Evaluating factor pricing models using high-frequency panels

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This paper develops a new framework and statistical tools to analyze stock returns using high-frequency data. We consider a continuous-time multifactor model via a continuous-time multivariate regression model incorporating realistic empirical features, such as persistent stochastic volatilities with leverage effects. We find that the conventional regression approach often leads to misleading and inconsistent test results when applied to high-frequency data. We overcome this by using samples collected at random intervals, which are set by the clock running inversely proportional to the market volatility. Our results show that the conventional pricing factors have difficulty in explaining the cross section of stock returns. In particular, we find that the size factor performs poorly in fitting the size-based portfolios, and the returns on the consumer industry have some explanatory power on the small growth stocks.

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JEL classification. C12, C13, C33.

1. Introduction

The empirical anomalies related to the capital asset pricing model (CAPM), typified by the size, value, and momentum effects, lie at the center of multifactor asset pricing models. Especially the model by Fama and French (1992, 1993) incorporates the excess returns on two portfolios, capturing the size and value premia as the additional factors. They estimated and tested this three-factor model using 25 equity returns from portfolios sorted by stocks' sizes and book-to-market ratios. The research in empirical finance has been focused on multifactor asset pricing models since then, mainly geared toward identifying asset pricing anomalies and, thereby, new pricing factors. Finding a new factor typically begins with grouping stocks by a characteristic, such as size, book-to-market ratio, or past return performances. Then econometric analyses follow, verifying if there exist significant, abnormal returns that are not explained by the incumbent pricing factors, and testing if a new model embedding an additional factor made from the anomaly variable is rejected. That is, empirical asset pricing involves the construction of panel data sets of returns and the ensuing statistical investigation of those data series with some economic restrictions.

In the paper, we develop a new framework and a new set of statistical tools for high-frequency panels and use them to reexamine Fama–French regressions.[1] Our approach utilizes some recent econometric research on models with high-frequency observations. Fama–French regressions have still been analyzed largely within the classical regression framework. There are at least two dimensions that we may look into for a new opportunity using our approach. First, asset return data sets are available at several different frequencies, for example, daily, monthly, and yearly. However, very few attempts have been made to address the issue of how to use these data sets provided at multiple frequencies. In modern financial markets, information flows almost in real time and assets are traded at high frequencies. Thus, a valid asset pricing model under the premise of well functioning markets must delineate relationships between asset returns and pricing factors at the (high) frequency of market clearing. This implies that a proper integration of higher frequency models is needed to accurately estimate and test asset pricing models at lower frequencies.

Second, at high frequencies, financial asset returns used in Fama–French regressions have volatilities that are excessive and heterogeneous. Unless appropriately taken care of, this excessive volatility introduces too much noise to make it meaningful to run regressions at high frequencies. Virtually all asset returns show strong evidence of time-varying and stochastic volatilities, and of leverage effects. The time-varying and stochastic volatilities would have only a second-order effect if they were asymptotically stationary. Unfortunately, however, all empirical research reported in the literature unanimously and unambiguously find that they are nonstationary, which is attributable

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[1]The empirical asset pricing literature often uses the term Fama–French regressions to refer to multifactor pricing models containing size or firm distress factors. For instance, a model including a momentum factor in addition to the three Fama–French factors is called a four-factor Fama–French model. See Carhart (1997) for details. Following this convention, we regard Fama–French regressions as multifactor models in contrast to the CAPM.

to structural breaks, switching regimes, and/or near-unit roots,[2] and are endogenous due to leverage effects. As shown in Chung and Park (2007), the nonstationary volatilities generally affect the limit distributions and invalidate the standard tests.[3] The negligence or misspecification of the time-varying and stochastic volatilities would therefore have a first-order effect. The presence of leverage effects introduces endogeneity in volatilities, which makes it more complicated to deal with the nonstationarity of volatilities.[4] It would certainly be a challenging problem to statistically analyze regressions with endogenous nonstationary stochastic volatilities.

To analyze Fama–French regressions, we derive a continuous-time multifactor pricing model and consider the corresponding panel regression. Our model is very general in the sense that it allows for time-varying and stochastic volatilities, which are both nonstationary and endogenous. The error term is just given as a general martingale differential, consisting of two components, namely, the common component and the idiosyncratic component, which are independent of each other. The common component is specified as having volatility driven by the market, but otherwise it is entirely unrestricted. We may of course permit the presence of endogenous nonstationarity in the volatility process of the common component. On the other hand, the idiosyncratic component is only assumed to be cross-sectionally independent and to have an asymptotically stationary volatility process. Our specification for the idiosyncratic component is therefore also very flexible and unrestrictive. In fact, the only meaningful restriction imposed on our error component model is that its nonstationary volatility component is generated exclusively by the market. This implies in particular that only the market risk is nondiversifiable over time.[5] Our specification of the error components is justified both theoretically and empirically in the paper.

For the statistical analysis of our model, we develop a new methodology that relies on the sampling at random intervals in lieu of fixed intervals, and uses the realized variance measure at a higher frequency to estimate the variance of the resulting sample. Our approach exploits a well known theorem in the theory of stochastic processes, due to Dambis, Dubins, and Schwarz, which is often referred to as the DDS theorem.[6] It implies that any realization from a continuous martingale can be regarded as a realization from Brownian motion if it is read using the clock running at a speed inversely proportional to its quadratic variation. At least on its continuous part, a martingale generated with an arbitrary volatility process can therefore be converted into a Brownian motion simply by a time change in sampling. Consequently, general martingale differentials now

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[2]The reader is referred to Jacquier, Polson, and Rossi (1994, 2004), So, Lam, and Li (1998), and Kim, Lee, and Park (2009) for the evidence of nonstationarity in stock return volatilities.

[3]They also show that the regressions may even become spurious in the case that the nonstationary volatilities are excessive. Their results essentially carry over to a continuous-time and high-frequency setting. See Jeong and Park (2013) and Kim and Park (2015).

[4]For more details of the leverage effects, the reader is referred to Harvey and Shephard (1996), Jacquier, Polson, and Rossi (2004), Yu (2005), and Kim, Lee, and Park (2009).

[5]As shown in Park (2002), the usual law of large numbers does not hold in the presence of nonstationary volatility.

[6]Readers are referred to, for example, Revuz and Yor (1994) for more details about the DDS theorem.

become Brownian differentials, which are independent and identically distributed normals.[7] The DDS theorem is not directly applicable if the error process is discountinuous and has jumps. However, our approach also remains valid for a wide class of discontinuous error processes with jumps, since we only require asymptotic normality—not normality in finite samples—of the regression errors after time change.

For our model, we may use the market volatility to set the required random sampling intervals. This is because only the market volatility drives the endogenous nonstationarity in our error component model. As long as the volatility in the common component is taken care of by sampling at proper random intervals, the errors become asymptotically normal. The variance of the errors collected at the random intervals is also determined by the idiosyncratic component, but its volatility is asymptotically stationary as in the standard regression model. Moreover, the error variance can be estimated readily by the realized variance obtained from higher frequency observations available at each random interval. Our methodology therefore utilizes observations at both high and low frequencies. We use observations at a high frequency to set the random sampling intervals and to estimate the variance of collected samples. On the other hand, samples collected at a low frequency are used to analyze the main regressions. They are analyzed at a low frequency to avoid distortions caused by excessive volatilities that exist in high-frequency observations. At the same time, however, we do not discard the available observations at the higher frequency, that is, we also use them to deal with time-varying and endogenous stochastic volatilities in observations collected at a low frequency.

With this new econometric methodology in hand, we revisit the classic issues in empirical asset pricing. We estimate and test the CAPM and various multifactor Fama–French models on four data sets of daily equity returns, which consist of two sets of decile portfolios sorted, respectively, by size and book-to-market ratio (B/M), a set of 25 portfolios sorted by size and B/M, and a set of 30 portfolios from different industries. For our random time regressions, we select the sampling intervals using the realized variance series of the daily excess market returns by setting the realized variance over each random sampling interval at the level comparable to the average realized variance of the monthly excess market returns. In the paper, we compare the results from our random time regressions with those from fixed-time regressions using monthly observations.

We find that conventional regressions on fixed-time intervals yield confusing test results. Specifically, with the fixed-time sampling, we cannot reject the CAPM on the size portfolios and the B/M portfolios even if the estimated market betas cannot explain the higher risk premium generated by small size or high book-to-market ratio.[8] This result is inconsistent with the vast amount of literature on the existence of size and value premia of stock returns. Furthermore, when we incorporate the B/M or the size factor into the CAPM regression on each of the corresponding data sets, the fixed-time monthly ordinary least squares (OLS) regressions cannot reject the two-factor models with even

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[7]For an application of this approach in the univariate setup, see Phillips and Yu (2005) and Andersen, Bollerslev, and Dobrev (2007). It has been more systematically and rigorously developed recently by Park (2009).

[8]For the industry portfolios, the CAPM is not rejected either.

higher p-values, stating that one cannot statistically reject either CAPM or the respective two-factor models on those portfolios. However, when all three factors are included, that is, when the Fama–French three-factor model is used for estimation on 25 portfolios sorted both by size and B/M, we have a flat rejection of the model. That is, this conventional method, which ignores time-varying volatilities, gives logically inconsistent test results.

Meanwhile, our random sampling approach based on time change decisively rejects the CAPM, reproducing the asset pricing anomalies that are compatible with the previous literature. Then we estimate the two-factor models on each corresponding portfolios to find that the B/M factor is indeed a valid pricing factor that explains variations in stock returns due to different book-to-market ratios. However, the test result shows that the size factor is not sufficient to capture the cross sectional variations of stock returns. Consistent with these findings, the three-factor model on 25 portfolios is rejected, and it turns out to be closely related to the small firm effect. Therefore, the random sampling approach offers a reliable and correct statistical method to estimate and test multifactor asset pricing models with high-frequency data. In a related matter, we find that the estimates of beta coefficients in most cases studied are not critically different across the two econometric procedures. Thus, their differences seem to come mainly from the estimates of constant terms representing the pricing errors and variance–covariance matrix of residual terms, implying the importance of properly treating highly persistent stochastic volatilities of residual terms.

Finally, when applied to the industry portfolios, we again obtain similar results: the conventional method cannot reject the CAPM, despite significant deviations of abnormal returns from zero, resulting in the rejection of the CAPM in the random sampling case. The main reason for the rejection turns out to be the portfolio returns from consumer product companies. This only prevails in the random sampling case. We find that the returns from the consumer goods industry help explain the size effect of the Fama–French portfolios, especially the returns of the microcap, growth firms. In sum, our empirical results coherently show that by appropriately handling stochastic volatilities, our method provides an accurate statistical procedure for both estimation and testing, without losing the attractive features of OLS regression. Thus, the good news that we want to convey is that empirical researchers can run OLS regressions of multifactor pricing models using data sets in any (especially high) frequencies and accurately test the adequacy of those models, provided that the persistent market stochastic volatilities are well treated using our time-change method. Another related point to be made from our empirical result is that we still need valid pricing factors to explain cross sectional behaviors of stock returns via factor models.

The rest of the paper is organized as follows. In Section 2, we develop a continuous-time multifactor model of asset returns with stochastic volatilities and propose a panel regression model based on our theoretical framework. Section 3 presents a statistical procedure to analyze our model and asymptotic theories. In so doing, we also provide a statistical toolkit necessary for our empirical analysis. Section 4 describes the data sets used in our empirical analysis and provides empirical evidence for various specifications of our model. We then employ our new methodology to reexamine the CAPM

and Fama–French regressions in Section 5, where empirical results from our analysis of Fama–French regressions are summarized and compared with other results reported in the literature. Section 6 concludes the paper. Useful lemmas and their proofs and the proofs of the main theorems are collected in the Appendix, available in a supplementary file on the journal website, http://qeconomics.org/supp/251/supplement.pdf.

2. The model and assumptions

2.1 *Theoretical background*

In this section, we derive a continuous-time beta model of asset returns on which our study of Fama–French regressions will be based. For the derivation of our model, we let π be the state price density given by

$$\frac{d\pi_t}{\pi_t} = v_t \, dt + \sum_{j=1}^{J} \tau_{jt} \, dV_{jt}, \tag{1}$$

where v and (τ_j) are, respectively, drift and volatility processes, and (V_j) are independent Brownian motions. Subsequently, we specify the price process (P_i) of security $i = 1, \ldots, I$ as

$$\frac{dP_{it}}{P_{it}} = \mu_{it} \, dt + \sigma_t \left(\sum_{j=1}^{J} \kappa_{ij} \, dV_{jt} + \sum_{k=1}^{K} \lambda_{ik} \, dW_k \right) + \omega_{it} \, dZ_{it}, \tag{2}$$

where (μ_i) and (σ, ω_i) are drift and volatility processes, $(\kappa_{ij}, \lambda_{ik})$ are nonrandom coefficients, and (Z_i) and (W_k) are independent Brownian motions. Throughout the paper, we make the following assumption.

Assumption 2.1. *We have that (Z_i), (V_j), and (W_k) are independent Brownian motions such that (ω_i, Z_i) and (σ, V_j, W_k) are independent of each other, and (W_k) are Brownian motions independent of (V_j) conditional on σ.*

State price density π is a process that makes (πP_i) a martingale for all $i = 1, \ldots, I$. It is well known since Harrison and Kreps (1979) that the existence of a state price density implies no arbitrage in the asset market. Throughout this section, we regard the instantaneous returns of a risky asset (dP_i/P_i) as the total returns from trading gains and the dividends paid between t and $t + dt$.

The drift term (μ_i) in (2) measures the risk–return trade-off, which will be determined below. For the specification of the diffusion term in (2), we introduce a component with a common volatility σ, as well as a component that represents an idiosyncratic volatility (ω_i) specific to asset i. The common volatility component is then further divided into two components: one involving (V_j) and the other consisting only of (W_k) that are independent of (V_j). In total, we have three terms describing the stochastic evolution of (dP_i/P_i). The first term involving (V_j) results from the covariations with the state price density π and is, therefore, closely related to the pricing factor. The coefficient

(κ_{ij}) measures the proportionality of the risk of security i relative to that of (V_j). Meanwhile, the second and third terms including (W_k) and (Z_i) have no bearing on π, and are not used to pin down the conditional mean component (μ_i) in (2). Instead, (W_k) and (Z_i) are viewed as fluctuations related to the part of a firm's cash flows that makes the dividend process volatile and, in turn, to the gross return process. It is alluded that the remainder of the firms' cash flows will matter for valuing the equity of these firms and these are already included in the first part $(\sigma \kappa_{ij}\, dV_j)$.[9] What (W_k) and (Z_i) capture, we believe, are the fluctuations of the dividends of these firms that do not affect investors' discount factors for pricing purposes. This is a sensible assumption based on the empirical evidence that realized sample paths of firms' dividends are much more volatile than those of aggregate consumption growth or other macroeconomic variables, which ought to be associated with the state price density process.[10] Our setup states that if individual assets' payoffs are not correlated with the state price density π, there will be no risk–return trade-off, which will be reflected via the terms in (μ_i) despite the volatility of asset returns.

Now we introduce pricing factors (Q_j), which we specify as

$$\frac{dQ_{jt}}{Q_{jt}} = \nu_{jt}\, dt + \rho_j \sigma_t\, dV_{jt} \tag{3}$$

for $j = 1, 2, \ldots, J$, where in particular (ν_j) are drift processes and (ρ_j) are nonrandom coefficients. In our specification, (Q_j) can be understood as the price of a portfolio made out of individual assets so that only the systematic diffusion part that is relevant for pricing will remain.[11] For instance, we can think of the price of a portfolio with a long position for small firms (or high book-to-market ratio) and a short position for large firms (or low book-to-market ratio) as a factor. In a similar context, the first factor Q_1 is set to be the market factor with the unit corresponding coefficient of ρs, that is, $\rho_1 = 1$.[12] The subsequent derivation of our model depends crucially on the existence of a common volatility movement σ, especially with *constant* proportionality of risk for all assets. When the assumption of constant proportion is relaxed, we obtain a conditional beta representation, leading to conditional factor models. With some additional assumptions on the structure of betas, we may also consider such models as in Ang and Kristensen (2012). However, given our emphasis on the Fama–French regressions on stock returns, we do not pursue this route in this paper.

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[9]It is an important task to quantify the relative contributions to explaining systematic return variations between the discount factor risk and cash flow risk. However, it is beyond the scope of our paper.

[10]Alternatively, one may consider that the asset market is incomplete in the sense that a source of shock (W_k) is either not priced or priced with a significantly downward bias via the conditional mean component.

[11]It is possible to allow for the presence of nonpricing factors (W_k) we introduced in (2). This will, however, make the Fama–French OLS regressions invalid, as we will explain later.

[12]That is, we regard the market as the asset that includes only the systematic component for pricing with the reference value of 1 for the beta, which will be introduced later.

We derive our main formula by invoking the definition of the state price density. Under a no arbitrage condition, we may easily deduce from (1), (2), and (3) that

$$\mu_{it} = -v_t - \sum_{j=1}^{J} \kappa_{ij}\sigma_t\tau_{jt},$$

$$\nu_{jt} = -v_t - \rho_j\sigma_t\tau_{jt}$$

(4)

holds for $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Note that the left-hand side of the first equation in (4) represents a conditional mean return for holding security i. If $\kappa_{ij} = 0$ for all j, that is, if there is no risk for this asset's payoffs, then $-v_t$ is the resultant return process, which thereby stands for the instantaneously riskless rate denoted as r_t^f. As mentioned earlier, this equation describes the important characteristics of risk–return trade-off via conditional covariation between an asset's return and the discount factor π. Upon setting $v_t = -r_t^f$, it follows immediately from (4) that

$$\mu_{it} - r_t^f = \sum_{j=1}^{J} \beta_{ij}(\nu_{jt} - r_t^f),$$

(5)

where $\beta_{ij} = \kappa_{ij}/\rho_j$ for $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

Now we have from (2), (3), and (5) that

$$\frac{dY_{it}}{Y_{it}} = \alpha_i\,dt + \sum_{j=1}^{J} \beta_{ij}\frac{dX_{jt}}{X_{jt}} + dU_{it}$$

(6)

with $\alpha_i = 0$ and

$$dU_{it} = \sigma_t \sum_{k=1}^{K} \lambda_{ik}\,dW_{kt} + \omega_{it}\,dZ_{it},$$

(7)

if we define

$$\frac{dY_{it}}{Y_{it}} = \frac{dP_{it}}{P_{it}} - r_t^f\,dt,$$

$$\frac{dX_{jt}}{X_{jt}} = \frac{dQ_{jt}}{Q_{jt}} - r_t^f\,dt$$

for $i = 1, \ldots, I$ and $j = 1, \ldots, J$.

Our subsequent empirical analysis will be based on the model given by (6) and (7). Imposing the loadings of all factors other than the market factor to be zero gives us the conventional CAPM regression in continuous time. One important restriction given in this model is that the constant coefficient α_i is not present for all i in our theoretical models. Since we only use excess returns for both factors and test assets, $\alpha_i = 0$ must hold for all i. In this vein, we call α the pricing errors throughout the paper, where we write $\alpha = (\alpha_1, \ldots, \alpha_I)'$. Testing the hypothesis of $\alpha = 0$ has been a focal point of empirical asset pricing literature. Unlike the conventional discrete time CAPM or multifactor

models, note that our continuous-time model offers an error structure derived from the underlying asset pricing model. Therefore, the estimation of model (6) and the related statistical inference require further elaboration. To tackle this, we develop our econometric method and procedure below.[13]

2.2 *Regression formulation*

Our model (6) is formulated as an instantaneous regression, where both the regressand and regressors are measured over an infinitesimal time interval. The regressions for observations collected at any prescribed time intervals may easily be obtained from (6). If time series observations are collected over the intervals defined by

$$0 \equiv T_0 < T_1 < \cdots < T_N \equiv T \tag{8}$$

over the time interval $[0, T]$, then we have the corresponding regression model

$$\int_{T_{n-1}}^{T_n} \frac{dY_{it}}{Y_{it}} = \alpha_i(T_n - T_{n-1}) + \sum_{j=1}^{J} \beta_{ij} \int_{T_{n-1}}^{T_n} \frac{dX_{jt}}{X_{jt}} + (U_{iT_n} - U_{iT_{n-1}}) \tag{9}$$

with

$$U_{iT_n} - U_{iT_{n-1}} = \sum_{k=1}^{K} \lambda_{ik} \int_{T_{n-1}}^{T_n} \sigma_t \, dW_{kt} + \int_{T_{n-1}}^{T_n} \omega_{it} \, dZ_{it} \tag{10}$$

for $n = 1, \ldots, N$. Herein, we consider both fixed and random sampling schemes. For the fixed sampling scheme, we set $T_n - T_{n-1}$ to be nonrandom and constant for all $n = 1, \ldots, N$, like a month or a year. Instead, we define (T_n) to be a sequence of stopping times, or a time change, for the random sampling scheme. In particular, we will use the time change given by the volatility process σ in the common volatility factor. As discussed earlier, σ is the volatility of the market factor introduced below (3).

For our random sampling scheme, we introduce a process S so that

$$dS_t = \frac{dX_{1t}}{X_{1t}} = \frac{dQ_{1t}}{Q_{1t}} - r_t^f \, dt \tag{11}$$

denotes the instantaneous market excess return, and we use its quadratic variation $[S]$ to define the time change (T_n) as

$$[S]_{T_n} - [S]_{T_{n-1}} = \int_{T_{n-1}}^{T_n} \sigma_t^2 \, dt = \Delta \tag{12}$$

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[13]In our model, the error process U is assumed to be a continuous process. However, the assumption can be relaxed and we may allow for the presence of jumps. This will be discussed in the next section.

for $n = 1, \ldots, N$, where Δ is a fixed constant.[14] This compares with the corresponding fixed sampling scheme (T_n) given by

$$T_n = (n/N)T \qquad (13)$$

for $n = 1, \ldots, N$. In particular, if we set

$$T_n = n\Delta = (n/N)[S]_T, \qquad (14)$$

then the random sampling scheme in (12) yields the same number of observations as the fixed sampling scheme in (13) for regression (9). In what follows, we will often simply refer to the sampling schemes in (12) with (14) and (13) as the random- and fixed sampling schemes, respectively.

The motivation for our random sampling scheme (12) is to effectively deal with the endogenous nonstationarity of market volatility σ in the common error component of (10). It is well known and clearly demonstrated in the literature that the market volatility has an autoregressive root that is very close to unity. Also, its leverage effect on the market excess return is quite strongly negative. The reader is referred to Jacquier, Polson, and Rossi (1994, 2004) and Kim, Lee, and Park (2009) for more discussions on the nonstationarity and leverage effect of market volatility. In this situation, the usual law of large numbers and central limit theory do not hold and hence the usual chi-square tests for inference in regression (9) are invalid as shown in, for example, Park (2002). This poses a serious problem in analyzing Fama–French regressions. Under the random sampling scheme, however, we have

$$\int_{T_{n-1}}^{T_n} \sigma_t \, dW_{kt} =_d \mathbb{N}(0, \Delta) \qquad (15)$$

for all $n = 1, \ldots, N$ and $k = 1, \ldots, K$, and that they are independent of each other. Here and elsewhere we use $\mathbb{N}$ to signify normal distribution. This is due to a theorem by Dambis, Dubins, and Schwarz, which will be called the DDS theorem in this paper. Of course, the normality in (15) only applies to the random sampling scheme.

The idiosyncratic error component of (10) is expected to behave much more nicely. Under Assumption 2.1, $(\int_{T_{n-1}}^{T_n} \omega_{it} \, dZ_{it})$ becomes independent across i and has variance

$$\mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it} \, dZ_{it} \right)^2 = \mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it}^2 \, dt \right) \qquad (16)$$

for each $i = 1, \ldots, I$. In what follows, we make the following assumption.

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[14]The choice of Δ is an important problem, and the reader is referred to Park (2009) for more discussions on this subject. For the empirical analysis in the paper, we simply set Δ so that the random sampling scheme has the same number of observations N as the fixed sampling scheme at monthly frequency. Strictly speaking, this is not allowed in our theoretical framework, which requires Δ to be a nonrandom constant.

ASSUMPTION 2.2. *For all $i = 1, \ldots, I$, we have*

$$\frac{1}{N} \sum_{n=1}^{N} \int_{T_{n-1}}^{T_n} \omega_{it}^2 \, dt \to_p \varpi_i^2$$

as $N \to \infty$, for some $\varpi_i^2 > 0$.

Assumption 2.2 is not stringent and should be satisfied widely. It holds under mild regularity conditions if, for instance, the volatilities generated by the idiosyncratic component over the random sampling intervals are asymptotically stationary. In particular, the presence of nonstationarity is not allowed in the idiosyncratic error component of our model. Note that we still permit endogeneity in (ω_i). In the special case where the idiosyncratic volatilities (ω_i) are independent of the driving Brownian motions (Z_i), we have $\int_{T_{n-1}}^{T_n} \omega_{it} \, dZ_{it} =_d \mathbb{MN}(0, \int_{T_{n-1}}^{T_n} \omega_{it}^2 \, dt)$, where $\mathbb{MN}$ denotes mixed normal distribution.[15]

For the statistical inference in our model, we need an estimate for the error covariance matrix for regression (9). It is easy to obtain the asymptotic error covariance matrix implied by our error component model (10). Under the random sampling scheme, note that we have

$$\mathbb{E}(U_{iT_n} - U_{iT_{n-1}})^2 = \Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \mathbb{E}\left(\int_{T_{n-1}}^{T_n} \omega_{it}^2 \, dt \right),$$

$$\mathbb{E}(U_{iT_n} - U_{iT_{n-1}})(U_{jT_n} - U_{jT_{n-1}}) = \Delta \sum_{k=1}^{K} \lambda_{ik} \lambda_{jk}$$

for all $0 \leq i \leq I$ and $0 \leq i \neq j \leq I$. Therefore, if we define $U_{T_n} - U_{T_{n-1}} = (U_{1T_n} - U_{1T_{n-1}}, \ldots, U_{IT_n} - U_{IT_{n-1}})'$, then we would expect to have

$$\mathbb{E}(U_{T_n} - U_{T_{n-1}})(U_{T_n} - U_{T_{n-1}})' \approx \Sigma$$

asymptotically, where Σ is a matrix with the ith diagonal entry $\Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \varpi_i^2$ and (i, j)th off-diagonal entry $\Delta \sum_{k=1}^{K} \lambda_{ik} \lambda_{jk}$. Subsequently, we call Σ the asymptotic error covariance matrix for our regression (9).[16]

The asymptotic error covariance matrix Σ can be estimated using two different approaches. As in the conventional approach, we may estimate Σ by

$$\hat{\Sigma} = \frac{1}{N} \sum_{n=1}^{N} (U_{T_n} - U_{T_{n-1}})(U_{T_n} - U_{T_{n-1}})'. \tag{17}$$

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[15]This will be the case, if there is no leverage effect on the asset return generated from the idiosyncratic error component.

[16]For K finite and known, our model imposes some restrictions on the asymptotic error covariance matrix Σ. However, these restrictions will not be exploited here, since the inference on K is beyond the scope of this paper.

Clearly, we have $\hat{\Sigma} \to_p \Sigma$ as $N \to \infty$, if we assume some extra regularity conditions to ensure that

$$\frac{1}{N} \sum_{n=1}^{N} \left(\int_{T_{n-1}}^{T_n} \omega_{it} \, dZ_{it} \right)^2 \to_p \varpi_i^2,$$

$$\frac{1}{N} \sum_{n=1}^{N} \left(\int_{T_{n-1}}^{T_n} \omega_{it} \, dZ_{it} \right) \left(\int_{T_{n-1}}^{T_n} \omega_{jt} \, dZ_{jt} \right) \to_p 0 \tag{18}$$

for all i and for all $i \neq j$. It is easy to see that (18) holds under appropriate assumptions, due in particular to (16), Assumption 2.2, and the independence of $(\int_{T_{n-1}}^{T_n} \omega_{it} \, dZ_{it})$ across $i = 1, \ldots, I$. Furthermore, we may estimate the asymptotic error covariance matrix Σ using

$$\tilde{\Sigma} = \frac{1}{N} \int_0^T [U, U']_t \, dt, \tag{19}$$

where $[U, U']$ is the matrix of quadratic variations and covariations of $U = (U_1, \ldots, U_I)'$. Note that

$$[U_i]_{T_n} - [U_i]_{T_{n-1}} = \Delta \sum_{k=1}^{K} \lambda_{ik}^2 + \int_{T_{n-1}}^{T_n} \omega_{it}^2 \, dt,$$

$$[U_i, U_j]_{T_n} - [U_i, U_j]_{T_{n-1}} = \Delta \sum_{k=1}^{K} \lambda_{ik} \lambda_{jk}$$

for all $1 \leq i \leq I$ and $1 \leq i \neq j \leq I$. Therefore, we have $\tilde{\Sigma} \to_p \Sigma$ as $N \to \infty$, which holds without any extra regularity conditions.

Due to Assumption 2.1, the usual condition for exogeneity of the regressors in (9) holds and the OLS procedure is valid for regression (9) for our random sampling scheme as well as the fixed sampling scheme. To see this more clearly, we let

$$\mathcal{F}_n = \sigma\big((U_{it}, i = 1, \ldots, I, t \leq T_n), (X_{jt}, j = 1, \ldots, J, t \leq T_{n+1}) \big),$$

$n = 1, \ldots, N$, for our fixed or random sampling scheme (T_n). Then we may easily see that the regressors $(\int_{T_{n-1}}^{T_n} dX_{jt}/X_{jt})$, $j = 1, \ldots, J$, are all $\mathcal{F}_{n-1}$-measurable, and the regression errors $(U_{iT_n} - U_{iT_{n-1}})$ satisfy the orthogonality condition

$$\mathbb{E}[U_{iT_n} - U_{iT_{n-1}} | \mathcal{F}_{n-1}] = 0$$

for $i = 1, \ldots, I$, as required for the validity of the OLS regression in (9). Recall in particular that we assume in Assumption 2.1 that (W_k) are Brownian motions independent of (V_j) conditional on σ.

Some discussions are in order on how we deal with the presence of jumps. Our theoretical development thus far assumes that the error process is given by a process with

a continuous sample path almost surely (a.s.). However, for the validity of our econometric methodology, we do not need to assume that the error process is continuous. Indeed, our random sampling scheme is well expected to remove endogenous and nonstationary volatilities even in the presence of jumps. In this case, the DDS theorem does not apply and the regression errors are not in general normally distributed. Nevertheless, our approach remains valid in general, since we only require asymptotic normality (not normality in finite samples) of the regression errors after time change. To be more consistent with our theoretical model, however, we assume that jumps are generated independently from the continuous part of the model and we do not include any information on the model parameters. Therefore, jumps are regarded as pure noise. Accordingly, for our empirical analysis, we simply get rid of the observations that appear to be contaminated with jumps.[17]

3. Implementation, statistical theory, and simulation

3.1 *Statistical procedure*

We introduce the actual statistical procedure to analyze our model. We assume throughout the section that a sample providing observations for

$$(Y_{i,m\delta}, X_{j,m\delta}) \tag{20}$$

is available for $m = 0, \ldots, M$ with δ time interval for each of $i = 1, \ldots, I$ and $j = 1, \ldots, J$. Moreover, from $(X_{i,m\delta})$ and $(r^f_{m\delta})$, we obtain the observations $(S_{m\delta})$ for the excess market return process S introduced in (11) as

$$S_{m\delta} = \frac{X_{1,m\delta} - X_{1,(m-1)\delta}}{X_{1,(m-1)\delta}} = \frac{Q_{1,m\delta} - Q_{1,(m-1)\delta}}{Q_{1,(m-1)\delta}} - r^f_{(m-1)\delta}\delta$$

for $m = 1, \ldots, M$. We let $M\delta = T$, so that T is the horizon of the sample with size M collected at δ time interval. Our subsequent procedure is based on the asymptotic theory requiring $\delta \to 0$ and $T \to \infty$. In particular, δ should be small relative to T.[18]

We rewrite our model (9) as

$$y_{ni} = \alpha_i c_n + \sum_{j=1}^{J} \beta_{ij} x_{nj} + u_{ni}, \tag{21}$$

---

[17]For our empirical analysis, we use a test by Lee and Mykland (2008) to find the locations of jumps. Once we find their locations, we identify the sampling intervals to which they belong and simply discard the corresponding regression samples. Of course, this pretesting on jumps would render the size of the subsequent test to deviate from its nominal. This is, however, ignored for simplicity.

[18]We use daily observations over approximately 45 years for the empirical analysis in this paper, for which we believe our asymptotics are highly suitable. Of course, our theory allows for observations collected at intraday ultrahigh frequencies. However, they appear to introduce much more noise—presumably due to market microstructure—than signal to our inference procedure, especially if used over a long sampling horizon.

where

$$
y_{ni} = \int_{T_{n-1}}^{T_n} \frac{dY_{it}}{Y_{it}}, \qquad c_n = T_n - T_{n-1},
$$

$$
x_{nj} = \int_{T_{n-1}}^{T_n} \frac{dX_{jt}}{X_{jt}}, \qquad u_{ni} = U_{iT_n} - U_{iT_{n-1}}
\tag{22}
$$

for $n = 1, \ldots, N$ and $i = 1, \ldots, I$. Clearly, we cannot run regression (21) directly in most cases, since we do not have observations in continuous time.

To implement our approach based on regression (21) under the random sampling scheme, we need to estimate the time change (T_n). First, we use

$$
[S]_t^\delta = \sum_{m\delta \leq t} (S_{m\delta} - S_{(m-1)\delta})^2
$$

to estimate $[S]_t$ for each $t \in [0, T]$, upon noticing that if δ is small enough relative to T, we have $[S]^\delta \approx [S]$ over $[0, T]$.[19] Once we obtain an estimate $[S]^\delta$ of $[S]$, the corresponding estimate of the time change (T_n) may easily be obtained, accordingly as in (12), for a prescribed value of Δ. We propose the estimate (T_n^δ) of (T_n), which is given by

$$
T_n^\delta = \delta \operatorname*{argmin}_{1 \leq \ell \leq M} \left| \sum_{m=1}^{\ell} (S_{m\delta} - S_{(m-1)\delta})^2 - n\Delta \right|,
\tag{23}
$$

and define $M_n = \delta^{-1} T_n^\delta$ for each $n = 1, \ldots, N$. For the fixed sampling scheme, we may set $M_n = \delta^{-1} T_n$ with T_n defined in (13).

Now we consider the discretized version of regression (21),

$$
y_{ni}^\delta = \alpha_i c_n^\delta + \sum_{j=1}^{J} \beta_{ij} x_{nj}^\delta + u_{ni}^\delta,
\tag{24}
$$

where (y_{ni}^δ), (c_n^δ), and (x_{nj}^δ) are, respectively, the discretized versions of (y_{ni}), (c_n), and (x_{nj}) defined in (22) that are given by

$$
y_{ni}^\delta = \sum_{m=M_{n-1}+1}^{M_n} \frac{Y_{i,m\delta} - Y_{i,(m-1)\delta}}{Y_{i,(m-1)\delta}}, \qquad c_n^\delta = T_n^\delta - T_{n-1}^\delta,
$$

$$
x_{nj}^\delta = \sum_{m=M_{n-1}+1}^{M_n} \frac{X_{j,m\delta} - X_{j,(m-1)\delta}}{X_{j,(m-1)\delta}},
\tag{25}
$$

and (u_{ni}^δ) is defined by $u_{ni}^\delta = U_{iT_n^\delta} - U_{iT_{n-1}^\delta}$ for $n = 1, \ldots, N$ and $i = 1, \ldots, I$. It is quite obvious that (y_{ni}^δ), (c_n^δ), and (x_{nj}^δ) get close to (y_{ni}), (c_n), and (x_{nj}) under appropriate

---

[19]See Foster and Nelson (1996) for some relevant asymptotics. Here we just use the integrated variance as an estimate for quadratic variation. In the presence of market microstructure noise, however, we may want to employ more sophisticated methods in, for example, Barndorff-Nielsen, Hansen, Lunde, and Shephard (2008), Zhang (2006), and Zhang, Mykland, and Aït-Sahalia (2005).

conditions as $\delta \to 0$ and $T \to \infty$. Note that we have a sample of size N to fit regression (24), which is formulated using a sample of size M in (20) with $M > N$. We call the latter *the original sample*, and the former *the regression sample*.

Our regression (24) can be analyzed exactly as the standard multivariate regression model. In particular, the single equation OLS estimators for (α_i) and (β_{ij}) are fully efficient asymptotically. Therefore, we may run the OLS regression on (24) for each $i = 1, \ldots, I$. This does not require the estimation of the asymptotic error covariance matrix Σ introduced earlier in the previous section. However, we need to estimate Σ for the test of a joint hypothesis involving multiple regression coefficients across $i = 1, \ldots, I$. For the estimation of Σ in regression (24), we may follow the usual two step procedure: In the first step, we estimate (α_i) and (β_{ij}) for each i by the single equation method. Then we use the fitted residuals in the second step to estimate Σ by

$$\hat{\Sigma}^\delta = \frac{1}{N} \sum_{n=1}^{N} \hat{u}_n^\delta \hat{u}_n^{\delta\prime}, \tag{26}$$

where $\hat{u}_n^\delta = (\hat{u}_{n1}^\delta, \ldots, \hat{u}_{nI}^\delta)'$ with $(\hat{u}_{ni}^\delta)$ being the fitted residual from regression (24) for equation i.

The error variance estimate $\hat{\Sigma}^\delta$ is expected to behave well only when $N \gg I$, that is, the size of the regression sample is substantially bigger than the number of cross sectional units.[20] In our approach, there is another way to estimate the asymptotic error variance Σ using the original sample. The estimator would be useful especially when N is small relative to I.[21] It is indeed well defined even if $N < I$, as long as the size M of the original sample is large enough. To introduce the estimator more explicitly, we let $(\hat{\alpha}_i^\delta)$ and $(\hat{\beta}_{ij}^\delta)$ be the OLS estimators of (α_i) and (β_{ij}) obtained from regression (24). Moreover, we define

$$\hat{U}_{i,m\delta} - \hat{U}_{i,(m-1)\delta} = \frac{Y_{i,m\delta} - Y_{i,(m-1)\delta}}{Y_{i,(m-1)\delta}} - \hat{\alpha}_i^\delta \delta - \sum_{j=1}^{J} \hat{\beta}_{ij}^\delta \frac{X_{j,m\delta} - X_{j,(m-1)\delta}}{X_{j,(m-1)\delta}}$$

and

$$\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta} = (\hat{U}_{1,m\delta} - \hat{U}_{1,(m-1)\delta}, \ldots, \hat{U}_{I,m\delta} - \hat{U}_{I,(m-1)\delta})'.$$

Then we may estimate Σ by

$$\tilde{\Sigma}^\delta = \frac{1}{N} \sum_{m=1}^{M} (\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta})(\hat{U}_{m\delta} - \hat{U}_{(m-1)\delta})' \tag{27}$$

as in (19), and use it in place of $\hat{\Sigma}^\delta$ in (26) that is based on (17).

---

[20]The estimator $\hat{\Sigma}^\delta$ defined in (26) even has a rank deficiency and becomes singular if $N < I$.

[21]Suppose, for instance, we run regressions at yearly frequency, when the data are available at daily frequency, for 40 years on the panel consisting of 25 cross sectional units.

In the CAPM and Fama–French regressions, it is one of the main interests to test for the hypothesis

$$\mathbb{H}_0 : \alpha_1 = \cdots = \alpha_I = 0. \tag{28}$$

The rejection of the hypothesis implies that the proposed model is not a true model and presumably requires a new factor. The Wald test for the hypothesis can be easily formulated in our model (24), which may simply be regarded as the classical multivariate regression. The test statistic $\tau^\delta(\alpha)$ is defined by

$$\tau^\delta(\alpha) = \left(c^{\delta\prime} c^\delta - c^{\delta\prime} X^\delta \left(X^{\delta\prime} X^\delta \right)^{-1} X^{\delta\prime} c^\delta \right) \hat{\alpha}^{\delta\prime} \bar{\Sigma}^{\delta-1} \hat{\alpha}^\delta, \tag{29}$$

where c^δ is an N-dimensional vector with c_n^δ as its nth component and X^δ is an $N \times J$ matrix with x_{nj}^δ as its (n, j)th element, $\bar{\Sigma}^\delta = \hat{\Sigma}^\delta$ or $\tilde{\Sigma}^\delta$, and $\hat{\alpha}^\delta = (\hat{\alpha}_1^\delta, \ldots, \hat{\alpha}_J^\delta)'$. As will be shown later, the test statistic $\tau^\delta(\alpha)$ has a chi-square limit distribution with I degrees of freedom. It is also possible to use F distribution after an appropriate adjustment for the degrees of freedom, as in Gibbons, Ross, and Shanken (1989).[22] We may similarly test the hypothesis

$$\mathbb{H}_0 : \beta_{1j} = \cdots = \beta_{Ij} \tag{30}$$

for some factor j, using the statistic

$$\tau^\delta(\beta_j) = \left(x_j^{\delta\prime} x_j^\delta - x_j^{\delta\prime} X_j^\delta \left(X_j^{\delta\prime} X_j^\delta \right)^{-1} X_j^{\delta\prime} x_j^\delta \right) \hat{\beta}_j^{\delta\prime} \bar{\Sigma}^{\delta-1} \hat{\beta}_j^\delta, \tag{31}$$

where x_j^δ is an N-dimensional vector with x_{nj}^δ as its nth component, X_j^δ is an $N \times J$ matrix defined by deleting the jth column from X^δ and adding c^δ as one of its columns, and $\hat{\beta}_j^\delta = (\hat{\beta}_{1j}, \ldots, \hat{\beta}_{Ij})'$. The test statistic $\tau^\delta(\beta_j)$ is also distributed asymptotically as chi-square with I degrees of freedom.

There are various methods developed in the literature that are comparable to our procedure in this paper. Andersen, Bollerslev, Diebold, and Wu (2006), Barndorff-Nielsen and Shephard (2004), and Todorov and Bollerslev (2010) all consider the inferential problem in a continuous time regression model similar to ours. Indeed, we may directly apply their methods to estimate (β_i) in our regression model (6).[23] However, their approach is different from ours in that they fix T and let $\delta \to 0$. They focus on the analysis of quadratic covariations of the regressands and regressors in continuous

---

[22]Strictly speaking, their test, often referred to as the GRS test in the literature, is not applicable in our context, since we do not assume normality. Of course, the estimation samples would be closer to normal under the random sampling scheme, and it would be more appropriate to use the random sampling scheme for the GRS test. We do not report their test in this paper, however, since in our case the degrees of freedom adjustment is negligible and their tests always yield the same results qualitatively as the Wald tests.

[23]As shown in Barndorff-Nielsen and Shephard (2004), (β_i) in (6) can be estimated consistently simply by the usual high-frequency regression without constant term, if $\delta \to 0$ with T fixed. It can be shown that the regression continues to yield a consistent estimate for (β_i) under our setup requiring $T \to \infty$. The inclusion of constant term (α_i) does not affect the consistency of the estimate for (β_i), as long as the integrated regressors ($\int_0^T dX_{jt}/X_{jt}$) are not exceedingly explosive.

time over a fixed time interval. It would therefore be more appropriate to apply their methods for ultrahigh frequency samples observed over a relatively short time horizon. In contrast, our methodology would be more useful to analyze continuous time regression models over longer time horizons, since we require $T \to \infty$ as well as $\delta \to 0$. For the inference on constant term (α_i) in regression (6), none of the aforementioned existing methods is applicable and it is absolutely necessary to utilize samples over long time horizons. In particular, all other existing methods are not applicable to test for the hypothesis (28).

The original Fama–French regressions and their variants have largely been analyzed in discrete time models using low-frequency observations spanning relatively long time horizons. It is possible to accommodate the presence of nonstationary stochastic volatilities in a discrete time framework. In fact, various discrete time regression models with nonstationary stochastic volatilities are suggested and studied by several authors including Hansen (1995), Chung and Park (2007), and Xu (2007). In particular, we may apply the methodologies developed in Hansen (1995) and Chung and Park (2007) to do inference in appropriate discrete time models corresponding to our continuous-time model (6). However, the form of nonstationary stochastic volatility we may consider in discrete time model is rather limited and somewhat unrealistic. The required statistical procedure to properly deal with the presence of nonstationary stochastic volatility is nevertheless quite complicated and difficult to implement. On the other hand, our continuous time approach permits truly general nonstationary stochastic volatility, and provides a very simple yet extremely powerful methodology to effectively deal with it.

3.2 *Asymptotic theory*

In this section, we develop the asymptotic theory for our statistical procedure. Under Assumptions 2.1 and 2.2, our choice of random sampling time (T_n) yields a regression model with errors, which are devoid of any potential endogenous nonstationarity in volatility and have asymptotically stationary volatilities. Note in particular that the regression errors (u_n), $u_n = (u_{n1}, \ldots, u_{nI})'$, are approximately multivariate normal with mild heterogeneity, even in the presence of a very general form of stochastic volatility on the underlying error process. Moreover, as discussed earlier, the usual exogeneity of regressors holds under Assumption 2.1. Consequently, assuming some mild and well expected technical conditions, we may easily show that the conventional asymptotics are applicable for our regression in continuous time. This will not be done in this paper. For expositional convenience, we just introduce necessary high-level assumptions instead of laying out the details of required technical conditions. In what follows, we let $y_n = (y_{n1}, \ldots, y_{nI})'$ and $z_n = (c_n, x_n')'$ with $x_n = (x_{n1}, \ldots, x_{nJ})'$.

ASSUMPTION 3.1. *We assume that* $N^{-1} \sum_{n=1}^N z_n z_n' \to_p \Lambda > 0$, $N^{-1} \sum_{n=1}^N u_n u_n' \to_p \Sigma > 0$, *and* $N^{-1/2} \sum_{n=1}^N z_n u_n' \to_d \mathbb{N}(0, \Lambda \otimes \Sigma)$ *as* $N \to \infty$.

Assumption 3.1 is completely standard for regression asymptotics, and it is straightforward to show that the OLS procedure yields the conventional asymptotics for our regression (21) defined with the random sampling scheme.

It is intuitively clear that the regression (24) constructed from discrete samples reduces to our original regression (21) based on continuous time samples if δ decreases to zero. Now we introduce a set of sufficient conditions to more formally establish the asymptotic equivalence of these two regressions.

ASSUMPTION 3.2. *We let* (a) $a_T(t-s) \leq \int_s^t \sigma_u^2\,du \leq b_T(t-s)$ *for all* $0 \leq s \leq t \leq T$ *with* (a_T) *and* (b_T) *depending only on* T, (b) $\sup_{t \geq 0} |\nu_{jt} - r_t^f| = O_p(1)$ *for all* $j = 1, \ldots, J$, (c) $\inf_t X_{jt} > 0$ *and* $\sup_{0 \leq t \leq T} X_{jt} = O_p(c_T)$ *for all* $j = 1, \ldots, J$ *with* (c_T) *depending only on* T, *and* (d) $\sup_{t \geq 0} \omega_{it} = O_p(1)$ *for all* $i = 1, \ldots, I$. *Furthermore, we set* (e) $\delta = O(T^{-4-\varepsilon}(a_T^2/b_T^7 c_T^4))$ *for some* $\varepsilon > 0$.

The conditions in Assumption 3.2 are very mild and expected to hold widely, perhaps except (e), which is not essential and is made to simplify the proofs.[24] Under Assumption 3.2, we have the following theorem.

THEOREM 3.1. *For all* $i = 1, \ldots, I$ *and* $j = 1, \ldots, J$,

$$\max_{1 \leq n \leq N} \left| c_n^\delta - c_n \right|, \ \max_{1 \leq n \leq N} \left| x_{nj}^\delta - x_{nj} \right|, \ \max_{1 \leq n \leq N} \left| u_{ni}^\delta - u_{ni} \right|, \ \max_{1 \leq n \leq N} \left| y_{ni}^\delta - y_{ni} \right| = o_p(N^{-1/2})$$

as $N \to \infty$.

It follows trivially from Theorem 3.1 that all the estimators and test statistics considered in this paper from regression (24) are asymptotically equivalent to those from regression (21).

Moreover, under Assumptions 3.1 and 3.2, we have the following corollary.

COROLLARY 3.2. *We have*

$$\hat{\Sigma}^\delta = \hat{\Sigma} + O_p(N^{-1/2}), \qquad \tilde{\Sigma}^\delta = \tilde{\Sigma} + O_p(N^{-1/2})$$

for all large N.

In particular, $\hat{\Sigma}^\delta$ and $\tilde{\Sigma}^\delta$ obtained from regression (24) are consistent. In fact, it is straightforward to show that the infeasible estimators $\hat{\Sigma}$ and $\tilde{\Sigma}$ based on continuous time samples are consistent under Assumptions 3.1 and 3.2.

Finally, under the null hypotheses (28) and (30), we may easily deduce the following corollary from Theorem 3.1 and Corollary 3.2.

---

[24]It is not meaningful to see whether or not this condition holds by directly comparing δ and T, since the comparison is dependent on the time unit. Suppose, for instance, we have daily observations over 1 year. If the annual time unit is used, we have $\delta \approx 1/250 \ll 1 = T^{-r}$ for any $r > 0$ and the condition appears to hold. If the daily time unit is used, however, we have $\delta = 1 \gg (1/250)^r \approx T^{-r}$ for any $r > 0$ and the condition clearly fails to hold. As shown in our simulation, we do not need δ to be very small. Our procedure works satisfactorily for $\delta = 1/250$, and acceptably well even for $\delta = 1/12$, corresponding, respectively, to daily and monthly observations, with $T = 50$ or larger.

COROLLARY 3.3. *We have*

$$\tau(\alpha) \to_d \chi_I^2, \qquad \tau(\beta_j) \to_d \chi_I^2$$

for $j = 1, \ldots, J$, *as* $N \to \infty$.

Therefore, under the null hypotheses, the Wald statistics $\tau(\alpha)$ and $\tau(\beta_j)$ have the usual chi-square limit distribution with I degrees of freedom.

3.3 *Simulation*

In this section, we execute a small scale simulation and present our simulation result on the relative performance of our approach based on the random sampling scheme and the conventional approach with the fixed sampling scheme. For our simulation, we consider the continuous time model given by

$$\frac{dY_{it}}{Y_{it}} = \alpha_i \, dt + \beta_i \frac{dX_t}{X_t} + dU_{it}$$

with

$$\frac{dX_t}{X_t} = \sigma_t \, dV_t \quad \text{and} \quad dU_{it} = \sigma_t \, dW_t + \omega_{it} \, dZ_{it}$$

for $i = 1, 2$, where (W_t), (V_t), (Z_{1t}), and (Z_{2t}) are independent standard Brownian motions, and where $\omega_{it} = v_i \sigma_t$ with $v_i = 0.1$ or 1.0 for $i = 1, 2$, and

$$\sigma_t = 1 + 5|V_t + cW_t|$$

with $c = 0, 1, 3$, or 5.[25],[26] The value of v_i for $i = 1, 2$ determines the relative magnitude of the idiosyncratic volatility process (ω_{it}) to the common volatility process (σ_t). We set $v_1 = v_2$ in our simulation and denote their common value by v in what follows. The common volatility process (σ_t) becomes exogenous in case $c = 0$, whereas it is endogenous in all other cases.

In our simulation, we consider the Wald tests for the null hypothesis of $\mathbb{H}_0 : \alpha_1 = \alpha_2 = 0$ to compare our approach based on the RT regression with the conventional approach based on the FT regression, where FT and RT refer, respectively, to the fixed time and

---

[25]We believe that our volatility model here generates realistic samples. The relative performance of our procedure depends only on the ratio $\max(\sigma_t)/\min(\sigma_t)$ of the volatility process (σ_t), not on its absolute level. The estimated value of this ratio using the daily realized volatilities obtained from the 5 minute Standard and Poors (S&P) data for the period 2000–2012 is approximately 41, while our volatility model yields the average values 22, 31, 68, and 110 of the ratio, respectively, for each value of c used in our simulation.

[26]Here we specify $\sigma_t = a + b|B_t|$ with some constants a and b and Brownian motion B. In our simulations, we also consider (σ_t) generated as $\sigma_t = a + b|A_t|$, where (A_t) is the Ornstein–Uhlenbeck process given by $dA_t = -\kappa A_t \, dt + dB_t$ with $\kappa = \bar{\kappa}/T$ for some constant $\bar{\kappa} > 0$ fixed. However, we do not report the details, since the simulation results from this volatility model were exactly the same as expected. Overall, the advantage of our approach relative to the conventional approach becomes less dominating as the local-to-zero parameter $\bar{\kappa}$ increases and the volatility process becomes more stationary, though our approach always outperforms the conventional approach.

random time sampling schemes. We assume that (dY_{1t}/Y_{1t}), (dY_{2t}/Y_{2t}), and (dX_t/X_t) are observed at time intervals of length δ over T years. For the reported results, we set $T = 20$ and 50, and $\delta = 1/252$ mimicking the daily sampling interval.[27] Therefore, in our simulation, the size of the original samples becomes $M = 252T$, while that of the estimation sample is given by $N = 12T$ for both fixed and random sampling schemes. For the fixed sampling scheme, we report the performances of two tests: one based on the monthly observations of size N and the other on the daily observations of size M, called, respectively, the FT1 and FT2 regressions. The FT1 regression uses the same number of regression samples as the RT regression, while the FT2 regression uses the same number of original samples as the RT regression. As shown in, for example, Jeong and Park (2013) and Kim and Park (2015), the powers of the Wald tests we consider in our simulation do *not* depend on the sample size N or M *but* on the sampling span T. Therefore, we may well expect that the tests based on the FT regression would not perform any better even if we used all M observations. Our simulation results are obtained from 10,000 iterations.

Table 1 compares the sizes of the tests. In the table, we report the actual rejection probabilities for the 1%, 5%, and 10% Wald tests, for each combination of $T = 20, 50$, $v = 0.1, 1.0$, and $c = 0, 1, 3, 5$. In case $c = 0$ and the volatility process (σ_t) becomes exogenous, both the conventional fixed sampling and our random sampling scheme work well and the Wald tests have no significant size distortions. The actual rejection probabilities of the tests are all reasonably close to their nominal values for all combinations of T and v we consider in our simulation. If $c \neq 0$ and we have endogeneity in the volatility process (σ_t), quite different pictures emerge. In all such cases, the Wald tests with the conventional fixed sampling scheme produce size distortions, which are generally nonnegligible and often serious. They tend to underreject the null hypothesis, and this problem remains and even gets slightly worse as we increase T. In sharp contrast, the

TABLE 1. Actual sizes of Wald tests.

| | | | $v = 0.1$ | | | | | | | $v = 1$ | | | |
| | | | 5 (%) | | | 10 (%) | | | 5 (%) | | | 10 (%) | |
T	c	FT1	FT2	RT	FT1	FT2	RT	FT1	FT2	RT	FT1	FT2	RT
20	0	5.8	5.3	6.0	10.8	10.4	11.1	5.7	5.5	6.1	11.0	10.4	11.0
	1	4.0	3.7	5.0	8.3	7.9	9.9	4.2	4.0	5.6	9.1	8.4	10.6
	3	3.3	3.1	4.7	7.0	6.3	10.0	3.8	3.5	5.1	8.0	7.3	10.0
	5	3.3	3.1	4.7	7.1	6.5	10.3	4.1	3.7	5.2	8.5	7.8	10.5
50	0	5.1	4.8	5.4	9.7	9.5	10.6	4.9	4.7	5.7	9.9	9.6	10.8
	1	3.9	3.8	5.1	7.9	7.6	10.3	4.1	4.1	5.3	8.8	8.6	10.3
	3	3.6	3.6	5.3	7.4	7.2	10.9	3.5	3.4	5.1	7.7	7.3	10.2
	5	3.1	3.0	5.1	6.8	6.6	10.4	3.8	3.7	5.4	7.8	7.8	10.7

[27]We also consider smaller sampling intervals including $\delta = 1/252 \times 1/6$ corresponding to the hourly sampling interval. Our simulation results do not change and remain very similar for any choices of δ once it becomes smaller than $1/252$. Observations at daily frequency seem to be good enough to implement our methodology.

actual rejection probabilities of the Wald tests based on our random sampling scheme are all close to their nominal values regardless of the values of v and c as well as T. As expected, the size performances of the Wald tests based on the fixed sampling scheme do not change much regardless of whether or not we use the entire sample.

Figures 1–4 plot the size-adjusted power functions of the 5% Wald test against the pricing errors α_i, where we set $\alpha_1 = \alpha_2$ in the range of $0 \leq \alpha \leq 50$ for $\alpha = \alpha_1 = \alpha_2$ as shown on the horizontal axis, and we consider each combination of $T = 20, 50$ and $v = 0.1, 1.0$. As before, RT and FT refer, respectively, to the Wald tests based on the RT and FT regres-
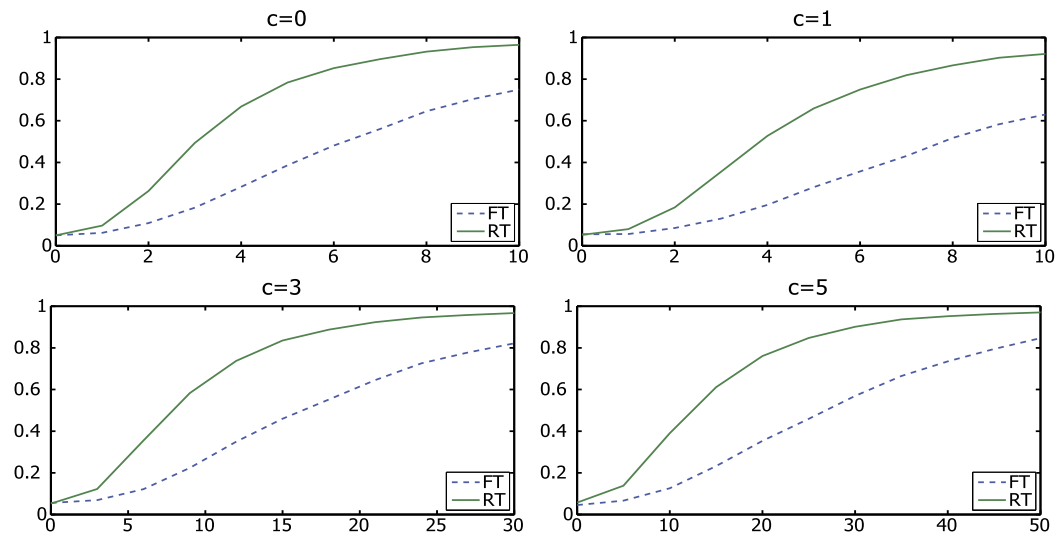
FIGURE 1. Size-adjusted powers of 5% Wald tests: $T = 20$, $v = 0.1$.
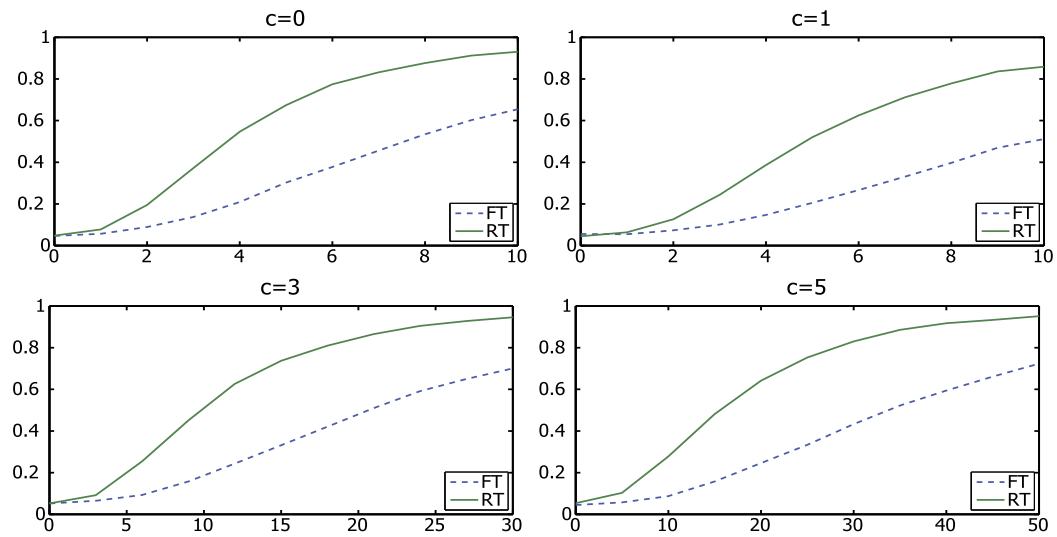
FIGURE 2. Size-adjusted powers of 5% Wald tests: $T = 20$, $v = 1$.
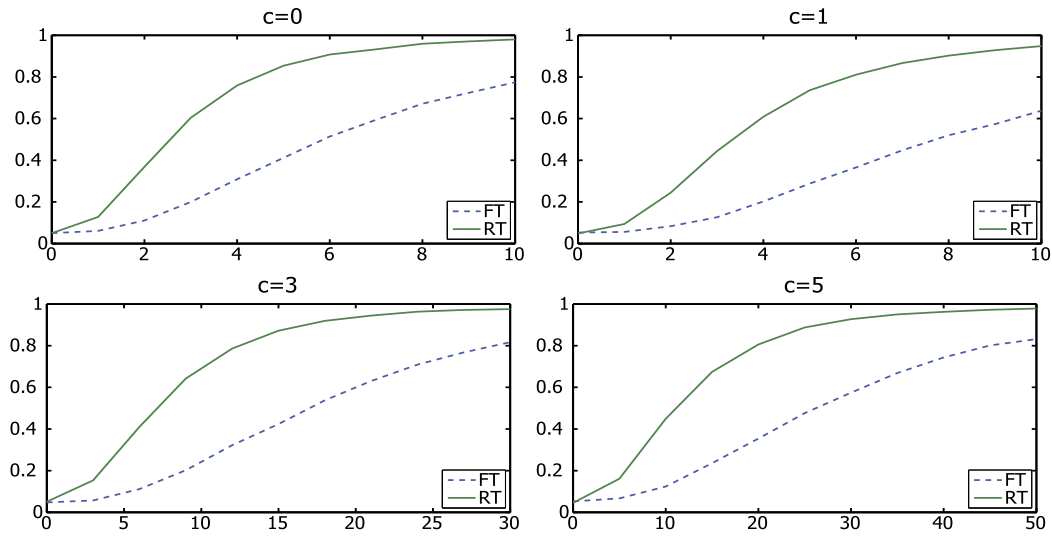
FIGURE 3. Size-adjusted powers of 5% Wald tests: $T = 50$, $v = 0.1$.
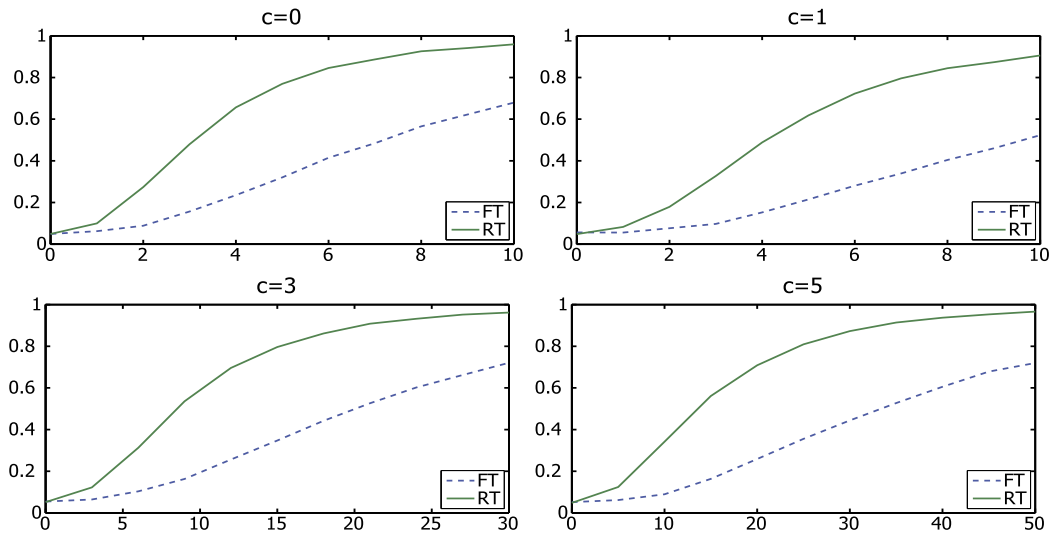
FIGURE 4. Size-adjusted powers of 5% Wald tests: $T = 50$, $v = 1$.

sions. The power functions obtained from the FT1 and FT2 regressions are virtually the same, and therefore, we just denote them commonly as FT. In all cases, the test based on our random time approach outperforms—unambiguously and substantially—the test with the conventional fixed time approach. It seems clear that the use of the random sampling scheme significantly improves the discriminatory power of the test. The relative power of the test using the random sampling scheme tends to increase as c gets bigger and as T becomes larger. For example, in the case that $\alpha = 10$, $v = 0.1$, and $T = 20$, the power of the test based on the random time sampling is 46%, 144%, and 210% bigger for

each value of $c = 1, 3$, and 5 than that based on the test with the fixed time sampling. The relative power performance of the random time test may become even more substantial if we increase T. Indeed, if we set $T = 50$, for the same values of $\alpha = 10$ and $v = 0.1$, its power becomes 49%, 193%, and 263% bigger, respectively, for $c = 1, 3$, and 5, compared with the fixed time test. The power of the test decreases uniformly as v increases to $v = 1$. However, the relative power performance between the tests with the random sampling and the fixed sampling schemes is largely unchanged or even becomes more favorable to the random time test. In fact, in the case that $\alpha = 10$, $v = 1$, and $T = 20$, the power of the random time test is 42%, 68%, 177%, and 219% bigger than that of the fixed time test, respectively, for $c = 0, 1, 3$, and 5.

4. Data and preliminary analysis

4.1 *Data*

This section describes the data sets used in our empirical analysis. We make use of decile portfolios stratified by sizes and book-to-market ratios (B/M). We also use 25 portfolios sorted by sizes and B/M, and 30 industry portfolios. All the data sets are available at French's web page.[28] For pricing factors, we adopt the market (MKT), the size (SMB (small minus big)), and the B/M (HML (high minus low)), often referred to as the Fama–French factors. The data sets cover the period from July 1963 to December 2008, and all of the returns in the data sets are of daily frequency and are annualized. Table 2 presents summary statistics of the factors and the corresponding portfolio returns. Specifically, Panel A reports means and standard deviations of the factors, together with correlations across each other. A high Sharpe ratio of HML states that buying and holding distressed firms would have been lucrative investment strategies during this period. In terms of correlations, both SMB and HML have moderately negative correlations with MKT. Correlation between SMB and HML is small. Panel B of Table 2 reports means and standard deviations of annualized returns stratified into 10 portfolios. The 11th row in each group refers to portfolio strategies with long positions of high returns and short positions of low returns, often called the hedged portfolio returns.[29] The size strategy yields about 1.6% per annum, while the book-to-market strategy earns about 5.6% per annum. These summary statistics suggest that they are good candidates for pricing factors, as discussed in the previous literature. How about the volatility structures of these portfolio returns? We delve into this issue in the next subsection.

4.2 *Preliminary analysis*

Our factor pricing model specified in (6) and (7) imposes some special error structure in the Fama–French regressions, which motivated us to invent a new methodology. Before

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[28]http://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html.

[29]This, respectively, corresponds to (i) the returns from the smallest size (1st group) of market equity minus the returns from the largest size (10th group) for the size strategy, and (ii) the returns from the highest B/M (10th group) minus the lowest B/M (1st group) for the book-to-market strategy.

TABLE 2. Summary statistics of factors and portfolio returns.

Panel A: Factors

	Mean	Stdev	Correlations		
			MKT	SMB	HML
MKT	0.0446	0.1515	1.0000	−0.2200	−0.4302
SMB	0.0152	0.0798	−0.2200	1.0000	−0.0623
HML	0.0511	0.0746	−0.4302	−0.0623	1.0000

Panel B: Decile Portfolio Returns

Size	Mean	Stdev	B/M	Mean	Stdev
1 Small	0.0562	0.1267	1 Growth	0.0316	0.1779
2	0.0559	0.1544	2	0.0449	0.1619
3	0.0646	0.1558	3	0.0499	0.1545
4	0.0619	0.1551	4	0.0499	0.1542
5	0.0656	0.1546	5	0.0477	0.1521
6	0.0581	0.1478	6	0.0572	0.1447
7	0.0595	0.1493	7	0.0665	0.1436
8	0.0561	0.1528	8	0.0724	0.1486
9	0.0507	0.1510	9	0.0831	0.1512
10 Big	0.0406	0.1608	10 Value	0.0878	0.1631
1–10	0.0156	0.1230	10–1	0.0562	0.1220

we reexamine the Fama–French regressions using our methodology, it is therefore necessary that we investigate whether various specifications of our model are empirically justifiable. For this purpose, we consider the conventional three-factor Fama–French regression, which uses 25 portfolio returns sorted by size and book-to-market ratio as regressands and Fama–French factors as regressors.

An important implication of Assumption 2.1 is that the error processes (dU_i) in (6) are correlated cross sectionally due to the presence of the common component (dW_k).[30] To see how much cross-correlations exist among the errors in a typical factor pricing model, we test for diagonality of the covariance matrix of fitted residuals estimated in the usual way from the aforementioned conventional Fama–French regression. We use the residuals from both fixed time regression and our new random time regression, which was formally introduced in (24), and apply the Lagrange Multiplier (LM) test of diagonality suggested by Breusch and Pagan (1980). As can be seen in Table 3, the null of diagonality is rejected in both cases, indicating that there exist cross-correlations among the errors, which may be generated by the common error component (dW_k), that is, the common error component that is not captured by the factors already included.

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[30]Independence of (ω_i, Z_i) and (ω_i) is not likely to be empirically testable by construction. However, whether there is a common component in the error terms is a fair empirical question, which we illustrate in Figures 5–6.

TABLE 3. Test of diagonality of the variance–covariance matrix.

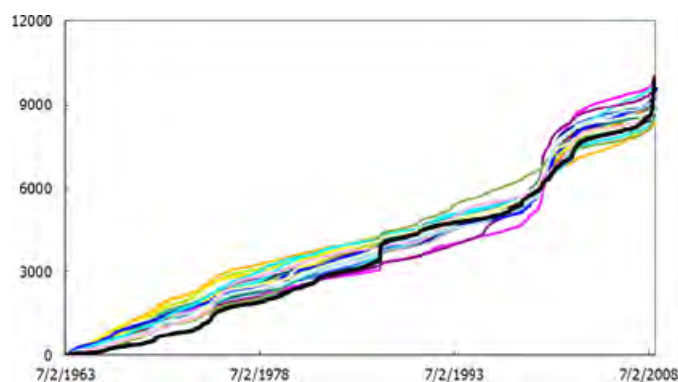| Model	LM Test	p-Value
Fixed time regression	7592.9883	0.0000
Random time regression	3530.9053	0.0000

FIGURE 5. Quadratic variations of market return and fitted residuals. Notes: There are 26 lines in the figure. The thick darker line represents the quadratic variation series, estimated by the realized variance series, of the daily excess market returns, and the remaining 25 lines represent the quadratic variation series of the 25 fitted residuals from Fama–French three-factor regression on 25 portfolios sorted by 5 size and 5 B/M ratio groups.

Our specification of the pricing formula given in (6) and (7) presumes the presence of common volatility factor σ in the diffusion terms (dV_j) of all pricing factors (dQ_j/Q_j) specified in (3) and more importantly in the common component (dW_k) of the errors (dU_i). Especially, we assume that the common volatility factor σ is given by the volatility of the excess market return, which is defined from the first pricing factor as in (11). To see if this assumption is empirically justified, we plot in Figure 5 the quadratic variation series of daily excess market return along with those of the 25 daily residuals recovered from Fama–French three-factor regression using the coefficient estimates obtained from running our random time regression of the same model with monthly observations. It is clear that the 25 residual quadratic variation series follow closely that of the excess market return signified by the dark thicker line, thereby strongly supporting our assumption that the volatility of excess market return represents the common component of individual residual volatilities.

To more carefully investigate the appropriateness of our assumption, we also estimate the instantaneous variances of the 25 fitted residuals and compare them with those of the excess market returns. Note that the quadratic variation series presented in Figure 5 can be regarded as the estimates for the integrated variances of the fitted residuals, and that the instantaneous variances are the time derivatives of integrated variances. For the actual estimation, we apply the local linear smoothing method to the quadratic variation series we obtained earlier and compute their time derivatives. We also conduct
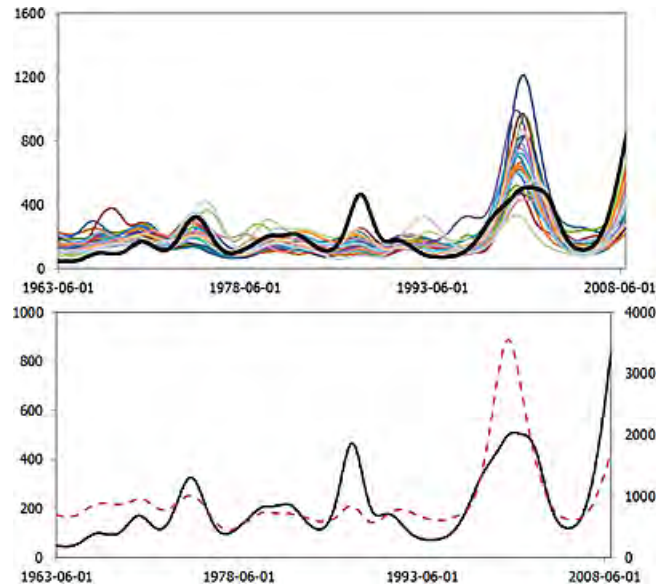
FIGURE 6. Instantaneous variances of market return and fitted residuals. Notes: The top panel presents 26 lines, of which the thick darker line signifies the estimated instantaneous variance series of the daily excess market returns, and the remaining 25 lines represent those of the 25 fitted residuals from the three-factor Fama–French regressions on 25 portfolios sorted by 5 size and 5 book-to-market ratio groups. The instantaneous variances are estimated by the derivatives of the quadratic variations that we obtain by using the local linear smoothing method with the rule of thumb bandwidth selection provided by Fan and Gijbels (1996). The bottom panel compares the estimated instantaneous variance series of the excess market returns (solid line) with that of the leading factor of the 25 fitted residuals (dashed line).

the principal component analysis to extract the leading factor from the estimates of the instantaneous variances for the fitted residuals. The leading factor is expected to represent the nonstationary volatility factor in the fitted residuals. Our results are provided in Figure 6. The magnitudes of the estimated instantaneous variances of the fitted residuals are not exactly identical to those of the market or those of the extracted leading factor. However, it is rather strongly suggested that they fluctuate together. In particular, their cycles are remarkably overlapped. For instance, the timings of peaks and troughs for the instantaneous variance series of the market and the extracted leading factor appear to coincide perfectly.

We also investigate whether the errors (u_{ni}) are orthogonal to the regressors (c_n) and (x_{nj}) in our regression (21), especially under random sampling scheme. Of course, this is crucial for the validity of the OLS procedure. Indeed, they may be correlated with each other. It happens, for instance, if the pricing factors (dQ_j/Q_j) in (3) have nonpricing volatility components (dW_k) as well as the pricing volatility components (dV_j). To see whether the orthogonality between the regressors and the regression errors is a plausible tenet, we run the fixed time regression using monthly observations to estimate the regression coefficients, and use the estimates to obtain the fitted residuals at the daily

TABLE 4. Correlations between residuals and regressors in random time regressions.

| (Size, B/M) | Correlation Coefficients | | | | |Correlation Coefficients| | | | |
	Alpha	MKT	SMB	HML	Alpha	MKT	SMB	HML
(1, 1)	0.0483	−0.0373	0.0862	0.0172	0.0483	0.0373	0.0862	0.0172
(1, 2)	−0.0982	−0.0320	−0.0842	0.0121	0.0982	0.0320	0.0842	0.0121
(1, 3)	−0.0459	−0.0204	0.1130	−0.0252	0.0459	0.0204	0.1130	0.0252
(1, 4)	−0.0667	0.0118	0.0811	−0.0685	0.0667	0.0118	0.0811	0.0685
(1, 5)	0.0937	0.0013	0.0884	0.0015	0.0937	0.0013	0.0884	0.0015
(2, 1)	−0.0940	−0.0267	0.0303	−0.0147	0.0940	0.0267	0.0303	0.0147
(2, 2)	−0.0639	0.0313	0.0904	−0.0956	0.0639	0.0313	0.0904	0.0956
(2, 3)	0.0895	0.0801	0.0412	−0.0841	0.0895	0.0801	0.0412	0.0841
(2, 4)	0.0035	−0.0104	−0.0062	−0.0369	0.0035	0.0104	0.0062	0.0369
(2, 5)	−0.1049	0.0581	−0.0854	−0.0761	0.1049	0.0581	0.0854	0.0761
(3, 1)	−0.0319	0.0096	0.0033	−0.0088	0.0319	0.0096	0.0033	0.0088
(3, 2)	0.0251	0.0488	0.1965	−0.1418	0.0251	0.0488	0.1965	0.1418
(3, 3)	0.0292	0.0116	0.1070	−0.0940	0.0292	0.0116	0.1070	0.0940
(3, 4)	0.0323	0.0749	0.1152	−0.1568	0.0323	0.0749	0.1152	0.1568
(3, 5)	−0.1181	0.0831	0.0676	−0.0624	0.1181	0.0831	0.0676	0.0624
(4, 1)	−0.1674	−0.0389	−0.0677	−0.0270	0.1674	0.0389	0.0677	0.0270
(4, 2)	−0.0564	−0.0432	0.0798	−0.0763	0.0564	0.0432	0.0798	0.0763
(4, 3)	−0.0025	−0.1101	0.0558	−0.0613	0.0025	0.1101	0.0558	0.0613
(4, 4)	0.0346	−0.0127	0.0803	−0.0653	0.0346	0.0127	0.0803	0.0653
(4, 5)	−0.0298	0.0638	0.0809	−0.0281	0.0298	0.0638	0.0809	0.0281
(5, 1)	−0.0185	0.0646	0.0378	−0.0837	0.0185	0.0646	0.0378	0.0837
(5, 2)	−0.0800	0.0139	−0.0186	−0.0580	0.0800	0.0139	0.0186	0.0580
(5, 3)	0.0559	0.0339	0.0561	0.0510	0.0559	0.0339	0.0561	0.0510
(5, 4)	0.0002	−0.0420	0.0635	−0.0376	0.0002	0.0420	0.0635	0.0376
(5, 5)	0.0497	0.0633	0.0699	−0.0855	0.0497	0.0633	0.0699	0.0855
Mean	−0.0206	0.0111	0.0513	−0.0522	0.0576	0.0409	0.0723	0.0588
Stdev	0.0682	0.0490	0.0655	0.0482	0.0405	0.0280	0.0398	0.0395

frequency. Then we obtain the time change and compute the sample correlations between the regressors (c_n) and (x_{nj}), and the regression errors (u_{ni}), for the random time regression. If the assumed orthogonality does not hold, then we must have at least some evidence of nonzero correlation between the regressors and the regression errors under the random sampling scheme. The results are reported in Table 4. The values of the actual sample correlations are quite low for all regressors, supporting the validity of OLS in the random time regressions.

Last, based on all the results, we are ready to check if the volatility structure we impose on error terms is plausible and well treated by the random sampling scheme proposed in the paper. To empirically evaluate this issue, we consider a stochastic volatility model to measure the degree of persistence in the stochastic volatilities of regression errors in (21). Therefore, we specify $u_{ni} = \sqrt{f_i(v_{ni})}\varepsilon_{ni}$ for $n = 1, \ldots, N$ and $i = 1, \ldots, I$ with $\mathbb{E}\varepsilon_{ni}^2 = 1$, where (v_{ni}) is the latent volatility factor generated as $v_{ni} = \gamma_i v_{n-1,i} + \eta_{ni}$ and (f_i) is the volatility function. We use the logistic function for the volatility function f_i, and allow for nonzero correlation ρ_i between (ε_{ni}) and (η_{ni}), which represents the leverage
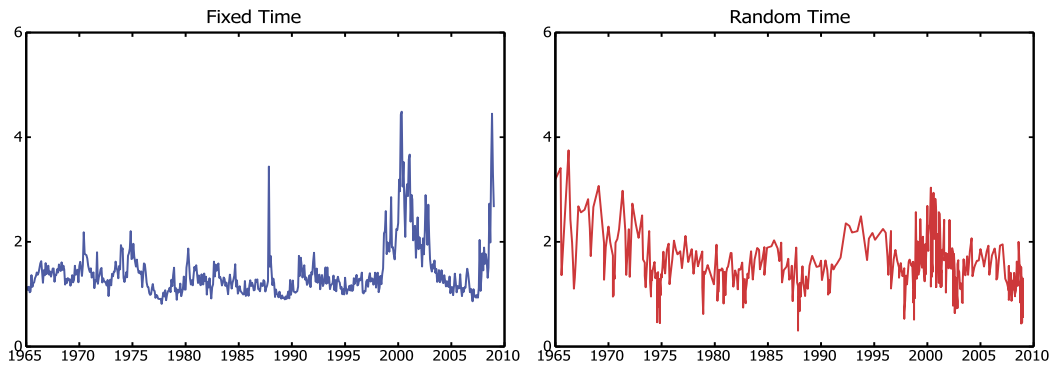
FIGURE 7. Average realized volatility for fitted residuals. Notes: The daily residuals for 25 portfolios formed with 5 size and 5 book-to-market ratio groups are recovered from Fama–French three-factor regression using the coefficient estimates obtained from running our random time regression with monthly equivalent observations. Then we calculate monthly (fixed time) and monthly equivalent (random time) realized volatility for 25 series to get the average realized volatility in both fixed and random time.

effect.[31] The stochastic volatility model is fitted for each i using the fitted residuals from regression (24) based on the random sampling scheme; the latent volatility factor is extracted using the conventional density-based Kalman filter method. For comparison, we also estimate the stochastic volatility model using the fitted residuals from the fixed time regression. The extracted volatility factors are given in Figure 7, and the estimated values of the autoregression (AR) coefficients (γ_i) and the leverage effects (ρ_i) of the extracted volatility factors are presented in Table 5.

It seems evident that the extracted volatility factors of the residuals from random time regressions are not persistent. Note that the volatilities of the individual portfolios consist of both the nonstationary common trend and stationary idiosyncratic components, and only the nonstationary common component is corrected via our random sampling method. Thus, one may expect that the estimated AR coefficients in this case reflect only the stationary component of the extracted volatilities. Indeed, the average of the estimated AR coefficients is around 0.804 in the random time, which is stationary. This is in sharp contrast with the volatility factors extracted from the fixed time residuals, most of which have the estimated AR coefficient very close to unity. In addition, the observed high persistence of the volatility factors extracted from the fixed time residuals is quite similar to that of the fixed time excess market return, as can be seen in the left-hand-side panel of Figure 8. The estimated AR coefficient of the fixed time market volatility factor is 0.6914.[32] On the other hand, the AR coefficient of the extracted volatility factor from the time changed market return is much smaller, indeed close to zero, and its sample path clearly shows no persistence as displayed in the right-hand-side panel of

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[31]The reader is referred to Kim, Lee, and Park (2009) for more details about the stochastic volatility model and estimation methodology we use in the paper.

[32]This is also consistent with the well known fact that the stochastic volatility of market return is highly persistent. See aforementioned references for the nonstationarity in stock return volatilities.

Table 5. AR coefficients and leverage effects of extracted volatility factors.

(Size, B/M)	Fixed Time		Random Time	
	Gamma	Rho	Gamma	Rho
(1, 1)	0.929 (0.014)	−0.140 (0.149)	0.636 (0.039)	−0.036 (0.138)
(1, 2)	0.975 (0.011)	0.163 (0.324)	0.964 (0.011)	0.409 (0.000)
(1, 3)	0.932 (0.018)	0.034 (0.190)	0.893 (0.023)	0.049 (0.153)
(1, 4)	0.988 (0.008)	−0.115 (0.273)	0.764 (0.028)	0.477 (0.130)
(1, 5)	0.960 (0.014)	−0.449 (0.395)	0.589 (0.037)	0.120 (0.216)
(2, 1)	0.937 (0.015)	−0.100 (0.172)	0.955 (0.013)	−0.181 (0.185)
(2, 2)	0.921 (0.024)	0.208 (0.167)	0.915 (0.019)	−0.065 (0.161)
(2, 3)	0.918 (0.022)	0.018 (0.172)	0.927 (0.019)	−0.175 (0.139)
(2, 4)	0.929 (0.023)	−0.460 (0.462)	0.636 (0.041)	−0.111 (0.145)
(2, 5)	0.905 (0.019)	−0.023 (0.203)	0.426 (0.038)	−0.033 (0.015)
(3, 1)	0.964 (0.012)	0.006 (0.241)	0.906 (0.020)	0.269 (0.236)
(3, 2)	0.980 (0.011)	−0.533 (0.543)	0.975 (0.011)	−0.210 (0.190)
(3, 3)	0.943 (0.017)	−0.231 (0.201)	0.881 (0.020)	−0.213 (0.183)
(3, 4)	0.908 (0.021)	−0.074 (0.150)	0.307 (0.049)	0.031 (0.117)
(3, 5)	0.946 (0.015)	0.101 (0.176)	0.725 (0.036)	−0.029 (0.117)
(4, 1)	0.961 (0.012)	−0.085 (0.177)	1.000 (0.005)	0.134 (0.207)
(4, 2)	0.941 (0.015)	−0.097 (0.145)	0.921 (0.019)	−0.114 (0.156)
(4, 3)	0.952 (0.013)	−0.345 (0.195)	0.727 (0.028)	−0.095 (0.138)
(4, 4)	0.871 (0.027)	0.170 (0.166)	0.804 (0.025)	0.007 (0.325)
(4, 5)	0.967 (0.009)	−0.330 (0.298)	0.811 (0.023)	0.281 (0.182)
(5, 1)	0.893 (0.019)	−0.378 (0.170)	0.963 (0.012)	−0.252 (0.177)
(5, 2)	0.942 (0.016)	0.276 (0.164)	0.835 (0.030)	0.313 (0.152)
(5, 3)	0.955 (0.012)	0.067 (0.186)	0.999 (0.008)	−0.177 (0.264)
(5, 4)	0.942 (0.016)	−0.056 (0.184)	0.696 (0.038)	−0.321 (0.146)
(5, 5)	0.952 (0.014)	−0.211 (0.163)	0.837 (0.024)	−0.117 (0.161)
Average	0.941 (0.016)	−0.103 (0.227)	0.804 (0.025)	−0.002 (0.161)

Figure 8. Putting things together, the empirical results confirm that our volatility setup is realistic and properly handled with the random sampling scheme.

5. Reexamination of Fama–French regressions

5.1 *Tests of the CAPM*

This section examines the CAPM regressions using daily portfolio returns. First two sets consist of 11 portfolios, 10 of which are sorted out by a firm characteristic (sizes or B/M ratios) and the 11th of which refers to the hedge portfolio explained in the previous section. The next set consists of 30 industry portfolios. Finally, the last data set comprises the traditional 25 portfolios sorted by sizes and B/M ratios. As discussed in Section 2, we run regression (9) under the two sampling schemes: fixed time and random time. In the case of the fixed time sampling, we construct monthly data by integrating portfolio returns over each month. For the random time sampling scheme, we follow (12) and set Δ at the level of quadratic variation comparable to the average, monthly excess market
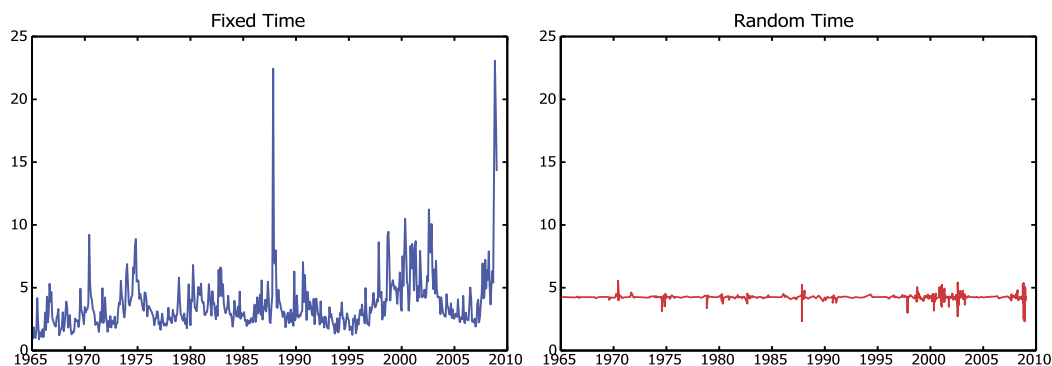
FIGURE 8. Realized volatility from excess market return. Notes: The excess market return series is first demeaned using nonparametric local linear regression before we estimate realized volatility. The AR coefficients of the realized volatility are 0.6914 under the fixed time and -0.0773 under the random time scheme.

TABLE 6. Test of CAPM on size portfolios.

Size	Fixed Time		Random Time	
	Alpha	Beta	Alpha	Beta
1 Small	0.0045 (0.0205)	1.0772 (0.0394)	0.0533 (0.0174)	1.0128 (0.0421)
2	0.0047 (0.0170)	1.1564 (0.0326)	0.0273 (0.0142)	1.1302 (0.0343)
3	0.0135 (0.0145)	1.1536 (0.0279)	0.0352 (0.0121)	1.1216 (0.0294)
4	0.0141 (0.0135)	1.1231 (0.0259)	0.0315 (0.0109)	1.1067 (0.0264)
5	0.0185 (0.0113)	1.1048 (0.0218)	0.0335 (0.0095)	1.0969 (0.0229)
6	0.0118 (0.0097)	1.0789 (0.0185)	0.0205 (0.0081)	1.0716 (0.0197)
7	0.0118 (0.0081)	1.0798 (0.0156)	0.0174 (0.0067)	1.0533 (0.0162)
8	0.0098 (0.0070)	1.0695 (0.0134)	0.0091 (0.0058)	1.0476 (0.0140)
9	0.0065 (0.0057)	0.9930 (0.0109)	0.0050 (0.0044)	0.9767 (0.0106)
10 Big	-0.0010 (0.0054)	0.9217 (0.0104)	-0.0074 (0.0044)	0.9403 (0.0106)
1–10 Size strategy	0.0055 (0.0247)	0.1555 (0.0475)	0.0607 (0.0209)	0.0725 (0.0505)
Wald	12.2747 (0.2671)		68.7681 (0.0000)	

return. Tables 6–8 report estimates of alphas and betas with standard errors for each portfolio, followed by the Wald statistic defined in (29) to test if the model is rejected.

Table 6 reports results for the decile size portfolios and the size strategy (1st–10th decile) portfolio. Beta estimates in both sampling schemes are close to each other and MKT mildly captures exposures to taking risks for small firms (i.e., beta is higher for small firms). However, comparing the alpha estimates, one can clearly see that there is a huge difference between the fixed time and the random time sampling schemes. There exists a significant risk component that is not captured by the market factor according to small firms' alpha estimates in the random sampling case, whereas the fixed sampling result is much weaker. Somewhat expected, the Wald test statistic states that the CAPM is not rejected in the case of the fixed sampling regression, while the p-value of the random

TABLE 7. Test of CAPM on book-to-market portfolios.

	Fixed Time		Random Time	
Book-to-Market	Alpha	Beta	Alpha	Beta
1 Growth	−0.0157 (0.0101)	1.0872 (0.0194)	−0.0199 (0.0085)	1.1211 (0.0207)
2	0.0019 (0.0075)	1.0101 (0.0143)	−0.0100 (0.0060)	1.0182 (0.0146)
3	0.0091 (0.0079)	0.9677 (0.0152)	0.0001 (0.0063)	0.9800 (0.0152)
4	0.0074 (0.0092)	0.9606 (0.0177)	0.0033 (0.0077)	0.9590 (0.0186)
5	0.0108 (0.0097)	0.8679 (0.0186)	0.0065 (0.0079)	0.8778 (0.0192)
6	0.0176 (0.0094)	0.8873 (0.0181)	0.0257 (0.0078)	0.8723 (0.0188)
7	0.0289 (0.0112)	0.8379 (0.0214)	0.0329 (0.0089)	0.8212 (0.0215)
8	0.0367 (0.0116)	0.8368 (0.0224)	0.0322 (0.0092)	0.8126 (0.0224)
9	0.0423 (0.0122)	0.8850 (0.0235)	0.0496 (0.0101)	0.8610 (0.0245)
10 Value	0.0409 (0.0165)	0.9898 (0.0316)	0.0427 (0.0134)	0.9992 (0.0324)
10–1 Book-to-market strategy	0.0566 (0.0231)	−0.0974 (0.0444)	0.0626 (0.0191)	−0.1219 (0.0462)
Wald	15.7134 (0.1081)		49.9838 (0.0000)	

TABLE 8. Test of CAPM on (Size, B/M) and industry portfolios.

	(Size, B/M)		Industry	
	Fixed Time	Random Time	Fixed Time	Random Time
Wald	121.7332 (0.0000)	344.1042 (0.0000)	44.2420 (0.0453)	79.3523 (0.0000)

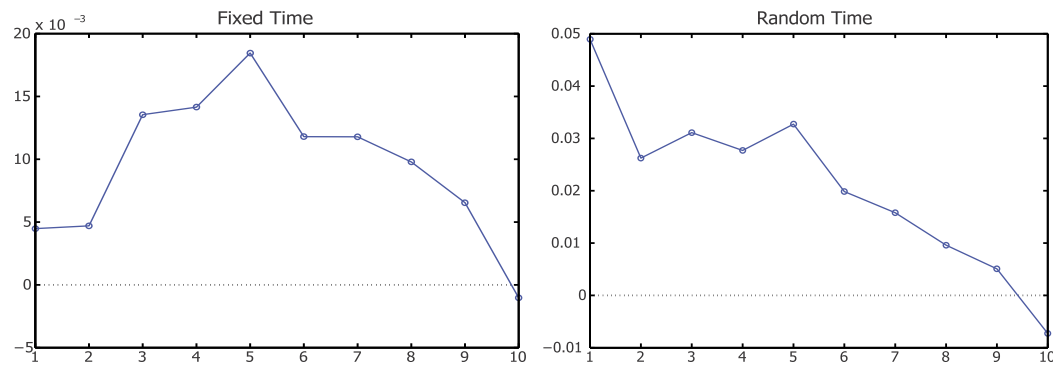FIGURE 9. Alphas of size portfolios.

sampling case is 0.0000, a clear rejection. Figure 9 displays this finding graphically. Fixed sampling results show a hump shape of alpha estimates, which is somewhat confusing, if the size effect does matter. On the contrary, the random sampling result with a proper treatment of stochastic volatilities shows a nice emergence of monotonically decreasing size premium.
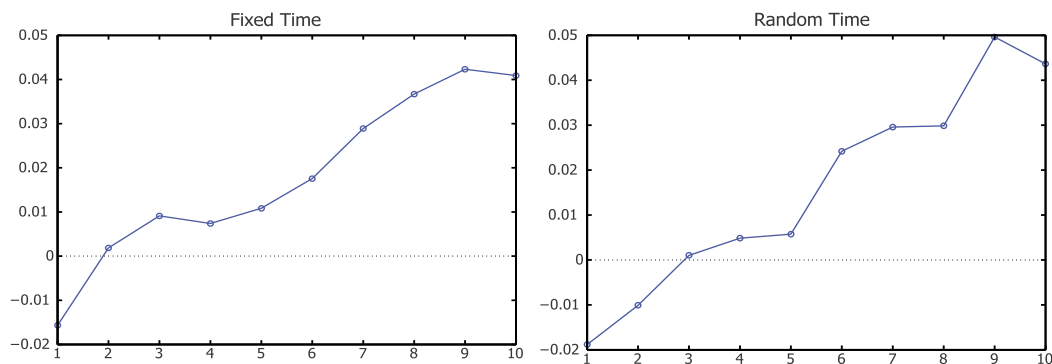
FIGURE 10. Alphas of book-to-market portfolios.

Table 7 reports basically the identical information for the book-to-market portfolios. However, this case shows other evidence that the conventional method fails in doing reliable statistical inferences. Unlike Table 6 with size-based portfolios, Table 7 report that both alphas and betas are similarly estimated, and the estimated amount of value premium is around 6% per annum. Figure 10 illustrates that the estimated alphas are similar across the two methods. However, when the Wald statistics are compared, the fixed sampling scheme cannot reject the CAPM, while the random sampling rejects the model with p-value of 0.0000. In addition, the estimated market betas dictate that the growth stocks are riskier than the value stocks, implying that the CAPM is not pricing these portfolios correctly, as shown by Fama and French (1993).

Based on the results, we suspect that the model is correctly rejected in the random time regression, and the conventional fixed time regression seems to have difficulty in doing this. However, at this stage, a natural question arises: Fama and French (1993) and many authors have used the conventional OLS with the Gibbons, Ross, and Shanken (GRS) tests to reject the CAPM and even various other Fama–French models. Why do our fixed sampling results differ from the previous OLS results? Recall that the main difference between the conventional OLS and our fixed sampling OLS is the way that data series are constructed. The conventional monthly return data use two data points of asset prices between two consecutive months, while our data are constructed by integrating the daily data over a month in fixed sampling cases. The two methods would produce the same monthly data if the instantaneous returns were defined as the differentials of logarithm of prices, namely $d\log(P_{it})$. However, our instantaneous returns are constructed as the ratios of the price differentials to previous prices, namely $dP_{it}/P_{i,t-1}$, and under this definition the two data construction methods can substantially differ.

But can we then achieve the same results by directly using the monthly return data with the conventional OLS machinery instead of using a random sampling scheme on a higher frequency data, because both will take a look at the data at a frequency comparable to monthly frequency after all? Note that our model is written in continuous time and then aggregated over time to make the model testable in discrete time environment. As shown in Section 3, the asymptotics and resultant test statistics of the model are different from those of the discrete time counterparts. If continuous-time diffusion models

better describe the actual market clearing processes, which we believe, then these differences are critical in evaluating the empirical asset pricing models. To further investigate this point, we run the OLS regressions on the conventional monthly returns with the same decile portfolios. We find two interesting results. First, in both size and B/M-based decile portfolios, the GRS statistics report p-values around 0.031 and 0.038, respectively. Therefore, the CAPM is not rejected at 3% despite the prevalent size or value effects. Second, as we vary the starting date of the data, p-values vary significantly between 0.002 and 0.208.[33] This result may be an indirect evidence of conditional factor models. But even in conditional models, a final test on whether a model is rejected would be to look at whether or not the long-run average of alphas is zero.[34] Thus, the use of low-frequency data does not necessarily give reliable and accurate test results. On the contrary, our random sampling results are quite robust to such variations. This is a subtle but important point: Conventional methods may fail to reject a model too easily, and often produce puzzling results.

Putting things together, the conventional testing procedure is inoperable and fails to reject a proposed model too often. As emphasized in our earlier discussions, the failure of the conventional testing procedure is due to the fact that the variance–covariance matrix of the error terms is very difficult to estimate in the presence of nonstationary stochastic volatilities. Indeed, existing empirical studies unequivocally show that they are nonstationary, though their sources may differ. And the models with time-varying and stochastic volatilities would yield misleading results when they are estimated using the conventional approach, as we discussed earlier. In addition, the presence of leverage effects, also prevalent in stock return data, brings about endogeneity in volatilities, which further complicates the treatment of the nonstationary volatilities. We also run a similar exercise for the 25 Fama–French portfolios sorted by size and B/M ratio, and we report the results in Table 8 under the (Size, B/M) column. Now, even the fixed sampling scheme rejects the CAPM with a p-value of 0.0000, which contradicts the test results with the decile portfolios. On the contrary, the random sampling method rejects the model, compatible with the results from the decile portfolios.

As a final exercise for the CAPM, we examine the unmanaged industry portfolios consisting of 30 groups and report the results in Table 8 under the Industry column. The familiar story prevails again. Despite the similar estimates of alphas and betas on average, Wald test statistics say that the conventional approach cannot reject the CAPM at 4%, while our approach rejects the model with zero p-value. To analyze what makes the difference between the two approaches in this case, we plot the alphas connecting each industry that belongs to one of the more broadly defined five groups of industries in Figure 11. Most conspicuous are the industries in the consumer goods group

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[33]Although not monotonic, CAPM on size portfolios is more difficult to reject when the sample period gets longer, while the opposite is likely to be true for the CAPM regressions on value portfolios. We do not report the results as a separate table since similar exercises have been performed in other studies. Nevertheless, our argument here is germane and new in the context of testing factor pricing models with nonstationary volatilities in a high-frequency setting.

[34]See Ang and Kristensen (2012) for more details. They test conditional factor pricing models using a nonparametric method.
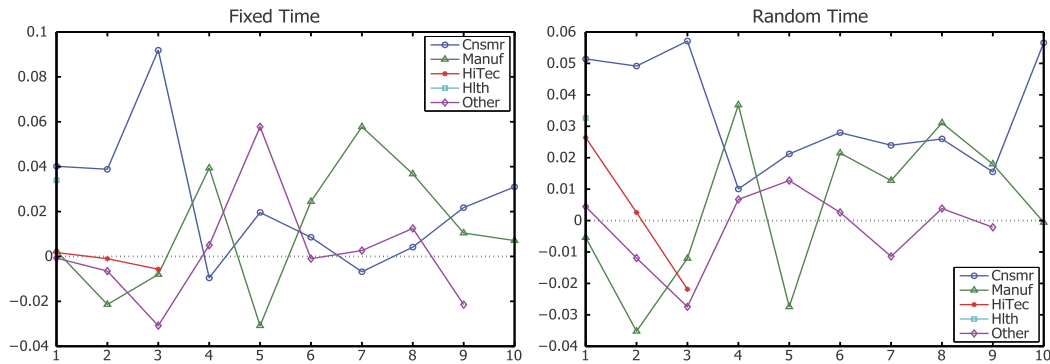
FIGURE 11. Alphas of industry portfolios. Notes: The graph presents the alphas connecting each industry that belongs to one of the more broadly defined five groups of industries (Cnsmr, Manuf, HiTec, Hlth, Other). Most conspicuous are the alphas of the industries in the consumer goods group (illustrated by the circles on the line for consumer industry) featuring consistently positive values in our random time regression (right), while the fixed time regression (left) produces a mixed bag of results.

featuring consistently positive alphas in our random time regression, while the fixed time regression produces a mixed bag of results. This suggests that consumption growth may be a valid pricing factor together with the financial market factor, which is reminiscent of consumption-based pricing models employing more flexible preferences such as Epstein and Zin (1989). Summing up, the random sampling method works reliably in a high-frequency environment, contrary to its fixed sampling counterpart. More importantly, all the test results for the CAPM based on the random sampling provide a strong case for multifactor models.

5.2 *Tests of the Fama–French models*

In this section, we investigate multifactor models of asset returns. Continued from the previous section, we begin with two-factor models, incorporating the size or B/M factor into the CAPM on each of the corresponding decile portfolio data sets. In Tables 9 and 10, like the CAPM, the fixed time OLS regressions cannot reject the two-factor models ((MKT, SMB), (MKT, HML)) with even higher p-values, stating that the size and B/M factors are relevant pricing factors, despite that the CAPM is not rejected on the same data sets. This is a contradicting result caused by the imprecise statistical method. Meanwhile, the random sampling result shows that the two-factor model with the B/M is not rejected at 11% of p-value for the 10 B/M-based portfolios, though the model with the size factor fails to explain the 10 size-based portfolios. That is, our method suggests that the B/M factor can be viewed as a valid pricing factor for explaining the variations of stock returns over the cross section of B/M ratio groups, while the size factor may be insufficient to account for the spectrum of asset returns in light of the firm sizes. In addition, we want to note that this is consistent and plausible with the random sampling CAPM results on size groups that are decisive rejections.

TABLE 9. Test of two-factor model on size portfolios.

Size	Fixed Time			Random Time		
	Alpha	Beta_MKT	Beta_SMB	Alpha	Beta_MKT	Beta_SMB
1 Small	−0.0038 (0.0095)	0.8466 (0.0189)	1.1876 (0.0272)	0.0123 (0.0075)	0.8096 (0.0185)	1.2391 (0.0277)
2	−0.0026 (0.0060)	0.9551 (0.0121)	1.0362 (0.0174)	−0.0074 (0.0050)	0.9586 (0.0123)	1.0465 (0.0185)
3	0.0074 (0.0056)	0.9832 (0.0112)	0.8771 (0.0161)	0.0056 (0.0044)	0.9751 (0.0108)	0.8932 (0.0163)
4	0.0086 (0.0057)	0.9682 (0.0114)	0.7975 (0.0164)	0.0052 (0.0042)	0.9765 (0.0104)	0.7942 (0.0156)
5	0.0139 (0.0056)	0.9799 (0.0112)	0.6434 (0.0162)	0.0116 (0.0044)	0.9881 (0.0109)	0.6634 (0.0163)
6	0.0085 (0.0064)	0.9872 (0.0128)	0.4718 (0.0184)	0.0038 (0.0051)	0.9890 (0.0126)	0.5039 (0.0189)
7	0.0093 (0.0061)	1.0119 (0.0122)	0.3494 (0.0175)	0.0051 (0.0048)	0.9927 (0.0119)	0.3697 (0.0179)
8	0.0082 (0.0060)	1.0249 (0.0120)	0.2296 (0.0173)	0.0016 (0.0051)	1.0104 (0.0125)	0.2266 (0.0188)
9	0.0063 (0.0057)	0.9852 (0.0113)	0.0402 (0.0163)	0.0041 (0.0044)	0.9724 (0.0109)	0.0262 (0.0164)
10 Big	0.0010 (0.0030)	0.9788 (0.0060)	−0.2942 (0.0087)	0.0021 (0.0025)	0.9872 (0.0061)	−0.2860 (0.0091)
1–10 Size strategy	−0.0049 (0.0099)	−0.1323 (0.0198)	1.4818 (0.0285)	0.0102 (0.0079)	−0.1776 (0.0195)	1.5251 (0.0293)
Wald		12.0613 (0.2810)			38.4651 (0.0000)	

One common feature in both of the two-factor models we consider here is that the abnormal returns of the hedged portfolios are not statistically different from zero, suggesting that pricing errors are small. What then drives the rejections of the model according to the Wald statistics in the random sampling case? A closer look at the Tables 9 and 10 reveals that portfolios other than those used to form the hedged portfolios, such as size 5, turn out to have significantly nonzero abnormal returns. Thus, the Wald test for all assets, compared to the test on a hedged portfolio alone, is a more stringent test verifying if a factor model can explain all the returns considered rather than just the ones specifically aimed at matching certain characteristics. Therefore, if the results between the two tests in a factor pricing model clearly disagree, then the proposed model may need new factors because it is likely to have difficulty in fitting the returns sorted by other characteristics. Consistent with this view, Table 10 shows that the medium value stocks as well as the hedged portfolio do not have significantly nonzero abnormal returns; hence the model is not rejected.

Based on this observation and following the tradition, now we estimate and test the three-factor Fama–French model on the data set with 25 portfolios. Table 11 shows that both fixed and random time regressions reject the model. This result is somewhat anticipated from the random sampling results on the two-factor model (MKT, SMB) in Table 9, where the size factor fails to explain the 10 size-based portfolios. Compared to the CAPM results on the 25 portfolios, the extent to which the model misbehaves appears to be smaller, yet the p-values based on the Wald statistic imply a clear rejection of the Fama–French model, which requires careful scrutiny. The upper panels for the fixed and random time regressions in Figure 12 display the deviations of alpha estimates from zero for the 25 portfolios. The first group consisting of the smallest stocks has the largest magnitude of deviations, which is a common feature in both the fixed sampling and the random sampling cases. Compared to the corresponding graphs for the CAPM (which we do not report to save space), the lines are much closer to the horizontal axis of zero value and even the slope is obviously reversed in some cases. However, one observation, corresponding to (size, B/M) = (1, 1) distinctively deviates from zero value, which seems to drive the rejection of the model. Note that this refers to small cap, growth stocks with low book-to-market ratios. To be more precise, we plot the average excess returns for each portfolio and the predicted returns from the Fama–French regressions in the lower panels of Figure 12 following Cochrane (2001, p. 441). It is easy to observe that the (1, 1) portfolio is quite off from the 45 degree line compared to other portfolios and it displays a significant premium within the smallest B/M group.[35] This is a part of the size premium, yet we must note that the graphs in the lower panel of Figure 12 display that the small growth stocks show a stark contrast to the typical pattern of the size premium, hinting that the conventional size factor may be not enough to capture this behavior.[36]

[35]This effect also appears in Cochrane (2001), but with a much weaker pattern. We suspect that the difference comes from the data period, which is between 1947 and 1996 in his case.

[36]There may be a common economic fundamental that affects both size and B/M portfolio returns in a different fashion than the conventional size and B/M factors do. Fama and French (1995) report that both the size and B/M premia are related to the earnings of the firms. They find that the small firms have persistently lower earnings and the growth stocks have persistently high earnings, though the former link

Table 10. Test of two-factor model on book-to-market portfolios.

Book-to-Market	Fixed Time			Random Time		
	Alpha	Beta_MKT	Beta_HML	Alpha	Beta_MKT	Beta_HML
1 Growth	0.0157 (0.0075)	0.9528 (0.0156)	−0.5052 (0.0239)	0.0122 (0.0061)	0.9684 (0.0160)	−0.5393 (0.0248)
2	0.0069 (0.0075)	0.9885 (0.0155)	−0.0811 (0.0239)	−0.0036 (0.0061)	0.9874 (0.0159)	−0.1085 (0.0246)
3	0.0052 (0.0080)	0.9845 (0.0165)	0.0631 (0.0254)	−0.0010 (0.0065)	0.9849 (0.0169)	0.0175 (0.0262)
4	−0.0084 (0.0087)	1.0281 (0.0180)	0.2538 (0.0276)	−0.0117 (0.0073)	1.0303 (0.0192)	0.2515 (0.0297)
5	−0.0092 (0.0088)	0.9537 (0.0181)	0.3224 (0.0279)	−0.0134 (0.0072)	0.9728 (0.0188)	0.3354 (0.0291)
6	−0.0063 (0.0080)	0.9894 (0.0165)	0.3837 (0.0254)	0.0024 (0.0066)	0.9831 (0.0172)	0.3912 (0.0266)
7	−0.0049 (0.0085)	0.9827 (0.0176)	0.5443 (0.0271)	0.0007 (0.0066)	0.9740 (0.0173)	0.5400 (0.0268)
8	−0.0061 (0.0070)	1.0201 (0.0145)	0.6890 (0.0223)	−0.0067 (0.0056)	0.9974 (0.0147)	0.6528 (0.0228)
9	−0.0001 (0.0081)	1.0665 (0.0168)	0.6825 (0.0258)	0.0095 (0.0068)	1.0517 (0.0178)	0.6733 (0.0275)
10 Value	−0.0126 (0.0118)	1.2189 (0.0243)	0.8612 (0.0374)	−0.0063 (0.0099)	1.2320 (0.0258)	0.8224 (0.0400)
10–1 Book-to-market strategy	−0.0284 (0.0141)	0.2661 (0.0290)	1.3663 (0.0446)	−0.0186 (0.0115)	0.2636 (0.0300)	1.3617 (0.0464)
Wald		9.8776 (0.4513)			15.6828 (0.1091)	

TABLE 11. Test of Fama–French three-factor model.

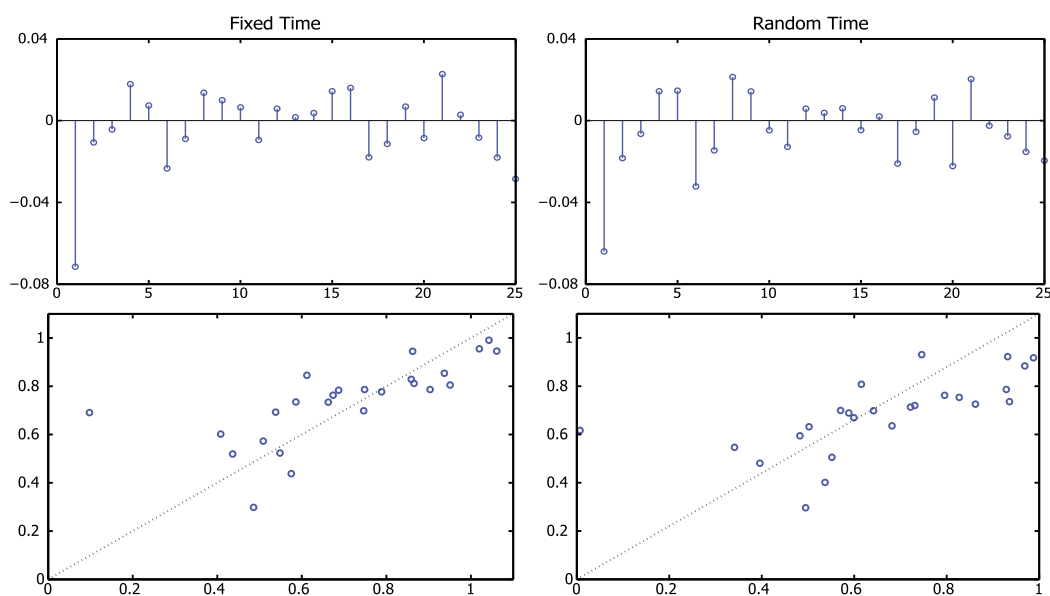|	Fixed Time	Random Time
Wald	95.5585 (0.0000)	161.2924 (0.0000)

FIGURE 12. Fama–French alphas and betas of (size, B/M) portfolios. Notes: The left panels display alphas (top) and betas (bottom) estimated from the fixed time regression, while the right panels present those from our random time regression. The betas here represent the predicted returns from Fama–French regressions and are plotted along with the average excess returns for each portfolio, following Cochrane (2001, p. 441).

This suggests that either an additional factor or a replacement factor may be needed to justify the premium related to buying large cap stocks and selling small cap stocks in the group of firms with small distress.[37]

We recall that the portfolio returns from the consumer goods industry feature significant abnormal returns when CAPM is used. Admittedly, there is no direct connection between the small growth stocks and the consumer goods industry. However, given the signifying role of consumption goods as a foundational link between a discount factor and asset prices, we believe that including the consumption sector returns as a factor is a worthy trial. Related, Lettau and Ludvigson (2001) found that a macroeconomic fac-

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is weak. If persistent high earnings imply low cash flow risk and the small firm effect is dominated by the value effect, this story may justify why a small, growth stock is a good asset to short sell. However, it still does not explain why the large cap within the smallest B/M ratio is a risky bet as illustrated in the graphs in Figure 12.

[37]As one of the usual suspects, we try the momentum factor, making a four-factor model. However, we find that the momentum factor is orthogonal to the size effect within low book-to-market ratios.

tor that captures the consumption wealth ratio can substantially improve the perfor-
mance of the consumption CAPM. Motivated by those findings, we form a consumer
goods industry factor, called the CMR factor, as the excess returns on the portfolio of
the firms producing consumer goods. Regarding the definition of the consumer goods
sector, we simply use the returns from the consumer goods sector out of the data with
the five-industry category available in the French's data library. Then we run regressions
of multifactor models incorporating the consumer goods industry factor (CMR) on the
10 size-based and B/M-based portfolios, and the 25 portfolios to see if a model with the
CMR factor can help explain the behaviors of asset returns, especially the small growth
stocks. We report the results from three-factor models, which include the market, the
CMR, and either the size or B/M factor.[38]

Table 12 displays the results from the three-factor models with the CMR factor on
the portfolios sorted by sizes and B/M ratios.[39] In comparison with the results in Ta-
bles 9 and 10, we observe that p-values increase in each case and the beta coefficients
for the CMR are mostly significant. Thus, it is inferred that the CMR factor helps explain
both the value-based and size-based portfolios. Especially, the model with the market,
B/M, and CMR factors is not rejected at 14%.[40] Figure 13 shows that the overall fit is
good for both of the three-factor models. Based on this positive result, we select the
three-factor model with the market, B/M, and CMR factors to investigate the 25 port-
folios. Unfortunately, the model is rejected at 0.0000 of p-value, but we find that in the
model with the market, B/M, and CMR factors, the pricing error for the $(1, 1)$ portfolio
gets significantly reduced and the overall fit appears to be generally comparable to that
of the Fama–French model. This is summarized in Table 13 and Figure 14. For the model
with the market, size, and CMR, overall fit is worse and the Wald statistic is higher; hence
it is clearly inferior to the model with the market, B/M, and CMR, as well as the tradi-
tional Fama–French model. The results suggest that the size factor is not entirely satis-
factory in terms of capturing the cross sectional behaviors of asset returns and that the
returns from the consumer goods industry are useful in complementing this deficiency.
However, further investigation and justification are necessary to incorporate industry-
specific return factors such as CMR into multifactor models, which we leave as future
work.

6. Conclusion

This paper develops a new econometric framework and tools to analyze multifactor as-
set pricing models. We consider a continuous-time factor model with a specific error
component structure consistent with an underlying asset pricing theory. We show that
our error structure is empirically supported as well. It is well known that asset returns

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[38]We also tried the two-factor model consisting only of the market and the CMR. To conserve space we
do not report the results here, but it appears that the CMR factor captures some of the size premia, but not
the value premia.

[39]From now on, we only report the random sampling results, because the fixed sampling results on the
industry portfolio do not pick up the CMR factor as shown in Table 8 and Figure 11.

[40]When we estimated a four-factor model, that is, the Fama–French three-factor model with the CMR
factor, we find that the p-values get lower to 0.0000 and 0.0003 for the B/M and size portfolios, respectively.

TABLE 12. Tests of three-factor models with consumer industry factor.

|	Alpha	Beta_MKT	Beta_HML	Beta_CMR
Panel A: On B/M Portfolios (Random Time)				
1 Growth	0.0099 (0.0057)	0.7365 (0.0323)	−0.5706 (0.0235)	0.2412 (0.0298)
2	−0.0064 (0.0055)	0.7104 (0.0309)	−0.1459 (0.0225)	0.2882 (0.0285)
3	−0.0036 (0.0060)	0.7274 (0.0339)	−0.0172 (0.0246)	0.2678 (0.0312)
4	−0.0127 (0.0073)	0.9340 (0.0411)	0.2386 (0.0299)	0.1001 (0.0379)
5	−0.0133 (0.0072)	0.9866 (0.0405)	0.3372 (0.0295)	−0.0143 (0.0374)
6	0.0027 (0.0066)	1.0079 (0.0371)	0.3945 (0.0270)	−0.0258 (0.0342)
7	−0.0001 (0.0066)	0.8941 (0.0371)	0.5292 (0.0270)	0.0832 (0.0342)
8	−0.0074 (0.0056)	0.9312 (0.0316)	0.6438 (0.0230)	0.0688 (0.0292)
9	0.0085 (0.0068)	0.9570 (0.0380)	0.6605 (0.0277)	0.0985 (0.0351)
10 Value	−0.0073 (0.0098)	1.1371 (0.0555)	0.8096 (0.0404)	0.0988 (0.0511)
10–1 Book-to-market strategy	−0.0172 (0.0114)	0.4006 (0.0643)	1.3801 (0.0468)	−0.1424 (0.0593)
Wald	14.6978 (0.1435)			
Panel B: On Size Portfolios (Random Time)				
1	0.0115 (0.0075)	0.7676 (0.0406)	1.2394 (0.0277)	0.0454 (0.0391)
2	−0.0092 (0.0049)	0.8656 (0.0266)	1.0472 (0.0182)	0.1004 (0.0256)
3	0.0037 (0.0043)	0.8785 (0.0233)	0.8939 (0.0159)	0.1044 (0.0224)
4	0.0031 (0.0041)	0.8686 (0.0222)	0.7950 (0.0151)	0.1164 (0.0213)
5	0.0098 (0.0043)	0.8962 (0.0235)	0.6641 (0.0160)	0.0992 (0.0226)
6	0.0016 (0.0050)	0.8726 (0.0270)	0.5047 (0.0184)	0.1258 (0.0260)
7	0.0033 (0.0047)	0.8966 (0.0257)	0.3704 (0.0175)	0.1038 (0.0247)
8	0.0004 (0.0051)	0.9504 (0.0274)	0.2271 (0.0187)	0.0648 (0.0264)
9	0.0022 (0.0043)	0.8719 (0.0234)	0.0270 (0.0160)	0.1086 (0.0225)
10	0.0016 (0.0024)	0.9604 (0.0133)	−0.2858 (0.0091)	0.0290 (0.0128)
1–10 Size strategy	0.0099 (0.0079)	−0.1928 (0.0430)	1.5252 (0.0293)	0.0165 (0.0413)
Wald	33.6686 (0.0002)			
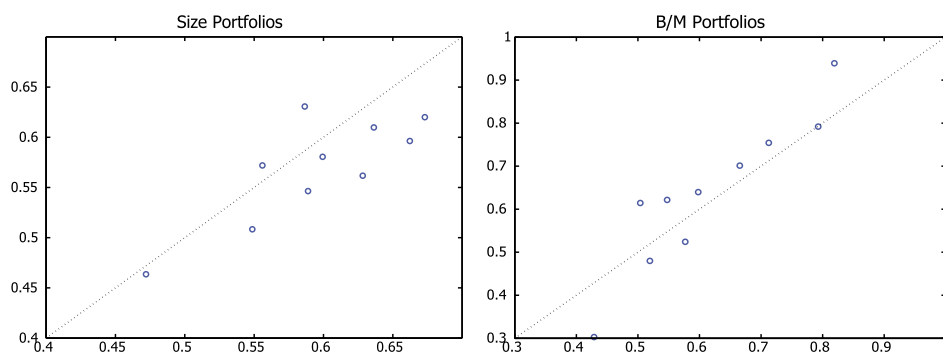
FIGURE 13. Betas of three-factor models with the consumer industry factor. Notes: The left panel plots the predicted returns estimated from our random time regression with the market, size, and CMR factors run on 10 size portfolios along with average excess returns for each portfolio. On the other hand, the graph on the right presents those returns obtained from our random time regression with the market, B/M, and CMR factors run on 10 B/M portfolios.

TABLE 13. Tests of three-factor models with consumer industry and B/M factors on (Size, B/M) portfolios: random time.

| (Size, B/M)	Alpha	Beta_MKT	Beta_HML	Beta_CMR
(1, 1)	0.001 (0.021)	1.354 (0.117)	−0.544 (0.085)	−0.122 (0.108)
(1, 2)	0.036 (0.017)	1.058 (0.098)	−0.148 (0.071)	0.063 (0.090)
(1, 3)	0.042 (0.016)	0.854 (0.089)	0.076 (0.065)	0.186 (0.082)
(1, 4)	0.059 (0.014)	0.834 (0.081)	0.212 (0.059)	0.163 (0.075)
(1, 5)	0.062 (0.015)	0.918 (0.086)	0.446 (0.063)	0.175 (0.079)
(2, 1)	0.012 (0.015)	1.166 (0.083)	−0.574 (0.061)	0.060 (0.077)
(2, 2)	0.024 (0.013)	0.905 (0.073)	−0.064 (0.053)	0.204 (0.067)
(2, 3)	0.054 (0.012)	0.898 (0.065)	0.200 (0.047)	0.162 (0.060)
(2, 4)	0.044 (0.010)	0.874 (0.058)	0.407 (0.042)	0.173 (0.053)
(2, 5)	0.029 (0.012)	0.991 (0.068)	0.563 (0.050)	0.177 (0.063)
(3, 1)	0.020 (0.012)	1.162 (0.066)	−0.577 (0.048)	0.019 (0.061)
(3, 2)	0.031 (0.010)	0.815 (0.057)	−0.017 (0.042)	0.271 (0.053)
(3, 3)	0.023 (0.009)	0.813 (0.050)	0.314 (0.036)	0.218 (0.046)
(3, 4)	0.023 (0.008)	0.832 (0.047)	0.466 (0.034)	0.199 (0.044)
(3, 5)	0.018 (0.010)	0.945 (0.057)	0.662 (0.041)	0.208 (0.053)
(4, 1)	0.017 (0.008)	1.066 (0.044)	−0.507 (0.032)	0.018 (0.040)
(4, 2)	−0.012 (0.008)	0.845 (0.044)	0.117 (0.032)	0.238 (0.041)
(4, 3)	0.001 (0.007)	0.835 (0.042)	0.359 (0.030)	0.213 (0.039)
(4, 4)	0.021 (0.007)	0.900 (0.041)	0.504 (0.030)	0.142 (0.038)
(4, 5)	−0.011 (0.009)	1.017 (0.052)	0.740 (0.038)	0.182 (0.048)
(5, 1)	0.006 (0.006)	0.653 (0.033)	−0.393 (0.024)	0.290 (0.030)
(5, 2)	−0.015 (0.007)	0.803 (0.039)	0.116 (0.028)	0.174 (0.036)
(5, 3)	−0.016 (0.008)	1.058 (0.045)	0.397 (0.033)	−0.106 (0.041)
(5, 4)	−0.025 (0.007)	0.884 (0.039)	0.640 (0.028)	0.075 (0.036)
(5, 5)	−0.024 (0.009)	0.986 (0.050)	0.744 (0.037)	0.078 (0.046)
Wald	177.2876 (0.0000)			
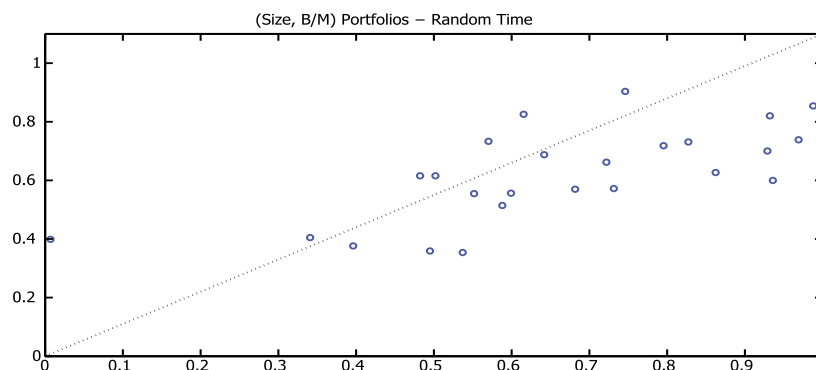
FIGURE 14. Betas of three-factor models with consumer industry and B/M factors. Notes: The graph displays the average excess returns for each portfolio and their predicted returns from the random time regression with the market, B/M, and CMR factors run on 25 portfolios sorted by 5 size and book-to-market ratio groups.

have highly persistent, time-varying, and stochastic volatilities that can substantially harm the reliability of estimation and testing of asset pricing models, especially when a higher sampling frequency is chosen. We overcome this difficulty by using samples collected at random intervals instead of those sampled at calendar time. Specifically, the clock is running inversely proportional to the market volatility. That is, a time interval is short when volatilities are high and vice versa. Under our random sampling scheme, Fama–French regressions may simply be regarded as the classical regressions having normal errors with variance given by the averaged quadratic variation of the martingale differential errors. Our method is quite simple: We run the usual OLS regressions on the time-changed data so that potential complexities from handling high-frequency data and nonstationary volatilities do not arise.

We apply our methods to various portfolios sorted by certain characteristics used to identify pricing factors. We find that the tests based on conventional regression models on fixed time intervals often produce invalid and contradicting results. These issues do not prevail in the random time regressions. Even in comparison with the conventional regressions on lower frequency data, our tests appear to yield more reliable results. Our additional empirical findings can be highlighted as follows. First, size premium is still an important part of cross sectional return variations. According to the fixed sampling scheme, size strategy produces around 0.6% annually, while our random sampling regression states around 6.1% per annum. In addition, even after including the size factor, the size-based portfolios are not fully explained. This problem is less severe for the value-based portfolios. Second, we also find that the three-factor Fama–French models cannot fully account for the size and value premia, and the rejection of the three-factor model appears to mainly come from the small firms with low book-to-market ratios in the case of the 25 portfolios sorted by the size and book-to-market ratios. Of course, this is well documented in Fama and French (1993, 1996). However, we point out that although Fama and French argue that their model still explains cross sectional variations very well despite this puzzling behavior, this effect not only survives over time, but appears to get even stronger according to our empirical results. Third, our CAPM and multifactor results on industry portfolios suggest some potential role to be played by an additional factor based on consumer goods industry sector. It is noteworthy that this anomaly does not prevail in the fixed sampling case.

In an attempt to find a better factor pricing model, we form a consumer industry factor using the returns from the consumer goods industry sector and test the model on the portfolios we consider. Interestingly, we find that this consumer factor has some explanatory power on the returns of the small growth stocks. This suggests that factors motivated by economic theories can shed light on the issue of explaining the cross sections of stock returns, because these factors are likely to be robust to alternative sets of portfolios to be explained. Related, a recent work by Fama and French (2008) shows that there are many other asset pricing anomalies related to net stock issues, accruals, asset growth, and profitability. Some of them are even robust across all size groups, and the conventional Fama–French model is not able to deliver satisfying performance. In this vein, a quest for valid pricing factors, and thereby a new and better asset pricing model, is still an important task to sharpen our understanding of how financial markets reward

taking systematic risks and uncertainties. We hope that our newly developed tool is a useful addition to this enterprise.

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