_{1,\ell}\right) \\ &= \frac{1}{1-\beta}\left(\mu\log\left(\frac{R_h^{\mathrm{C}}}{R_h^{\mathrm{B}}}\right)+(1-\mu)\log\left(\frac{R_\ell^{\mathrm{C}}}{R_\ell^{\mathrm{B}}}\right)\right).\end{aligned} \tag{206}$$

Having defined our welfare measures, we now use our previous results to determine the welfare implications of a marginal increase in the wealth tax.

Workers gain from an increase in wealth taxes because wages increase. For entrepreneurs, the welfare effects of the increase in wealth taxes come from changes in after-tax returns. There are two effects. First, higher wealth taxes reduce the current returns of low-productivity entrepreneurs and increase those of high-productivity entrepreneurs. Second, (log-)average of returns decrease with wealth taxes, decreasing entrepreneurs' expectations over future returns and reducing their welfare. The net result of these effects is a lower welfare for the low-productivity entrepreneurs and for entrepreneurs as a group.

The welfare gain for the high-productivity entrepreneurs depends on the magnitude of the decrease in average returns, that in turn depends on the initial return dispersion. The upper bound on the dispersion of returns (κ_R) ensures that the loss from lower expected returns is low relative to the increase in R_h. The upper bound is a function of only β and ρ and does not change with wealth taxes.[21]

Proposition 13. ***(Welfare Gain by Agent Type)*** *For all $\tau_a < \overline{\tau}_a$, if Assumption 2 holds and $\rho > 0$, a marginal increase in wealth taxes (τ_a) increases the welfare of workers ($CE_{1,w} > 0$) and decreases the welfare of low-productivity entrepreneurs ($CE_{1,\ell} < 0$) and the average welfare of entrepreneurs ($CE_1^e < 0$). Furthermore, there exists an upper bound on the dispersion of returns (κ_R) such that an increase in wealth taxes increases the welfare of high-productivity entrepreneurs ($CE_{1,h} > 0$) if and only if $R_h - R_\ell < \kappa_R$.*

Proof. For the workers' welfare note that:

$$\frac{d \log\left(1 + \mathrm{CE}_{1,w}\right)}{d\tau_a} = \frac{d \frac{\alpha}{1-\alpha} \log\left(Z_a/Z_k\right)}{d\tau_a} = \frac{\alpha}{1-\alpha}\frac{1}{Z}\frac{dZ}{d\tau_a} > 0 \longleftrightarrow p > 1/2$$

The welfare gain is positive if and only if productivity is persistent because of Proposition (12).

The welfare of the low-productivity entrepreneurs decreases unambiguously:

$$\frac{d \log\left(1 + \mathrm{CE}_{1,\ell}\right)}{d\tau_a} \propto \frac{1-\beta}{R_\ell}\frac{dR_\ell}{d\tau_a} + \frac{\beta\left(1-p\right)}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} < 0$$

[21] The $\mathrm{CE}_{1,h}$ welfare measure we consider here ignores the effects of the increase in the assets of high-productivity entrepreneurs brought about by the increase in wealth taxes. Taking the change in assets into account makes the welfare change unambiguously positive for them.

which follows from Lemma (11) $\left(\frac{dR_\ell}{d\tau_a}, \frac{dR_\ell R_h}{d\tau_a} < 0\right)$.

The welfare of entrepreneurs as a group also decreases unambiguously.

$$\frac{d\log\left(1+\mathrm{CE}_1^{\mathrm{e}}\right)}{d\tau_a} = \frac{\beta\left(1-p\right)}{1-\beta}\frac{1}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} < 0$$

Finally, for the high-productivity entrepreneurs:

$$\begin{aligned}
\frac{d\log\left(1+\mathrm{CE}_{1,h}\right)}{d\tau_a} &\propto \frac{1-\beta}{R_h}\frac{dR_h}{d\tau_a} + \frac{\beta\left(1-p\right)}{R_\ell R_h}\frac{dR_\ell R_h}{d\tau_a} \\
&= \left[\left(1-\beta\right) - \frac{1}{R_\ell}\frac{\left(2s_h-1\right)p\left(1-p\right)}{\left(1-s_h\right)^2\left(2p-1\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a} \\
&= \left[\left(1-\beta\right) - \frac{\beta\left(2s_h-1\right)p\left(1-p\right)}{\left(p-s_h\right)\left(1-s_h\right)}\right]\frac{\left(1-p\right)}{\beta\left(2p-1\right)s_h^2 R_h}\frac{ds_h}{d\tau_a}
\end{aligned}$$

We maintain the assumption that $p \geq 1/2$, and from Lemma (11) we know that $\frac{ds_h}{d\tau_a} > 0$. So, the sign of derivative of interest depends on the sign of the term in square brackets.

$$\frac{d\log\left(1+\mathrm{CE}_{1,h}\right)}{d\tau_a} \geq 0 \longleftrightarrow 1-\beta \geq \frac{\beta\left(2s_h-1\right)p\left(1-p\right)}{\left(p-s_h\right)\left(1-s_h\right)}$$

It is easy to verify that in steady state $s_h < p$, which together with Lemma (10) implies that the right hand side of the inequality is always positive. To verify that $s_h < p$ holds in steady steady note that this condition is equivalent to $Z < pz_\lambda + \left(1-p\right)z_\ell$, then evaluate function h defined in (61) at $pz_\lambda + \left(1-p\right)z_\ell$. The value of h (the residual of the quadratic equation) is always positive, so it must be that $Z < pz_\lambda + \left(1-p\right)z_\ell$ and thus $s_h < p$.

Then, the high-type entrepreneurs' welfare gain is positive if and only if

$$g\left(s_h\right) \equiv \left(1-\beta\right)\left(p-s_h\right)\left(1-s_h\right) - \beta\left(2s_h-1\right)p\left(1-p\right) \geq 0. \tag{207}$$

Evaluating at $s_h = 1/2$

$$g\left(s_h\right) = \left(1-\beta\right)\left(p-\frac{1}{2}\right)\frac{1}{2} > 0.$$

Evaluating at $s_h = p$

$$g\left(s_h\right) \equiv -\beta\left(2p-1\right)p\left(1-p\right) < 0.$$

Moreover, g is continuous for $s_h \in [1/2, p]$ and monotonically decreasing:

$$g^{'}\left(s_h\right) = -\left(1-\beta\right)\left[\left(1-s_h\right)+\left(p-s_h\right)\right] - 2\beta p\left(1-p\right) < 0$$

So, there exists an upper bound $\overline{s}_h$ such that

$$\frac{d\log\left(1+\mathrm{CE}_{1,h}\right)}{d\tau_a} \geq 0 \longleftrightarrow s_h \in \left[\frac{1}{2}, \overline{s}_h\right]$$

The upper bound for z_ℓ is characterized by the solution to

$$(p-\overline{s}_h)(1-\overline{s}_h)-\beta(2\overline{s}_h-1)\,p\,(1-p)=0$$

Alternatively, we can make us of the link between s_h and the dispersion of returns:

$$R_h-R_\ell=\frac{(1-p)(2s_h-1)}{\beta(2p-1)(1-s_h)\,s_h}$$

So the high-productivity entrepreneurs benefit from an increase in wealth taxes if and only if the dispersion of returns is low enough:

$$\frac{d\log(1+\text{CE}_{1,h})}{d\tau_a}\geq 0\longleftrightarrow s_h\in\left[\frac{1}{2},\overline{s}_h\right]\longleftrightarrow R_h-R_\ell\in[0,\kappa_R]$$

where $\kappa_R\equiv\frac{(1-p)(2\overline{s}_h-1)}{\beta(2p-1)(1-\overline{s}_h)\overline{s}_h}$. Note that $\overline{s}_h$ depends only on p and β, therefore the upper bound for the dispersion of returns is also a function of p and β alone.

□

We now characterize the optimal tax combination $(\tau_a^\star,\tau_k^\star)$ that maximizes the utilitarian welfare measure CE_1. Proposition 13 makes clear the key tradeoff when considering the welfare effects of wealth taxation: Higher wealth taxes increase the welfare of workers by increasing wages through productivity gains, but they reduce the welfare of entrepreneurs by increasing the dispersion of returns and decreasing their expected value. As we show in Proposition 14 below, the tradeoff is captured by the elasticities of wages and returns to changes in productivity. The welfare gain of workers is proportional to the wage elasticity with respect to productivity, $\xi_w=\frac{\alpha}{1-\alpha}$, while the welfare loss of entrepreneurs is proportional to the average elasticity of returns with respect to productivity, $\mu\xi_{R_h}+(1-\mu)\,\xi_{R_\ell}$.

Proposition 14. ***(Optimal*** $\boldsymbol{CE_1}$ ***Taxes)*** *Under Assumption 2, there exist a unique tax combination* $(\tau_a^\star,\tau_k^\star)$ *that maximizes the utilitarian welfare measure* CE_1*. An interior solution,* $\tau_a^\star<\overline{\tau}_a$*, is the solution to:*

$$0=n_w\xi_w+\frac{1-n_w}{1-\beta}\left(\mu\xi_{R_h}+(1-\mu)\,\xi_{R_\ell}\right),\tag{208}$$

where $\xi_x\equiv\frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. Furthermore, there exist* *two cutoff values for* α, $\underline{\alpha}$ *and* $\overline{\alpha}$, *such that* $(\tau_a^\star, \tau_k^\star)$ *satisfies the following properties:*

$$\begin{aligned} \tau_a^\star &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau_k^\star > \theta && \text{if } \alpha < \underline{\alpha} \\ \tau_a^\star &\in \left[0, \frac{\theta(1-\beta)}{\beta(1-\theta)}\right] \text{ and } \tau_k^\star \in [0,\theta] && \text{if } \underline{\alpha} \leq \alpha \leq \bar{\alpha} \\ \tau_a^\star &\in \left(\frac{\theta(1-\beta)}{\beta(1-\theta)}, \tau_a^{\max}\right) \text{ and } \tau_k^\star < 0 && \text{if } \alpha > \bar{\alpha}, \end{aligned}$$

where $\underline{\alpha}$ *and* $\overline{\alpha}$ *are the solutions to equation (208) with* $\tau_a = 0$ *and* $\tau_a = \tau^{TR} \equiv \frac{\theta(1-\beta)}{\beta(1-\theta)}$, *respectively. Recall that* $\xi_w = \xi \equiv {}^{\alpha}/_{1-\alpha}$.

As an alternative to the CE_1 measure used above, we consider the welfare gain of a stand-in representative entrepreneur of each type. We compare the values assigned by a type-i entrepreneur of being in the capital income or wealth tax economy while holding the average type-i wealth level in that economy. We denote this welfare measure as $\mathrm{CE}_{2,i}$:

$$\log(1+\mathrm{CE}_{2,i}) = (1-\beta)(V_a(A_{i,a}, i) - V_k(A_{i,k}, i)) = \log(1+\mathrm{CE}_{1,i}) + \log\left({}^{A_{a,i}}/_{A_{k,i}}\right). \quad (209)$$

We can also ask each entrepreneur how much they value being in the wealth tax economy with its average wealth (K_a) relative to being in the capital income tax economy with its average wealth (K_k). The welfare gain for a type-i entrepreneur is

$$\log\left(1+\widetilde{\mathrm{CE}}_{2,i}\right) = (1-\beta)(V_a(K_a, i) - V_k(K_k, i)) = \log(1+\mathrm{CE}_{1,i}) + \log\left({}^{K_a}/_{K_k}\right), \quad (210)$$

and the aggregate (or expected) welfare is

$$\log\left(1+\widetilde{\mathrm{CE}}_2\right) = \sum_i n_i \log\left(1+\widetilde{\mathrm{CE}}_{2,i}\right) = \log(1+\mathrm{CE}_{1,i}) + \log\left({}^{K_a}/_{K_k}\right). \quad (211)$$

This is actually not unlike the result in our benchmark model. The optimal taxes are similarly given as:

Proposition 15. ***(Optimal*** $\widetilde{CE}_2$ ***Taxes)*** *Under Assumption 2, there exist a unique tax combination* $\left(\tau_{a,2}^\star, \tau_{k,2}^\star\right)$ *that maximizes the utilitarian welfare measure* $\widetilde{CE}_2$, *an interior solution* $\tau_{a,2}^\star < \overline{\tau}_a$ *is the solution to:*

$$0 = n_w \xi_w + (1-n_w)\xi_K + \frac{1-n_w}{1-\beta}\left(\mu \xi_{R_h} + (1-\mu)\xi_{R_\ell}\right) \quad (212)$$

where $\xi_x \equiv \frac{d\log x}{d\log Z}$ *is the elasticity of variable* x *with respect to* Z*. Furthermore, there exist two cutoff values for* α*,* $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$*, such that* $\left(\tau^\star_{a,2}, \tau^\star_{k,2}\right)$ *satisfies the following properties:*

$$\begin{aligned}
\tau^\star_{a,2} &\in \left[1-\frac{1}{\beta}, 0\right) \text{ and } \tau^\star_{k,2} > \theta && \text{if } \alpha < \underline{\alpha}_2 \\
\tau^\star_{a,2} &\in \left[0, \frac{\theta\,(1-\beta)}{\beta\,(1-\theta)}\right] \text{ and } \tau^\star_{k,2} \in [0,\theta] && \text{if } \underline{\alpha}_2 \le \alpha \le \bar{\alpha}_2 \\
\tau^\star_{a,2} &\in \left(\frac{\theta\,(1-\beta)}{\beta\,(1-\theta)}, \tau^{\max}_{a,2}\right) \text{ and } \tau^\star_{k,2} < 0 && \text{if } \alpha > \bar{\alpha}_2
\end{aligned}$$

where $\underline{\alpha}_2$ *and* $\overline{\alpha}_2$ *are the solutions to equation (212) with* $\tau_a = 0$ *and* $\tau_a = \frac{\theta(1-\beta)}{\beta(1-\theta)}$*, respectively. Recall from Lemma (**??**) that* $\xi = {}^{\alpha}/_{1-\alpha}$*.*

Proof. From (211) we obtain the first order condition to maximize $\widetilde{\mathrm{CE}}_2$:

$$\begin{aligned}
\frac{d\log\left(1+\mathrm{CE}_1\right)}{d\tau_a} + \left(1-n_w\right)\frac{d\log K}{d\tau_a} &= 0 \\
\left[\frac{d\log\left(1+\mathrm{CE}_1\right)}{d\log Z} + \left(1-n_w\right)\frac{d\log K}{d\log Z}\right]\frac{d\log Z}{d\tau_a} &= 0 \\
\left[n_w\xi_w + \frac{1-n_w}{1-\beta}\left(\mu\xi_{R_h} + \left(1-\mu\right)\xi_{R_\ell}\right) + \left(1-n_w\right)\xi_K\right]\frac{d\log Z}{d\tau_a} &= 0
\end{aligned}$$

As in the proof of Proposition 14 this leads to the optimality condition:

$$n_w\xi_w + \left(1-n_w\right)\xi_K = -\frac{1-n_w}{1-\beta}\left(\mu\xi_{R_h} + \left(1-\mu\right)\xi_{R_\ell}\right).$$

Further, we know that $\xi_w = \xi_K = \xi = {}^{\alpha}/_{1-\alpha}$. The right hand side of the equation is the same as in Proposition 14 and an explicit formula can be found in the proof to that proposition. The uniqueness of the solution and the definition of the thresholds for α and its implications for the optimal taxes follow from the same arguments as in Proposition 14.

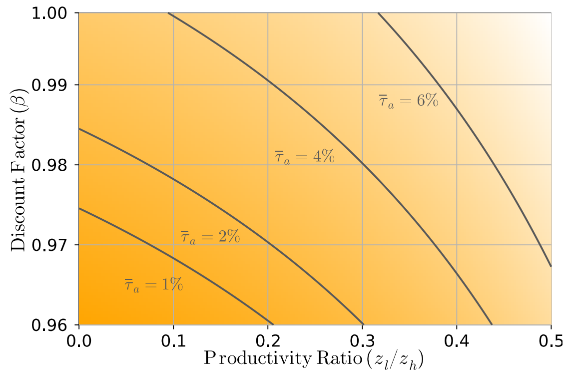
□

Taking into account the role of capital accumulation results in a higher optimal tax level, and lower thresholds $\underline{\alpha}$ and $\overline{\alpha}$:

Corollary 4. ***(Comparison of CE$_1$ and CE$_2$ Taxes)*** *Optimal wealth taxes are higher when taking the wealth accumulation into account* $\left(\tau^\star_{a,2} > \tau^\star_{a}\right)$*. Moreover, the* α*-thresholds are lower* $\underline{\alpha}_2 < \underline{\alpha}$ *and* $\overline{\alpha}_2 < \overline{\alpha}$*.*

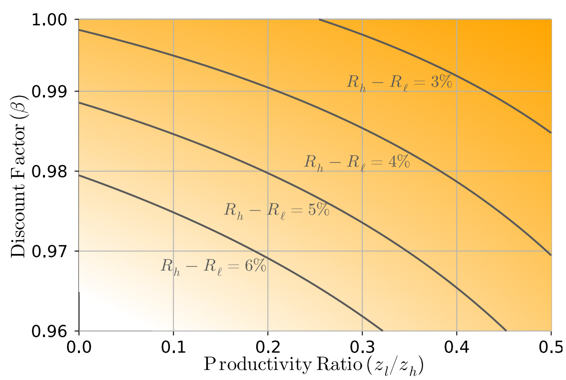
F Additional Figures

Figure 9: Upper Bound Wealth Tax $\overline{\tau}_a$



Note: *The figure reports the upper bound on wealth taxes from Corollary 1 for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$. λ is such that the debt-to-output ratio in our baseline calibration is 1.5.*

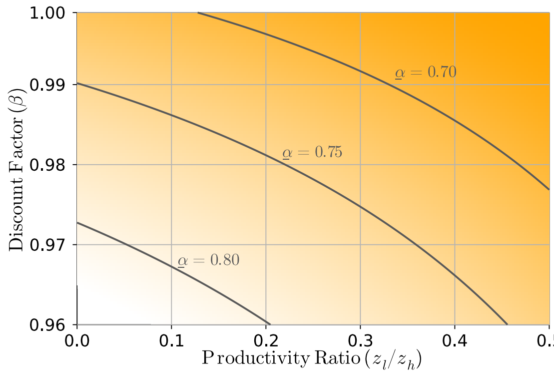
Figure 10: Dispersion of Return in Steady State $R_h - R_\ell$



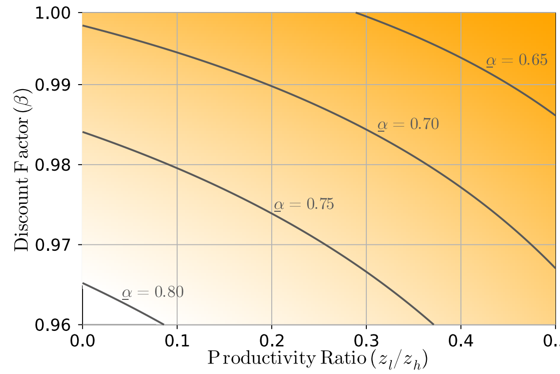
Note: *The figure reports the value return dispersion in steady state for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

Figure 11: α Thresholds for Entrepreneurial Welfare Gains

(a) Low-Productivity Entrepreneurs $dV_\ell/d\tau_a > 0$

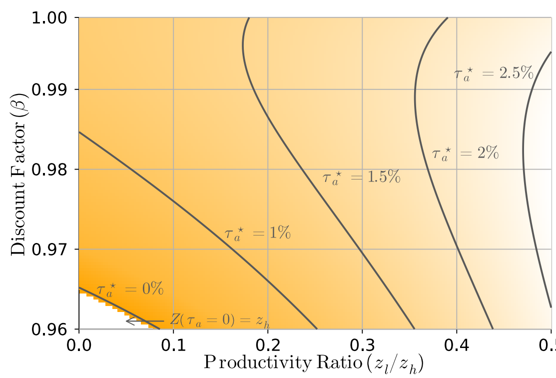


(b) Entrepreneurs $dV_E/d\tau_a > 0$



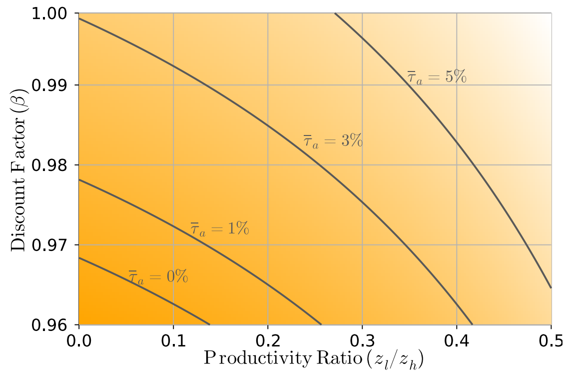
Note: *The figures report the threshold value of α above which entrepreneurial welfare increases after an increase in wealth taxes for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

Figure 12: Optimal Wealth Tax



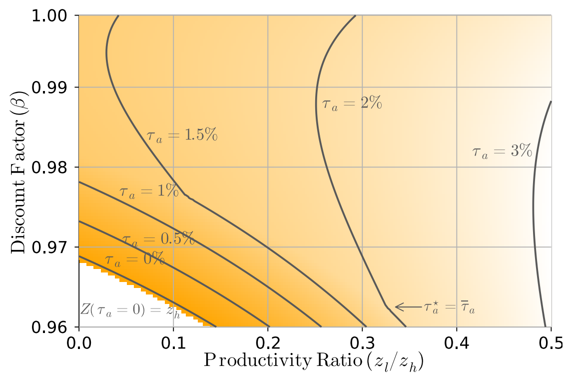
Note: *The figure reports the value of the optimal wealth tax for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

Figure 13: Upper Bound Wealth Tax with Endogenous Innovation



Note: *The figure reports the upper bound on wealth taxes from Corollary 1 when innovation is endogenous tax for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

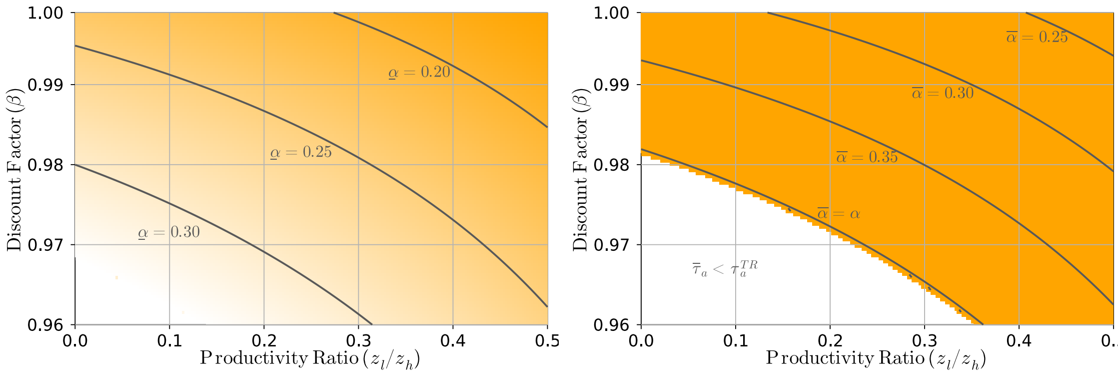
Figure 14: Optimal Wealth Tax with Innovation



Note: *The figure reports the value of the optimal wealth tax for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*

Figure 15: α Thresholds for Optimal Wealth Taxes with Innovation

(a) Threshold $\underline{\alpha}$ for $\tau_a^\star > 0$ with Innovation (b) Threshold $\overline{\alpha}$ for $\tau_k^\star < 0$ with Innovation



Note: *The figures report the threshold value of α for the optimal wealth taxes to be positive (left) and capital income taxes to be positive (right) for combinations of the discount factor (β) and productivity dispersion (z_ℓ/z_h). We set the remaining parameters as follows: $\delta = 49/50$, $\beta\delta = 0.96$, $\mu = 0.10$, $z_h = 1$, $\tau_k = 25\%$, and $\alpha = 0.4$.*