Recursive Utility for Thompson Aggregators: Least Fixed Point, Uniqueness, and Approximation Theories

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Prelude: On Functional Equations

- Many economics, operations research problems (e.g. dynamic programming), math (e.g. integral equations) come down to solving a functional equation of the form

$$u(x) = \varphi(x, u(x)), \tag{1}$$

  where X is a complete metric space, $\varphi : X \times \mathbb{R} \to \mathbb{R}$ is a continuous function. Does this equation have a solution? Depends, in part, on what is meant by a solution. More than one solution? Depends on properties of φ.

- Most popular method: Reduce the existence and uniqueness to an application of the Contraction Mapping Theorem where a solution is an element in a particular Banach space: a complete metric space in its norm topology.

- Here's how this works. Let $C_b(X)$ be the **Banach** space of all bounded, real-valued norm-continuous functions defined on X with the $\|v\| = \sup_{x \in X} |v(x)| < \infty$.
- **Assume** $\exists \beta, 0 < \beta < 1$, such that for each $x \in X$ and $y_1, y_2 \in \mathbb{R}$:

$$|\varphi(x, y_1) - \varphi(x, y_2)| \leq \beta |y_1 - y_2|.$$

- That is: for each x, φ satisfies a Lipschitz condition in y with modulus $\beta < 1$.
- Convert the existence/uniqueness problem to a nonlinear operator problem and find a fixed point of that operator and note it solves the functional equation.

Apply the Contraction Mapping Theorem

- Define a nonlinear operator $T : C_b(X) \to C_b(X)$: for each $x \in X$, given $u \in C_b(X)$,

$$(Tu)(x) = \varphi(x, u(x)). \tag{2}$$

- T is a contraction mapping; $\exists$ a unique fixed point, $u^* = Tu^*$:

$$u^*(x) = \varphi(x, u^*(x)).$$

- For each $u_0 \in C_b(X)$, the **sequence of iterates** $\{T^n u_0\}$ converges to u^* with $\|T^n u_0 - u^*\| \leq \frac{\beta^n}{1-\beta} \|Tu_0 - u^*\|$.
- Great!...*Provided you can verify T is a contraction mapping*.
- Challenge is in figuring out HOW?
- Contractive property of φ in y assumed — what if this FAILS?

Monotone Contraction Mapping Theorem and Order Structure

- Use **all** available math structure of $C_b(X)$ and T: *Focus: Order structures!*
- **Blackwell's Theorem** is a popular way to verify T contracts — uses the usual pointwise partial order property of $C_b(X)$, $u \geq v$ iff $u(x) \geq v(x)$ for each x. One of Blackwell's conditions is that the operator be **monotone:** $Tu \geq Tv$ whenever $u \geq v$. His second condition, is a **discounting condition:** $\exists \beta, 0 < \beta < 1$, such that for each $x \in X$, and scalar $a \geq 0$ (set $a(x) = a$):

$$(T(u+a))(x) \leq (Tu)(x) + \beta a. \tag{3}$$

- Many operators in economic models are monotone, but fail the discounting condition.
- What other properties of economic model might supplement monotonicity to solve existence/uniqueness problems?
- **Present a case study today.**

The Economic Problem Situation

- Many dynamic capital theory, growth, and macro models assume exponential discounting. Results may be sensitive to the discounting function's structure (Epstein and Hynes *JPE 1987*, Becker & Boyd 1997). Recursive utility theory may overcome some issues, or at least, broaden the robustness of conclusions, such as in turnpike theory.
- Recursive utility preserves the stationary, time consistency, & time invariance (Halevy) properties of the stationary exponential model. Time preference MAY depend on the consumption sequence.
- Foundations of recursive utility are consequently important: Use deterministic models with discrete time and an infinite horizon.
- Our paper **reworks a Thompson aggregator existence & uniqueness questions** (Marinacci & Montrucchio [MM], 2010;2019). Use **Du's (1989; 1990) mathematical theory**. Exploit the model's order and topological properties without a detour through Thompson metric space theory.

Goals & our paper's contribution

- Recursive methods in macrodynamics: solve functional equations for value functions, policy or equilibrium functions, or recursive utility functions (to choose a few examples).
- Use monotone and order-concave operator methods when the corresponding nonlinear operators are not contractions.
- Multiple solutions may exist. Here are 2 important problems:
  - Find solutions by iterative methods or successive approximations;
  - Find error bounds connecting an approximate solution and the theoretical solution & show geometric convergence rate.
- **Goal:** Provide a **constructive** (iterative), intuitive and direct **existence & uniqueness theory**. Find **computable error bounds** in terms of the primitive concepts: the (restricted) commodity space and the aggregator's functional form/parameter values.
- Employ **order theoretic and topological methods** derived the model's economic structure. *Today's focus: Least Fixed Point theory & pointers to the general uniqueness case.*

Background References:

- *Existence theory* only RAB/JPRZ *Math Soc. Sci.* 2021.
- *Existence & uniqueness theories* are in: MM 2010 & 2019, RB/JPRZ 2017 WP, Balbus 2020, Bloise & Vailakis 2018.
- Latter paper focuses on dynamic programming (**DP**)as well. MM's papers also feature DP as well as recursive utility existence and uniqueness foundational results.
- Guanlong Ren's ANU Ph.D. dissertation (2019) & joint papers w/Stachurski apply Du's theorems to stochastic recursive utility setup;
- **Supply chain models apply Du's results:Yu and Zhang (2019, JME) and Kikuchi, Nishimura, Stachurski, and Zhang (TE 2021).**
- Math Reference:

Yihong Du, "Fixed Points of Increasing Operators in Ordered Banach Spaces and Applications," *Applicable Analysis* , Vol. 38:01-02, (1990), pp. 1-20. Du's related Chinese language (1989) paper is presented in:

Zhitao Zhang, *Variational, Topological, and Partial Order Methods with Their Applications*, Springer-Verlag, Berlin, 2013.

Recursive Utility Background

- Koopmans' (1960s) preference order's axiomatics yields a **recursive** utility representation & an **aggregator** function, W, which together form the **Koopmans equation** & embody **recursive separability**:

$$U(C) = W(c_1, U(SC)), \tag{4}$$

  where $C = \{c_t\}_{t=1}^{\infty} \in X \subseteq \ell_{\infty}^{+}$ is the **commodity space** $(X \neq \varnothing)$and S is the **shift operator**: $SC = \{c_2, c_3, \ldots\}$.
- $W(x, y)$ is the **aggregator**: x — present consumption; y — future utility — captures trade-offs.
- Multiple utility function "solutions..." may exist! Or a unique one! It just depends on what is meant by a SOLUTION: Change X and utility space changes.
- Multiple solutions, *given* W, means the utility representation is *ambiguous*! Which is the "correct" representation? Use Uniqueness Theory...this depends on *what is meant by a solution*! (Hilbert's Dictum via LC Young).

Recursive Separability Follows From Weak Separability & Stationarity

- **Partition** $C = \{c_1, SC\}$ — form **commodity groups** $\{c_1\}$ & $\{SC\}$ (a continuation sequence).
- U represents preferences satisfies: (I) **future consumption is weakly separable from the present**, i.e., there are **subutility functions** $u(c_1)$ and $V(SC)$ such that $U(C) = W(u(c_1), V(SC))$.
- The *aggregator aggregates* (links) the **felicity function** $u(c_1)$ and **future utility**, $V(SC)$ into $U(C)$.
- And (II) **stationary utility — Utility rankings are independent of calendar time:** fix c_1 and consider two continuations: SC and SC^*. Stationarity says $U(c_1, SC) \geq U(c_1, SC^*)$ if and only if $V(SC) \geq V(SC^*)$. That is, shifting the date of SC starting at $t = 2$ to start at $t = 1$ preserves the utility ranking, implying $U = V$ and Koopmans equation holds: $U(C) = W(c_1, U(SC))$. Incorporate u in W's. Obtain *recursively separable* U.

- Koopmans' equation links the X, W, utility space, and T_W. *Lucas & Stokey* (1984): W is the **primitive concept**. Find a recursive utility by solving the Koopmans equation in utility space.
- **Koopmans' operator** is a self-map on the utility space: $T_W U(C) = W(c_1, U(SC))$. A recursive utility function is a **fixed point (FP) of** T_W. That is, $U_\infty = T_W U_\infty$ holds (pointwise).
- **Thompson aggregators** debuted in **MM** *JET* 2010) are **not contractive** in the natural utility/Banach space. They show T_W *is contractive* using Thompson metric space techniques. *Roundabout approach — more capital intensive than ours?*
- **Du's theory combines the existence and uniqueness theories** — applies *Monotone Order-Concave Methods* to the noncontractive Koopmans operator. *Commodity space restrictions impact utility space choice to yield a unique solution exists. Use available Banach lattice features of commodity and utility spaces.*

Concave Thompson Aggregator Assumptions

$W : \mathbb{R}_+^2 \to \mathbb{R}$ is a **concave Thompson aggregator** provided it satisfies:

(T1) $W(x, y) \geq 0$, *continuous, and monotone in* (x, y);

(T2) $W(x, y) = y$ has at least one nonnegative solution for each $x \geq 0$;

(T3) W is a concave function of (x, y);

(T4) $W(x, 0) > 0$ for each $x > 0$;

(T5) W *is* $\gamma-$ *subhomogeneous:* $\exists \gamma > 0$ *s.t.* $\forall \mu \in (0, 1]$ *:*

$$W(\mu^\gamma x, \mu y) \geq \mu W(x, y).$$

(T6) W satisfies: $\lim_{t\to\infty} (W(1, t)/t) < 1$ with $t > 0$.

- The Koopmans, Diamond and Williamson (KDW) aggregator:

$$W(x, y) = \frac{\delta}{d} \ln\left(1 + ax^b + dy\right); \;\; a, b, d, \delta > 0. \tag{5}$$

with $b, \delta < 1$, and $\delta \geq 1$ *is a concave Thompson aggregator with* $\gamma = b^{-1}$; **Lipschitz constant** ≥ 1 (compute $\sup_y W_2(x, y)$).

The Commodity Space (Details)

- $(\ell_\infty, \|\bullet\|_\infty)$ — is a **Banach lattice:** $\|x\|_\infty = \sup_n (x_n) < \infty$. Let $0 = \{0, 0, 0 \ldots\}$& $e = \{1, 1, 1, \ldots\}$.
- Say x is **strictly positive** if $\inf_n (x_n) > 0$, write $x = \{x_n\} >> 0$. NB $e >> 0$.
- e is an **order unit**: for each $x \in \ell_\infty$ there is a $\lambda > 0$ such that $|x| \leq \lambda e$.
- ℓ_∞^+ — the **positive cone** IS the **commodity space**. It's a **normal cone** $\|\bullet\|_\infty$: the lattice norm property yields $0 \leq x \leq y$ implies $\|x\| \leq \|y\|$ (normality constant is 1).
- The **norm-interior** is $\ell_\infty^{++} \neq \varnothing$ — consists of order units: points $x >> 0$.
- Denoted consumption sequences by $C = \{c_t\} \in \ell_\infty^+$ with $t \in \mathbb{N} = \{1, 2, \ldots\}$.

Multiple Solution Problem vs A Restricted Commodity Space

- All known examples of non-unique solutions to the Koopmans equation arise when a solution U is evaluated at $C \in \partial_0 \ell_\infty^+$, the topological boundary of the positive cone ℓ_∞^+ with $\partial_0 \ell_\infty^+ = \left\{ C \in \ell_\infty^+ : \inf_{C \in \ell_\infty^+} \{c_t\} = 0 \right\}$. Note $c_0^+ \subset \partial_0 \ell_\infty^+$ (Bloise and Vailakis (2018) example).
- Impose minimum and maximum possible consumption sequences: ae and be where $0 < a < b < \infty$.
- Interpret ae as the *minimum positive consumption necessary to support life*;
- Interpret be as the *maximum potential consumption* (tied to the maximum sustainable capital stock in a one or two-sector growth model — rules out an Ak growth model).
- The order interval $\langle ae, be \rangle \subset \ell_\infty^{++}$ is the **restricted commodity space**. Note $\ell_\infty^{++} = \cup_{0<a<b<\infty} \langle ae, be \rangle$.
- **The utility space's specification depends on the domain** $\langle ae, be \rangle$.

Utility Space

- Focus TODAY on possible **bounded utility functions** defined on $\langle ae, be \rangle$:

$$B \equiv B(a,b) = \left\{ U : \langle ae, be \rangle \to \mathbb{R} : \|U\| = \sup_{C \in \langle ae, be \rangle} |U(c)| < \infty \right\}.$$

- B is a **Banach lattice**. The **positive cone** is B^+, i.e. $U \geq \theta$ and $\theta(C) = 0$ is the zero function.
- B^+ — **norm-interior,** $B^{++} = \{U \in B : U >> \theta\} \neq \emptyset$: constant function $U_e(C) = 1$ is an **order unit**, $U_e \in B^{++}$ & $U >> \theta$ for $U \in B^{++}$.
- $B^+$is a norm-closed, convex set, & **normal cone** as $\|\bullet\|$ is a lattice norm.
- $\partial_0 B^+ = \left\{ U \in B^+ : \inf_{C \in \langle ae, be \rangle} U(C) = 0 \right\}$ — is norm-closed.

Monotone Koopmans Operator

- W a Thompson aggregator implies T_W is a **monotone** self-map on B^+, i.e. $U \geq V \in B^+$ implies $T_W U \geq T_W V$. In particular, $T_W \theta \geq \theta$ — $(T_W \theta \neq \theta)$ and $T_W U^T \leq U^T$ — $(T_W U^T \neq U^T)$.
- Define $U^T \in B^+$ (pointwise) by the formula

$$U^T(C) = W(b, y_b) = y_b \quad (6)$$

where $y_b > 0$ is the unique solution to $W(b, y) = y$. U^T is an order unit, so $U^T \in B^{++}$.

- Define the **order interval** $\langle \theta, U^T \rangle \subset B^+$: a **complete lattice** (in the induced partial order).
- **MM 2010 prove** $T_W : \langle \theta, U^T \rangle \to \langle \theta, U^T \rangle$.

Existence Theorem Redux (MM 2010 & RAB/JPRZ 2021)

- $\text{fix}(T_W) = \{U \in \langle \theta, U^T \rangle : T_W U = U\}$. Utility space: B^+.

Theorem

There are functions $U_\infty \leq U^\infty$ in $\text{fix}(T_W)$. Moreover, $\text{fix}(T_W) \subseteq \langle U_\infty, U^\infty \rangle \subset \langle \theta, U^T \rangle$.

- The proof is **constructive** (holds on ℓ_∞^+) & iterates **order converge** (our *MSS* paper):
- $T_W^N \theta \nearrow U_\infty = \sup_N (T_W^N \theta)$ is the **least fixed point (LFP)** of T_W;
- $T_W^N U^T \searrow U^\infty = \inf_N (T_W^N U^T)$ is the **greatest fixed point (GFP)** of T_W.
- NOTE: if $U_\infty = U^\infty$ then $\text{fix}(T_W)$ is a singleton in $\langle \theta, U^T \rangle$!
- *Do $\{T_W^N \theta\}$ and $\{T_W^N U^T\}$ also NORM converge to U_∞ and U^∞, respectively? Du's answer: YES!*

Du's Comparability Condition

- Du's (1989 & 1990) uniqueness theorems turn on the showing:

$$\exists \varepsilon \in (0,1) \text{ such that } \varepsilon U^T \leq T_W \theta. \tag{7}$$

- (7) implies $T_W \theta$ is **comparable** (*linked*) to U^T: scalars $\alpha > 0$ and $\beta > 0$ (depending on $T_W \theta$) exist s.t.

$$\alpha U^T \leq T_W \theta \leq \beta U^T.$$

  Du's condition implies this holds for $\alpha = \varepsilon$ & $\beta = 1$ since $T_W U^T \leq U^T$. Call (7) **Du's Comparability Condition:** implies each $U \in \langle T_W \theta, U^T \rangle$ is comparable to U^T.

- Proving this condition obtains rests on joint conditions defining the commodity and utility spaces.
- U^T is an order unit in B. Therefore, $U^T >> \theta$. Hence, if Du's condition obtains, then $T_W \theta >> \theta$ is also an order unit in B.

- Du's Condition DOES NOT HOLD for the commodity space $\langle 0, b \rangle \subset \ell_\infty^+$ *and corresponding* B^+. Then, $\inf \{ T_W \theta (C) : C \in \langle 0, b \rangle \} = 0$ as $T_W \theta (C) = W(0,0) = 0$ when $C = 0$. Thus,
- $T_W \theta \in \partial_0 B^+ \implies T_W^N \theta \in \partial_0 B^+$. **HYPOTHETICAL:** $\{ T_W^N \theta \} \overset{\|\bullet\|}{\to} U_\infty \in \partial_0 B^+$, then $T_W U_\infty = U_\infty$ is a boundary solution. *It can co-exist with a distinct GFP. CAVEAT: Norm convergence does not follow from order convergence in this setup!* Directly verify $U_\infty (0) = 0$ & $U_\infty \in \partial_0 B^+$.
- *However*, $\inf \{ T_W \theta (C) : C \in \langle ae, be \rangle \} = T_W \theta (ae) = W(a, 0) > 0$. Du's condition follows: choose $\left(\text{for } U^T (C) = y_b \right)$:

$$0 < \varepsilon = \frac{W(a,0)}{y_b} < 1. \qquad (8)$$

- Then each $T_W^N \theta$ is an **order unit & comparable to** U^T; $T_W^N \theta \nearrow \sup_N \left(T_W^N \theta \right) = U_\infty \in B^{++}$. The LFP is an order unit — not a "boundary" FP!
- MM's 2019 theorems require $T_W U \neq U$ for any $U \in \partial_0 B^+$. We have this property!

The Koopmans Operator is Order-Concave

- Du requires T_W be *monotone* and **order-concave** on $\langle \theta, U^T \rangle$: The latter property is : For $U \geq V$ and each $0 \leq \alpha \leq 1$,

$$T_W\left(\alpha U + (1-\alpha) V\right) \geq \alpha T_W U + (1-\alpha) T_W V. \qquad (9)$$

- Order concavity is weaker than concavity as it only applies to pairs (U, V) with $U \geq V$. Summarizing:

The Koopmans operator corresponding to a concave Thompson aggregator satisfies:

- *the order-concavity property on* $\langle \theta, U^T \rangle$;
- $T_W \theta$ *is comparable to* U^T *for the choice of* ε *given by (8).*

LFP Lemma

Lemma

(Based on Du's LFP Lemma) The Koopmans operator corresponding to a concave Thompson aggregator has a LFP $U_\infty \in \langle \theta, U^T \rangle$. *The approximate solution* $T_W^N \theta$ *based on iteration of the the Koopmans operator with initial seed* θ, *satisfies the error bound:*

$$\begin{aligned} \left\| T_W^N \theta - U_\infty \right\| &\leq (W(b,0))\, \varepsilon^{-2} (1-\varepsilon)^N ; \\ \left\| T_W^N \theta - U_\infty \right\| &\to 0 \ (N \to \infty), \end{aligned}$$

where ε *is specified by (8). Moreover,* $\left\{ T_W^N \theta \right\}$ *is monotone and converges uniformly (geometric convergence) and order converges to* U_∞. *The LFP is also comparable to* U^T *and is an order unit in* B.

- Du assumes the underlying partially ordered Banach space has a normal positive cone. No other order or topological properties are assumed.
- We exploit stronger conditions available to us since B is a Banach lattice. Du's hypotheses do NOT have sufficient structure to prove $U_\infty = \sup_N \left(T_W^N \theta\right)$ even though the norm limit found by iteration is monotone & uniquely determined in a *Banach* space. Its underlying partial order need not be $\sigma -$ *Dedekind complete! The* $\sup_N(T_W^N \theta)$ may not exist! *However, in a Banach lattice, monotonicity of the sequence and its norm convergence imply order convergence from the uniform continuity of lattice operations (Aliprantis, et al references).*
- Our error bounds use the Thompson aggregator's properties and the parameters defining the restricted commodity space.

Uniqueness Theorem (Restricted Commodity Space)

Theorem

Given the Lemma's hypotheses, the LFP is the unique fixed point of the Koopmans operator in the order interval $\langle \theta, U^T \rangle$. *Moreover,*

1. *For each* $U_0 \in \langle \theta, U^T \rangle$, $T_W U_0$ *is comparable to* U^T *and there is a number* $M > 0$ *such that for each natural number* N,

$$\begin{aligned} \left\| T_W^N U_0 - U_\infty \right\| &\leq M (1 - \varepsilon)^N ; \\ \left\| T_W^N U_0 - U_\infty \right\| &\rightarrow 0 \ (N \rightarrow \infty) . \end{aligned}$$

2. *In particular, if* $U_0 = U^T$, *then* $\{ T_W^N U^T \}$ *norm and order-converges to* $U_\infty >> \theta$.
3. *The number* $M = \| U^T \| + 2 \| T_W \theta \| = y_b + 2W(b, 0)$.

Uniqueness Theorem & Strictly Positive Consumption

- The choice of the restricted commodity space is basically arbitrary, so it is possible to extend the uniqueness result so that

$$T_W U_\infty(C) = U_\infty(C) \text{ for each } C \in \ell_\infty^{++}.$$

Nonuniqueness may still hold when $C \in \partial_0 \ell_\infty^{+}$, as is the case with all currently known examples.

- Since $C \in \ell_\infty^{++}$ is an order unit, C is comparable to e, i.e. $\exists\, \lambda > 0$ & $\mu > 0$ such that

$$\frac{1}{\mu} e \leq C \leq \lambda e.$$

Apply the Theorem on this order interval: $U_\infty(C) = U^\infty(C)$.

- The error bounds only hold on a given $\langle ae, be \rangle$ and vary as the parameters a and b change.

Concluding Comments

- Ours and MM's techniques have some common elements — may be disguised — e.g. the comparability relation manifests itself in the proofs — we don't need to work with the Thompson metric or translate the result of the fixed point problem into that metric structure & then back into the original norm structure.
- Du's Condition clearly prevents the Koopmans operator from mapping a $U \in \langle \theta, U^T \rangle$ into $\partial_0 B^+$. Both approaches (their 2019 paper) use the same restricted commodity space for the uniqueness theory.
- No problem selecting between the LFP and GFP for "interior" commodity bundles in applications as long as the restricted commodity space is built into the applied model. Hence, qualitative properties demonstrated for the iterative schemes such as continuity of each iterate (with respect to a suitable topology) imply the LFP is lower semicontinuous and the GFP is upper semicontinuous. When the LFP equals the GFP, continuity of the unique fixed point follows.
- Questions

Comment on Links to Order Convergence

$\left\{ T_W^N \theta \right\}$ **approximates** U_∞ **from below starting with no information & consistent with information partial ordering for each** C **:**

$$T_W^N \theta\left(C\right) = W\left(c_1, W(c_2, W(c_{3,\ldots} \ldots, W\left(c_N, 0\right)\right) \nearrow U_\infty\left(C\right).$$

- This is also consistent with convergence in the non-Hausdorff Scott topology: $T_W^N \theta$ is eventually in each Scott-neighborhood of the LFP. **Qualitative property – not metric or norm based.**
- No information about the rate of convergence or an approximate solution's error bounds.
- Du's theory **on the restricted commodity space** remedies this by showing the LFP is in each norm-neighborhood of the LFP and provides error bounds which we show are computable in terms of the given W, a, & b.
- The full uniqueness theorem yields the LFP is the **unique solution** and $T_W^N U_0$ **norm-converges** to U_∞ for ANY initial seed in $\left\langle \theta, U^T \right\rangle$. In particular, $T_W^N \theta \nearrow U_\infty$ and $T_W^N U^T \searrow U^\infty = U_\infty$ are also order convergent.