On the Inconsistency of Cluster-Robust Inference and How Subsampling Can Fix It

Harold D. Chiang† Yuya Sasaki‡ Yulong Wang§

Abstract

Conventional methods of cluster-robust inference are inconsistent in the presence of unignorably large clusters. We formalize this claim by establishing a necessary and sufficient condition for the consistency of the conventional methods. We find that this condition for the consistency is rejected for a majority of empirical research papers. In this light, we propose a novel score subsampling method which is robust even under the condition that fails the conventional method. Simulation studies support these claims. With real data used by an empirical paper, we showcase that the conventional methods conclude significance while our proposed method concludes insignificance. (98 words)

Keywords: cluster-robust inference, score subsampling, unignorably large cluster

JEL Code: C12, C18, C46
1 Introduction

Cluster-robust (CR) standard errors account for within-cluster correlations. Such correlations often arise by construction within an industry (Hersch, 1998) or within a state (Bertrand, Duflo, and Mullainathan, 2004), to list a couple of the earliest examples. Today, even if a model may not induce cluster dependence by construction, applying CR methods by observable group identifiers is quite common in practice.

The initial theory (White, 1984; Liang and Zeger, 1986; Arellano, 1987) for CR inference methods assumes small cluster sizes $N_g$ (uniformly bounded above by $\bar{N} < \infty$) under a large number $G \to \infty$ of clusters, where $N_g$ denotes the number of entities in the $g$-th cluster for $g \in \{1, 2, \ldots, G\}$. The procedure based on this theory is implemented by the ‘cluster()’ and ‘vce(cluster)’ options by Stata and is used by almost all, if not all, empirical papers that report CR standard errors.

It has been known that a large cluster size $N_g$ could lead to a large CR standard error (e.g., Cameron and Miller, 2015, p. 324). More recent theory (Djogbenou, MacKinnon, and Nielsen, 2019; Hansen and Lee, 2019; Hansen, 2022b) accommodates larger cluster sizes $N_g$. They no longer require $N_g \leq \bar{N}$ and thus widen the scope of applications in which the ‘cluster()’ and ‘vce(cluster)’ options, among others, work. With this said, they still impose the restriction $\max_g N_g^2/N \to 0$ of vanishing maximum cluster size relative to the whole sample size $N = \sum_{g=1}^{G} N_g$ as $G \to \infty$.

A natural question is whether this relaxed condition $\max_g N_g^2/N \to 0$ accommodates a wide range of data sets. To answer it, we analyze 31 published articles. They all use the aforementioned Stata options for CR standard errors, and hence they implicitly assume $\max_g N_g^2/N \to 0$. Table 1 summarizes the number of articles with $\max_g N_g^2/N$ falling in each of the bins in the logarithmic scale. Observe that 55 percent (respectively, 39, 29 and 16 percents) of them use data sets entailing $\max_g N_g^2/N \geq 1$ (respectively, $\geq 10$, $\geq 100$ and $\geq 1000$). In other words, the condition $\max_g N_g^2/N \to 0$ for the validity of the conventional

\footnote{We studied all the articles published in American Economic Review and Econometrica between 2020 and 2021. Among them, we extracted a list of those papers that report estimation and inference results based on regressions, IV regressions, and their variants. Furthermore, we focus on those articles that use publicly available data sets for replication. See Section 3 for further details of this study.}
The Distribution of $\max_g N_g^2/N$ in Empirical Economic Research: 2020–2021

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<th>0.1– 1</th>
<th>1– 10</th>
<th>10– 100</th>
<th>100– 1000</th>
<th>≥1000</th>
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<td>1</td>
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<td><strong>Econometrica</strong></td>
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<td>2</td>
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</tr>
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<td><strong>Total</strong></td>
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<td>5</td>
</tr>
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<td>(19%)</td>
<td>(26%)</td>
<td>(15%)</td>
<td>(10%)</td>
<td>(13%)</td>
<td>(16%)</td>
</tr>
</tbody>
</table>

Table 1: Number of articles with $\max_g N_g^2/N$ falling in each of the bins $[0, 0.1)$, $[0.1, 1)$, $[1, 10)$, $[10, 100)$, $[100, 1000)$ and $[1000, \infty)$ in the logarithmic scale. The articles are drawn from those papers published in *American Economic Review* and *Econometrica* during the period of 2020-2021. We focus on those papers that report CR standard errors for regression and IV regression estimates with publicly available data sets for replication. For each paper running more than one regression, we take the largest $\max_g N_g^2/N$ among the multiple regressions.

CR inference may well fail for a nontrivial portion of the list of these published articles.

In this paper, we first establish a necessary and sufficient condition under which the conventional CR methods of inference are valid. Specifically, the limiting distribution is shown to be normal if and only if the score of the largest cluster is ignorable. In the presence of *unignorably large* clusters, regression estimates have non-normal limiting distributions such as those illustrated in Figure 1. We then conduct formal statistical tests based on Sasaki and Wang (2023) to conclude that the limiting distribution is non-normal for 24 out of those 31 papers (i.e., 77 percent) reported in Table 1.

Non-normal limiting distributions imply that the conventional critical values, such as 1.96, and bootstrap critical values are invalid. For instance, erroneously using the critical value of 1.96 would result in the sizes of 0.053, 0.087, and 0.250 (as opposed to the desired size of 0.050) when the limiting distribution takes $\alpha = 1.75, 1.50, \text{ and } 1.25$, respectively, in the left panel of Figure 1. The empirical bootstrap also fails in these scenarios where the score has an infinite variance. Furthermore, we show that the wild cluster bootstrap is inconsistent in these scenarios. In light of this negative discovery, one may ask how to fix the problem. We propose a novel score subsampling for clustered data, and formally show that it is consistent."

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2 We provide details of the non-normal distributions illustrated in this figure in Section 4.2 after presenting our formal theory.
yields correct critical values regardless of whether the limiting distribution is normal or not. In particular, this new method robustly delivers valid inference under any of the limiting distributions illustrated in Figure 1. Simulation studies support this theoretical claim.

**Related Literature:** The literature of cluster robust inference has a long history dating back to White (1984), Liang and Zeger (1986), and Arellano (1987). For a thorough review of the literature, we refer the readers to Cameron and Miller (2015) and MacKinnon, Nielsen, and Webb (2023). The sampling frameworks in which cluster sizes are treated as a random variable have been recently investigated by Bugni, Canay, Shaikh, and Tabord-Meehan (2022), Cavaliere, Mikosch, Rahbek, and Vilandt (2024), and Bai, Liu, Shaikh, and Tabord-Meehan (2022). We consider a model-based perspective with an increasing number of clusters and unrestricted intra-cluster dependence, as the vast majority of the papers did in this literature. An alternative framework is a fixed number of clusters with growing cluster sizes and manage to derive asymptotic normality under some extra assumptions on weak intra-cluster dependence following Canay, Santos, and Shaikh (2021), as well as design-based asymptotics\(^3\) under some stronger treatment assignment rules, such as randomized experiments, considered by Abadie, Athey, Imbens, and Wooldridge (2023).

In an insightful recent work, Kojevnikov and Song (2023) obtain an impossibility result

\(^3\)See Reichardt and Gollob (1999) for an in-depth philosophical discussion on the model-based versus design-based perspectives.
on consistent estimation for asymptotic variance when there is only a single large cluster in the sample. They also provide a necessary and sufficient condition for the cluster structure that the asymptotic variance is consistently estimable. Our findings complement their impossibility result of normal approximation for t-statistics in the presence of unignorably large clusters. Our proposed procedure overcomes such an impossibility as it does not require consistent estimation of the variance. Indeed, we show that the variance estimator, after normalization by an unknown rate, is convergent in distribution and formally derive its limiting stable distribution in such scenarios. In addition, such unknown rate is not necessary for the implementation of the proposed score subsampling inference procedure, due to the self-normalizing nature of the test statistics.


2 The Model

While the idea extends to a general class of econometric models, we consider the linear model

\[ Y_{gi} = X'_{gi} \theta + U_{gi} \quad \mathbb{E}[U_g|X_g] = 0 \]

for ease of exposition, where \( X_g = (X_{g1}, \ldots, X_{gN_g})' \), \( U_g = (U_{g1}, \ldots, U_{gN_g})' \), \( g \in \{1, \ldots, G\} \) indexes clusters, and \( N_g \) denotes the size of the \( g \)-th cluster. Define the OLS estimator and
its cluster-robust (CR) variance estimator by

\[
\hat{\theta} = \left( \sum_{g=1}^{G} \sum_{i=1}^{N_g} X_{gi}X_{gi} \right)^{-1} \sum_{g=1}^{G} \sum_{i=1}^{N_g} X_{gi}Y_{gi} = \left( \sum_{g=1}^{G} X_{g}'X_{g} \right)^{-1} \sum_{g=1}^{G} \left( X_{g}'X_{g}\hat{\theta} + S_g \right) \quad \text{and}
\]

\[
\hat{V}^{CR} = a_G \left( \sum_{g=1}^{G} X_{g}'X_{g} \right)^{-1} \left( \sum_{g=1}^{G} \hat{S}_g\hat{S}_g' \right) \left( \sum_{g=1}^{G} X_{g}'X_{g} \right)^{-1}
\]

respectively, for some finite sample adjustment factor \( a_G \) such that \( a_G \to 1 \) as \( G \to \infty \), where \( S_g = \sum_{i=1}^{N_g} X_{gi}U_{gi} \), \( \hat{S}_g = \sum_{i=1}^{N_g} X_{gi}\hat{U}_{gi} \), and \( \hat{U}_{gi} = Y_{gi} - X_{gi}'\hat{\theta} \). For simplicity of writing, we set \( a_G = 1 \) throughout.

Consider a linear transformation \( \delta = r'\theta \), such that \( r \in \mathbb{R}^{\text{dim}(\theta)} \) and \( ||r|| = 1 \), as the parameter of interest. Let the corresponding estimator and its CR standard error be denoted by

\[
\hat{\delta} = r'\hat{\theta} \quad \text{and}
\]

\[
\hat{\sigma}^2 = r' \left( \sum_{g=1}^{G} X_{g}'X_{g} \right)^{-1} \left( \sum_{g=1}^{G} \hat{S}_g\hat{S}_g' \right) \left( \sum_{g=1}^{G} X_{g}'X_{g} \right)^{-1} r,
\]

respectively. We are interested in conducting inference for \( \delta \) using the t-statistic

\[
\frac{(\hat{\delta} - \delta)}{\hat{\sigma}} = \frac{r'\hat{\theta} - \theta}{\sqrt{r' \left( \sum_{g=1}^{G} X_{g}'X_{g} \right)^{-1} \left( \sum_{g=1}^{G} \hat{S}_g\hat{S}_g' \right) \left( \sum_{g=1}^{G} X_{g}'X_{g} \right)^{-1} r}}
\]

based on the CR standard error.

To state our model assumption, we introduce a few definitions. A random variable \( \eta \) is said to be stable if it has a domain of attraction in that there exists a sequence of i.i.d. random variables \( \xi_1, \xi_2, \ldots \) and sequences of positive numbers \( A_G \) and real numbers \( D_G \) such that

\[
\frac{\sum_{g=1}^{G} \xi_g}{A_G} \overset{d}{\to} \eta \quad \text{as} \quad G \to \infty.
\]

A function \( L(\cdot) \) is said to be slowly varying at \( \infty \) if \( \lim_{t \to \infty} \frac{L(yt)}{L(t)} = 1 \) for all \( y > 0 \).
If $\eta$ is stable, then $A_G$ takes the form of $G^{1/\alpha}L(G)$ for some $\alpha \in (0, 2]$ and some slowly varying function $L(\cdot)$ at $\infty$ (cf. Proposition 2.2.13 in Embrechts, Klüppelberg, and Mikosch 1997). If $\alpha \in (1, 2]$, then $D_G$ can be chosen to be $G \cdot \mathbb{E}[\xi_g]$. The number $\alpha$ is called the index of stability, and $\eta$ is said to be $\alpha$-stable. In such a case, $\xi_g$ is said to belong to the domain of attraction of an $\alpha$-stable distribution. Although this concept may look esoteric to some readers, it essentially states that a sum of i.i.d. random variables, after being suitably centered and normalized, converges in distribution to a limiting random variable, and it, in particular, encompasses the standard cases where central limit theorems (CLTs) hold.

We shall focus on $\alpha \in (1, 2]$, since even the first moment of $\xi_g$ would not be well-defined otherwise.

**Assumption 1.** $(X_g'X_g, S_g)_g^{G}$ are i.i.d., $\mathbb{E}[N_g] = c \in (0, \infty)$, and the design matrix satisfies

$$
\left( \frac{1}{G} \sum_{g=1}^{G} X_g'X_g \right)^{-1} = Q^{-1} + o_p(1)
$$

for a finite and positive definite matrix $Q$. For $v = r'Q^{-1}$ and for all $u_1, u_2 \in \mathbb{R}^{\dim(\theta)}$ with unit length, $v'S_g$ and $u'_1X'_gX_gu_2$ belong to the domain of attraction of stable laws with an index of stability $\alpha \in (1, 2]$.

Assumption 1 accommodates a broad class of both standard and non-standard cases considered in econometrics. First, the case of $\alpha = 2$ encompasses the conventional assumption under which $r'(\hat{\theta} - \theta)$ enjoys the standard convergence rate of $\sqrt{G}$ through CLTs. In this case, the limiting $\alpha$-stable distribution must be normal (cf. Geluk and de Haan 2000, Theorem 2). Furthermore, it also covers some non-standard cases with a normal limiting distribution but without a finite variance, e.g., a Pareto random variable with the shape parameter (Pareto exponent) of 2.

Second, on the other hand, the case of $\alpha < 2$ entails the power law (de la Peña, Lai, and Shao 2009, Theorem 2.24), i.e.,

$$
\mathbb{P}(|v'S_g| > t) = t^{-\alpha}L_1(t) \quad \text{and} \quad \mathbb{P}(|u_1X'_gX_gu_2| > t) = t^{-\alpha}L_2(t)
$$

(2.2)
for some slowly varying functions, \(L_1(\cdot)\) and \(L_2(\cdot)\), where \(L_2(\cdot)\) may depend on \(u_1\) and \(u_2\).

In this case of \(\alpha < 2\), the index \(\alpha\) of stability coincides with the tail exponent \(\alpha_T\) in the sense that \(\alpha = \min\{\alpha_T, 2\}\). Thus, the case of \(\alpha < 2\) implies infinite variance of the score. See Theorem 3 in Appendix A.1 for more precise details. In this case, unignorably large clusters are literally unignorable because the sample sum of the (scaled) scores becomes asymptotically proportional to the (scaled) score of the largest cluster - see Remark 4 in Appendix A.2 for more discussions. Hence, the asymptotic distribution cannot be normal.

To simplify the writings, we focus on the case where \(v'S_g\) and \(u_1'X_g'X_gu_2\) share the common index \(\alpha\) of stability. This simple setting is rationalized if the tail shape of their distributions are driven by the tail shape of the distribution of cluster sizes \(N_g\) – see Sasaki and Wang (2022, Theorem 1). With this said, we emphasize that this setting can be relaxed only at the cost of more cumbersome writing.

The i.i.d. requirement in Assumption 1 is standard in this literature, (cf. Bugni et al. 2022; Cavaliere et al. 2024; Bai et al. 2022). It is mild because 1) the conditional distributions of \(S_g\) and \(X'_gX_g\) given \(N_g = n_g\) can be heterogeneous across \(n_g\); and 2) the distributions of individuals within each cluster can be non-identical. In addition, \(S_g\) and \(X_g\) can be arbitrarily correlated with the cluster size \(N_g\).

3  Non-Robustness of the Conventional CR Methods

In this section, we argue that the conventional methods of CR inference work if and only if \(\alpha = 2\). In other words, they are doomed to fail if \(\alpha < 2\). We then discuss how often researchers encounter cases with \(\alpha < 2\) in empirical economic studies.

**Theorem 1** (Necessary and sufficient condition). *Suppose that Assumption 1 is satisfied for an \(\alpha \in (1, 2]\), then the t-statistic (2.1) is asymptotically Gaussian if and only if \(\alpha = 2\).*

A proof is found in Appendix B.3. This theorem implies that the conventional inference based on the common variance estimators, such as CR1, CR2, CR3, and jackknife, together with the Gaussian critical values (e.g., \(\approx 1.96\) for the 97.5-th percentile) fails if \(\alpha < 2\).
There is certainly another class of conventional approaches, namely the bootstrap. However, it is well known that the empirical bootstrap is inconsistent when the variance of the score is infinite (cf. Athreya [1987], Knight [1989]). In light of the power law characterization (2.2), therefore, the empirical cluster bootstrap is inconsistent under Assumption 1 with $\alpha < 2$. Furthermore, we formally show in Appendix A.3 that the wild cluster bootstrap is also inconsistent under Assumption 1 with $\alpha < 2$.

Provided that the case of $\alpha < 2$ fails all these conventional methods of CR inference, our natural question now is how common it is to encounter $\alpha < 2$ in empirical studies in economics. We analyzed all the articles published in a couple of journals (American Economic Review and Econometrica) between 2020 and 2021. Among them, we extracted a list of those papers that report estimation and inference results based on regressions, IV regressions, and their variants. Furthermore, we focus on those articles that use publicly available data sets for replication.

For these articles, we test the null hypothesis $H_0 : \alpha = 2$ against the alternative hypothesis $H_1 : \alpha < 2$ for the score. Such a test can be conducted via the likelihood ratio test (Sasaki and Wang, 2023) of the surrogate null hypothesis $H_0 : \alpha_T \geq 2$ against the alternative $H_1 : \alpha_T < 2$ in light of (2.2), where $\alpha_T$ denotes the tail exponent of the score.

Table 2 summarizes the list of all the papers we studied. The first two columns list the journals and years of publication. The following column “All #” indicates the total number of eligible articles according to the above selection criteria. The column group under “Cluster” collects articles in which CR inference is used for at least one regression result. Under this column group, the column “#” shows the numbers of articles, and the column “Test $\alpha < 2$” shows the fractions of those articles for which the test rejects the null hypothesis for at least one regression specification. The final row displays the summary of each column.

During 2020–2021, American Economic Review published 30 articles meeting our selection criteria. Out of them, 21 articles report CR standard errors. We reject the null hypothesis for 16 of these 21 articles. In other words, the inference results may be mislead-

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4 The test of $H_0 : \alpha_T \geq 2$ against $H_1 : \alpha_T < 2$ is implemented with the Stata command “testout y x1 x2 ..., cluster(cid)” for regressions and “testout y x1 x2 ..., iv(z) cluster(cid)” for IV regressions based on Sasaki and Wang (2023).
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<th>Year of Publication</th>
<th>All #</th>
<th>Cluster #</th>
<th>Test $\alpha &lt; 2$</th>
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<td><strong>American Economic Review</strong></td>
<td>2020</td>
<td>15</td>
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<td>7/10 (70%)</td>
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<tr>
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<tr>
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</tr>
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<td>7/8 (88%)</td>
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<tr>
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<td>2</td>
<td>1/2 (50%)</td>
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<tr>
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<td><strong>31</strong></td>
<td><strong>24/31 (77%)</strong></td>
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</tbody>
</table>

Table 2: The column “All – #” indicates the total number of eligible articles that use regressions or IV regressions with publicly available data for replication. The column “Cluster – #” indicates the number of the eligible articles that use CR inference. The column “Cluster – Test $\alpha < 2$” indicates the rate of rejecting the null hypothesis $\alpha = 2$ among those articles that use CR inference. The tests of the null hypothesis $\alpha = 2$ against the alternative hypothesis $\alpha < 2$ is implemented with the Stata command \texttt{testout y x1 x2 ...}, \texttt{cluster(cid)} for regressions and \texttt{testout y x1 x2 ...}, \texttt{iv(z) cluster(cid)}” for IV regressions based on Sasaki and Wang (2023).

During 2020-2021, \textit{Econometrica} published 14 articles meeting our selection criteria. Out of them, 10 articles report CR standard errors. We reject the null hypothesis for 8 of these 9 articles. In other words, the inference results may be misleading for 80% of those articles that employ the conventional CR method of inference.

Combining the two journals together, we suspect potentially misleading inference results for as many as 77% of those articles that employ the conventional CR method of inference. Hence, problematic practice is prevalent even in these highly influential journals\footnote{Spreadsheets of all the test results with specific papers and specific equations are available upon request from the authors under certain conditions.}.

## 4 Score Subsampling as a Robust Solution

In light of the issue with the conventional methods of CR inference reported in the previous section, we now propose a novel method of score subsampling as a robust solution. We
first present the proposed method without theoretical details in Section 4.1. Its theoretical support follows in Section 4.2. Finally, Section A.4 presents practical details.

4.1 The Method

Our objective is to conduct statistical inference for $\delta$ using the self-normalized t-statistic (2.1). Let the CDF $J_G^\circ$ of the sampling distribution of the t-statistic be given by

$$J_G^\circ(t) = \mathbb{P}\left( \frac{\hat{\delta} - \delta}{\hat{\sigma}} \leq t \right).$$

We will show that it converges to the CDF $J^*$ of a limiting distribution under suitable conditions. Consider a sequence of subsample sizes $b = b_G$ that grows with $b/G = o(1)$ as $G \to \infty$. Let $B_G = \binom{G}{b}$ denote the total possible number of subsamples of $b$ clusters. For a given $b$ and $j \in \{1, ..., B_G\}$, let $S_j \subset \{1, ..., G\}$ be one of the $B_G$ subsamples of the cluster indices with $|S_j| = b$, and define the score-subsampled estimators

$$\hat{\delta}_{b,j} = r^T \hat{\theta}_{b,j} = \left( \frac{G}{b} \right)^r \left( \sum_{g=1}^{G} X'_g X_g \right)^{-1} \sum_{g \in S_j} X'_g Y_g$$

and

$$\hat{\sigma}^2_{b,j} = \left( \frac{G}{b} \right)^{2r} \left( \sum_{g=1}^{G} X'_g X_g \right)^{-1} \left( \sum_{g \in S_j} \hat{S}_{g,j} \hat{S}_{g,j}' \right) \left( \sum_{g=1}^{G} X'_g X_g \right)^{-1} r,$$

where $\hat{S}_{g,j} = X'_g (Y_g - X_g \hat{\theta}_{b,j})$. Observe that the inverse factor $(\sum_{g=1}^{G} X'_g X_g)^{-1}$ is calculated based on the full sample while the linear component and its variance are computed based on the subsample $S_j$. We discuss practical motivations for this feature in Remark 2 below.

Define the empirical CDF $L^*_{G,b}$ of $(\hat{\delta}_{b,j} - \hat{\delta})/\hat{\sigma}_{b,j}$ based on all possible $B_G$-subsamples by

$$L^*_{G,b}(t) = \frac{1}{B_G} \sum_{j=1}^{B_G} 1 \left( \frac{\hat{\delta}_{b,j} - \hat{\delta}}{\hat{\sigma}_{b,j}} \leq t \right).$$

It will be shown that $J^*$ can be approximated by $L^*_{G,b}$ uniformly as the number $G$ of clusters grows under suitable conditions. In practice, however, $L^*_{G,b}$ is computationally infeasible when $G$ and $b$ are both large. Thus, we randomly draw $M$ such subsamples of $b$ clusters
with replacement, and define

\[
\hat{L}_{G,b}(t) = \frac{1}{M} \sum_{j=1}^{M} \mathbf{1}_{\left(\frac{(\hat{\delta}_{b,j} - \hat{\delta})}{\hat{\sigma}_{b,j}} \leq t\right)}.
\]

As \(M\) grows with the number \(G\) of clusters, this \(\hat{L}_{G,b}\) can be used in place of \(L^*_G\).

For any \(a \in (0, 1)\), define the critical value

\[
\hat{c}_{G,b}(1 - a) = \inf \left\{ t \in \mathbb{R} : \hat{L}_{G,b}(t) \geq 1 - a \right\}.
\]

Since \(J^*(\cdot)\) has no point mass as we shall show, it follows that

\[
\mathbb{P} \left( \frac{(\hat{\delta} - \delta)}{\hat{\sigma}} \leq \hat{c}_{G,b}(1 - a) \right) \to 1 - a
\]
as \(G \to \infty\). Therefore, this critical value leads to theoretically valid tests. In addition, a confidence region can be obtained by test-inversion.

**Practical Implication:** For the estimate \(\hat{\delta}\), one can continue to use the conventional CR “standard error” \(\hat{\sigma}\). However, instead of using the conventional critical values, \(\Phi^{-1}(0.025) \approx -1.96\) and \(\Phi^{-1}(0.975) \approx 1.96\), one should use \(\hat{c}_{G,b}(0.025)\) and \(\hat{c}_{G,b}(0.975)\) obtained by the score subsampling to construct a 95% confidence interval for example.

**Remark 1** (Practicality of the method). The convergence rate of \(\hat{\delta} - \delta\) is unknown, but the inference is robust to the unknown rate due to the use of the self-normalized statistic. In particular, this implies that our inference procedure does not require an estimation of the unknown tail exponent. Furthermore, it is not necessary to estimate the unknown slowly varying function either. These features are practical advantages of our proposed method. ▲

**Remark 2** (Finite sample non-invertibility of other cluster-based resampling methods). In comparison with the (conventional) subsampling, the score subsampling has two major advantages. First, as it does not require to recompute the inverse factor for each subsample, the score subsampling is computationally more efficient than the subsampling. Second, in finite samples, when regressors contain a cluster-specific binary treatment variable or other
dummies variables that are highly correlated within a cluster, $\sum_{g \in S_j} X'_g X_g$ can be often singular especially for small $b = |S_j|$, and thus the subsampled OLS may not behave well for a non-negligible proportion of subsamples. This issue is also faced by other cluster-based resampling methods, such as the jackknife and bootstrap. In practice, several ad hoc ‘fixes,’ such as the use of generalized inverse or dropping such realizations, are employed. However, their theoretical implications remain unclear. Our cluster-robust score subsampling procedure avoids such an issue in a theoretically supported manner. ▲

4.2 Theory

The following theorem formally justifies that the subsampling method of inference presented in Section 4.1 is asymptotically valid even if $\alpha < 2$, as well as when $\alpha = 2$.

**Theorem 2** (Cluster robust inference by score subsampling). *Suppose that Assumption \(\text{I}^{23}\) is satisfied for $\alpha \in (1, 2]$. If $b \to \infty$ and $b/G = o(1)$ as $G \to \infty$, then*

$$
\sup_{t \in \mathbb{R}} |L^*_{G,b}(t) - J^*(t)| \xrightarrow{p} 0
$$

*and the limiting distribution $J^*(\cdot)$ is continuous. In addition, if $M \to \infty$, then*

$$
\sup_{t \in \mathbb{R}} |\hat{L}_{G,b}(t) - J^*(t)| \xrightarrow{p} 0,
$$

*and thus*

$$
\mathbb{P}\left(\frac{(\hat{\delta} - \delta)}{\hat{\sigma}} \leq \tilde{c}_{G,b}(1 - a)\right) \to 1 - a.
$$

The proof branches into two cases. First, we focus on the pathological case with $\alpha < 2$. The statement for this case is presented as Lemma \(^{1}\) in Appendix A.2, which is proved in Appendix B.1. Appendix B.2 proves the statement for the case with $\alpha = 2$, and combines the two cases to establish Theorem 2.

The limiting distribution, which is approximated by our proposed method of score subsampling, is not unique. It varies with the values of two parameters. One is the index $\alpha$ of
stability, and the other is

\[ p = \lim_{t \to \infty} \frac{\mathbb{P}(v'S_g > t)}{\mathbb{P}(|v'S_g| > t)}, \]

where \( v \) is defined in Assumption 1. See the proof of Lemma 1 in Appendix B.1 for details. This second parameter \( p \) measures the tail asymmetry of the distribution of \( v'S_g \). One need not know the true limiting distribution in practice as the score subsampling can be used to approximate it. With that said, we can now discuss Figure 1 shown in the introductory section. Namely, the left, middle, and right panels of Figure 1 illustrate the limiting distributions under \( p = 0.25, 0.50, \) and \( 0.75, \) respectively. In each of these three panels, three non-normal limiting distributions under \( \alpha = 1.25, 1.50, \) and \( 1.75, \) are drawn with distinct line styles, along with the normal reference \( (\alpha = 2.00) \). The main takeaway is that the conventional CR inference based on the normal approximation becomes less precise as \( \alpha \) becomes smaller and \( p \) deviates from 0.5.

**Choice of \( b \):** We close this section with discussions on the choice of \( b \) in practice. While the theory requires \( b \to \infty \) and \( b/G = o(1) \) as \( G \to \infty \), a researcher needs to choose some value of \( b \) in a finite sample. We suggest to adapt the minimum volatility method (Algorithm 9.3.3 in Politis et al. 1999, Section 9.3.2) to our framework. Appendix A.4 provides a detailed algorithmic procedure that a practitioner can readily implement. We also employ this method to choose \( b \) in the numerical studies presented below.

## 5 Simulations

In this section, we present simulation studies to evaluate the finite sample performance of our proposed score subsampling method of CR inference in comparison with the conventional CR methods. The data-generating design is defined as follows. We consider the cluster treatment model with individual covariates

\[ Y_{gi} = \theta_0 + \theta_1 T_g + \sum_{j=1}^{K} \theta_j X_{g, i, j+1} + U_{gi} \]
following MacKinnon, Nielsen, and Webb (2022, Equation (40)) among others. The binary treatment variable $T_g$ takes the value of one for $[0.2G]$ clusters and zero for the remaining clusters $G-[0.2G]$, where $[a]$ denotes the smallest integer greater than or equal to $a$. We draw cluster sizes $N_g \sim [\text{Pareto}(1, \alpha)]$ independently for $g \in \{1, \cdots, G\}$. For each $g \in \{1, \cdots, G\}$, we independently draw $N_g$-variate random vectors, $(\tilde{X}_{g1j}, \cdots, \tilde{X}_{gN_g})' \sim \mathcal{N}(0, \Omega)$ for $j \in \{1, \cdots, K\}$ and $(\tilde{U}_{g1}, \cdots, \tilde{U}_{gN_g})' \sim \mathcal{N}(0, \Omega)$ in the baseline design, where $\Omega$ is an $N_g \times N_g$ variance-covariance matrix such that $\Omega_{ii} = 1$ for all $i \in \{1, \cdots, N_g\}$ and $\Omega_{ii'} = 1/2$ whenever $i \neq i'$. The controls are constructed by $X_{gij} = 0.2F_{\text{Beta}(2,2)}^{-1}(\Phi(\tilde{X}_{gij}))$, where $F_{\text{Beta}(2,2)}$ and $\Phi$ denote the CDFs of the Beta(2, 2) and standard normal distributions, respectively. The errors are heteroskedastically constructed by $U_{gi} = 0.2\tilde{U}_{gi}$ if $T_g = 0$ and $U_{gi} = \tilde{U}_{gi}$ if $T_g = 1$.

We vary values of the exponent parameter $\alpha \in \{1.1, 1.2, \cdots, 1.9, 2.0\}$ across sets of simulations. The regression coefficients are fixed to $(\theta_0, \theta_1, \theta_2, \cdots, \theta_{K+1})' = (1, 1, 1, \cdots, 1)'$ throughout, whereas the dimension $K$ of covariates vary as $K \in \{0, 5, 10\}$. We set the sample size (i.e., the number of clusters) to $G = 50$ across sets of simulations, which is close to the number of states in the U.S. we discussed as an example earlier. Each set of simulations consists of 5,000 Monte Carlo iterations.

Figure 2 illustrates the Monte Carlo coverage frequencies. The horizontal axis measures the value of $\alpha$, and the vertical axis measures the coverage frequency. In the legend, ‘SUB’ (respectively, ‘WCB’, ‘JACK’ and ‘CR1’) indicates the score subsampling (respectively, wild cluster bootstrap, jackknife standard error with normal critical value, and CR1 standard error with normal critical value). The nominal coverage probability of 95% is indicated by the horizontal gray line at 0.95.

When $\alpha$ is small, say $\alpha < 1.6$, the score subsampling performs the best, followed by the jackknife, the WCB, and the CR1. When $\alpha$ is larger, say $\alpha > 1.6$, the score subsampling still performs the best, followed by the WCB, the jackknife, and the CR1. Overall, the score subsampling robustly yields the coverage frequencies closest to the nominal probability of 95% across various values of $\alpha$. All the conventional methods suffer from sever under-coverage especially for small values of $\alpha$. 

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6 An Empirical Illustration

Akhtari, Moreira, and Trucco (2022) study the effects of political turnover on a range of outcomes measuring the quality of public services in Brazil. In the original paper (Table 3, Column 5), they estimate the following linear model by the OLS:

\[
score_{gi+1} = \theta_0 + \theta_1 \mathbb{1}\{IVM_g < 0\} + \theta_2 IVM_g + \theta_3 \mathbb{1}\{IVM_g < 0\} IVM_g + \theta_4 score_{gi} + U_{gi}.
\]

The dependent variable \(score_{gi+1}\) is the test score of fourth-grade students in the next year after an election. The main explanatory variable \(IVM_g\) is the incumbent vote margin. Thus, \(\mathbb{1}\{IVM_g < 0\}\) equals 1 when the incumbent party loses the election. The parameter of interest is \(\delta = \theta_1\), which measures the effects of political turnover on the test score\(^7\).

\(^7\)Effectively, it implements a sharp regression discontinuity design, but the original paper estimates the...
Table 3: p-values for the effect $\delta = \theta_1$ of political turnover on the test score of fourth-grade students based on the conventional methods of inference (HC1, CR1, JACK, WCB) and our proposed method of score subsampling (SUB).

<table>
<thead>
<tr>
<th></th>
<th>HC1</th>
<th>CR1</th>
<th>JACK</th>
<th>WCB</th>
<th>SUB</th>
</tr>
</thead>
<tbody>
<tr>
<td>p-value</td>
<td>0.000</td>
<td>0.006</td>
<td>0.007</td>
<td>0.006</td>
<td>0.129</td>
</tr>
</tbody>
</table>

The original paper considers alternative bandwidths, we focus on the bandwidth 0.110 for the maximum sample size following the prior work (MacKinnon et al., 2022, Section 7.2) which replicates the above regression.

The original paper clusters the standard errors at the municipality level, and we follow this definition of a cluster unit. There are $G = 2101$ municipalities in the data, and $\max_{1 \leq g \leq G} N_g^2 / N \approx 26$. Thus, the assumption $\max_{1 \leq g \leq G} N_g^2 / N \to 0$ under which the conventional methods of CR inference are guaranteed work is hard to justify for this application.

Table 3 reports the p-values for $\delta = \theta_1$ based on the alternative inference method. Column HC1 reports the p-value with the conventional method inference without clustering, i.e., the HC1 standard error. Columns CR1, JACK, and WCB report the p-values with the conventional methods of CR inference, namely the CR1 standard error, the jackknife standard error, and the wild cluster bootstrap, respectively, along with the normal approximation. Finally, column SUB reports the p-value based on our proposed inference method with the score subsampling. We use the same code used for simulation studies in Section 5, including the choice of $b$ based on the minimum volatility method.

Observe that the p-value is zero up to the third digit when standard errors are not clustered (HC1). Conventional methods of CR inference (CR1, JACK, and WCB) with the normal approximation certainly yield larger p-values, but do not change the statistical significance. Now, observe that our proposed method of inference with the score subsampling yields a much larger p-value, and entails statistical insignificance of the effect $\delta = \theta_1$ of interest, unlike any of the conventional methods. These results showcase that not correctly accounting for the potentially non-normality in the limiting distributions in the presence of effect by the OLS for this linear estimating equation.

While there are four alternative methods of WCB (cf. MacKinnon et al., 2022, Section 7.2), they all produce the same p-values up to the displayed digits, and hence we summarize them in a single column.
unignorably large clusters could lead to erroneous statistical conclusions.

Appendix

A Omitted Details

This appendix section collects technical details that are omitted from the main text.

A.1 Alternative Characterization of $\xi_g$ Belonging to the Domain of Attraction of an $\alpha$-Stable Distribution for $\alpha < 2$

Citing a result from the existing literature, this section presents complete details about the power law characterization (2.2) discussed in Section 2 in the main text.

**Theorem 3** (de la Peña et al., 2009, Theorem 2.24). Suppose $\alpha < 2$. Then, $\xi_g$ belongs to the domain of attraction of an $\alpha$-stable distribution if and only if

$$
\mathbb{P}(|\xi_g| > t) = t^{-\alpha}L(t) \quad \text{and} \quad \lim_{t \to \infty} \frac{\mathbb{P}(\xi_g > t)}{\mathbb{P}(|\xi_g| > t)} = p, \quad p \in [0, 1],
$$

for some slowly varying function $L(\cdot)$.

The first condition means that the tail limit of the absolute value of the random variable of interest has an approximately Pareto tail, or so-called power law. Known as the balancing condition, the second condition in this alternative characterization imposes a mild restriction on the existence of limiting ratios of one-sided tail probabilities over the two-sided tail probability; it rules out some irregular, infinitely oscillating type situations such that these limiting ratios do not exist. This condition only imposes restrictions in the limit and accommodates a wide range of tail behaviors as $p$ are permitted to be either 0 or 1.
A.2 Auxiliary Theory Focusing on Cases with $\alpha < 2$

This section presents a lemma that we state and prove on the way to proving Theorem 2 in Section 4.2 in the main text. Namely, for ease of writing, we state our main result focusing on cases with $\alpha \in (1, 2)$. An extension of this result to the general cases with $\alpha \in (1, 2]$ follows as Theorem 2 with additionally accounting for the case with $\alpha = 2$.

**Lemma 1** (Cluster robust inference by score subsampling). *Suppose that Assumption 7 is satisfied for $\alpha \in (1, 2)$. If $b \to \infty$ and $b/G = o(1)$ as $G \to \infty$, then*

$$
\sup_{t \in \mathbb{R}} |L_{G,b}^*(t) - J^*(t)| \xrightarrow{p} 0
$$

*and the limiting distribution $J^*(\cdot)$ is continuous. In addition, if $M \to \infty$, then*

$$
\sup_{t \in \mathbb{R}} |\hat{L}_{G,b}(t) - J^*(t)| \xrightarrow{p} 0,
$$

*and thus*

$$
\mathbb{P}\left( (\hat{\delta} - \delta)/\hat{\sigma} \leq \bar{c}_{G,b}(1 - a) \right) \to 1 - a.
$$

A proof is provided in Appendix B.1.

**Remark 3** (Heavy-tailed cluster sums). In this lemma, we essentially assume that the tails of the distributions of $\|S_g\|$ and $\|X_g\| \|X_g\|$ both follow the power law with the shape parameter (Pareto exponent) in $(1, 2)$, which implies that the variances of $S_g$ and $(X'_g X_g)$ do not exist. See Appendix A.1. This is a rather general condition in the sense that the heavy tail can come from the distribution of cluster sizes $N_g$, the distribution of individuals' $(X_{gi}', U_{gi})$, or both. ▲

**Remark 4** (Unignorability and impossibility of normal approximation). An inspection of the proof of Lemma 1 combined with Remark 2 in LePage et al. (1981), unveils that, when $\alpha < 2$, the tails of the first component of representation (B.3) satisfies

$$
\mathbb{P}\left( |\epsilon_1 Z_1 - (2p - 1)\mathbb{E}[Z_1 1(Z_1 < 1)]| > t \right) \sim \mathbb{P}\left( \left| \sum_{k=1}^\infty \epsilon_k Z_k - (2p - 1)\mathbb{E}[Z_k 1(Z_k < 1)] \right| > t \right)
$$

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as $t \to \infty$. Since the term $|\epsilon_1 Z_1 - (2p - 1)\mathbb{E}[Z_1 1(Z_1 < 1)]|$ corresponds to the limiting distribution of the absolute value of the scaled score of the largest cluster, it has an asymptotically unignorable influence on the limiting $\alpha$-stable distribution – see also Section 1.4 in Samorodnitsky and Taqqu (1994). For ease of illustration, suppose that the regressor and error distributions are uniformly bounded and $\text{Cov}(X_{g_1 U_g}, X_{g_1 U_g'}, N_g) \geq \zeta > 0$ for all $i = 1, \ldots, N_g$ with probability one. This then implies

$$\frac{\max_{g=1,\ldots,G} \| S_g \|}{G} \overset{p}{\sim} \frac{\max_{g=1,\ldots,G} N_g}{G} \gg 0.$$ 

This is in contrast to the conventional assumption

$$\frac{\max_{g=1,\ldots,G} N_g^2}{G} = o_p(1),$$

which is used in the literature of CR inference based on the normal approximation.

In addition, a necessary and sufficient condition for the limiting distribution of sums of independent random variables to be normal is the uniform asymptotic negligibility condition, i.e., the largest summand in absolute value has an asymptotically negligible contribution to the sum (cf. Davidson, 1994, Theorem 23.13). Thus, it is impossible to derive a theoretically valid normal-approximation-based procedure of inference in the presence of unignorably large clusters without imposing restrictions on within-cluster dependence. ▲

**Remark 5** (On CR standard error estimation). The test statistic we consider is the standard $t$-statistic used in the literature. Its denominator consists of a CR standard error without imposing a null hypothesis. When $\alpha < 2$, the asymptotic variance does not exist, and nor is this “standard error” consistent but remains random asymptotically. This is similar in spirit to the fixed-$b$ asymptotics (e.g., Kiefer and Vogelsang, 2002) in the literature of long-run variance estimation, although the underlying theory is completely different as the fixed-$b$ asymptotics crucially relies on normal approximation and the functional central limit theorem. Showing that this “standard error” with estimated residuals has negligible impact on

---

9It is assumed in the literature of CR inference based on the normal approximation that $\frac{\max_{g=1,\ldots,G} N_g^2}{N} = o_p(1)$. When $\mathbb{E}[N_g] = c > 0$ exists, this assumption is equivalent to $\frac{\max_{g=1,\ldots,G} N_g^2}{G} = o_p(1)$. 

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the asymptotic distribution requires a completely different proof strategy from the conventional approach of those taken in the proof of Theorem 7.6 in [Hansen (2022a)] for example.

### A.3 Inconsistency of the Wild Cluster Bootstrap under $\alpha < 2$.

The wild cluster bootstrap ([Cameron, Gelbach, and Miller (2008)] is a popular alternative resampling method of CR inference. It has been shown in various simulation studies to behave well under $\alpha = 2$. Validity of the wild cluster bootstrap in cases of $\alpha = 2$ has been shown in [Djogbenou et al. (2019)] under fairly general conditions. As their proof relies crucially on Lyapunov’s CLT, however, their arguments do not hold under $\alpha < 2$ – see Remark 4. A remaining and potentially more interesting question is whether one can prove its validity using an alternative argument. The following result suggests that such efforts are ill-fated when $\alpha < 2$.

For simplicity of illustration, consider the case of a univariate regression with only the intercept, i.e. a cluster sampled mean $\hat{\theta} = N^{-1} \sum_{g=1}^{G} \sum_{i=1}^{N_g} Y_{gi}$ with the cluster specific population mean normalized to $\theta = E \left[ \sum_{i=1}^{N_g} Y_{gi} \right] = 0$ without loss of generality. Suppose that the parameter of inference is $\theta$. Under the null hypothesis $H_0 : \theta = 0$, the standard CR $t$-statistic can be formed as

$$T_G = \frac{\sum_{g=1}^{G} \sum_{i=1}^{N_g} Y_{gi}}{\sqrt{\sum_{g=1}^{G} \left( \sum_{i=1}^{N_g} (Y_{gi} - \hat{\theta}) \right)^2}}.$$

The wild-cluster-bootstrap version of the estimator is defined by $\hat{\theta}^* = N^{-1} \sum_{g=1}^{G} v_g^* \sum_{i=1}^{N_g} Y_{gi}$, where $(v_g^*)_{g=1}^{G}$ are i.i.d. Rademacher auxiliary random variables generated by a researcher independently from the observed data $\{(Y_{gi})_{i=1}^{N_g} \}_{g=1}^{G}$. The null-imposed wild cluster bootstrap test statistic is defined by

$$T_G^* = \frac{\sum_{g=1}^{G} v_g^* \sum_{i=1}^{N_g} Y_{gi}}{\sqrt{\sum_{g=1}^{G} \left( v_g^* \sum_{i=1}^{N_g} (Y_{gi} - \hat{\theta}^*) \right)^2}}.$$

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We introduce the notation $Y_{1:G} = (Y_{gi} : g = 1, ..., G, i = 1, ..., N_g)$ for convenience. As Lemma 1 implies continuity of the limiting distribution of $T_G$, the wild cluster bootstrap is consistent if

$$ \sup_{t \in \mathbb{R}} \left| \mathbb{P}(T_G^* \leq t | Y_{1:G}) - \mathbb{P}(T_G \leq t) \right| = o_p(1) \quad \text{as } G \to \infty. $$

**Theorem 4** (Inconsistency of the wild cluster bootstrap). *Under the above setup and Assumption 1 if $\alpha \in (1, 2)$, then the wild cluster bootstrap with Rademacher auxiliary random variables is inconsistent.*

A proof can be found in Appendix B.4

**A.4 Choosing the Number $b$ of Cluster Subsamples**

For the choice of $b$ in practice, we adapt the minimum volatility method (Algorithm 9.3.3 in Politis et al. [1999] Section 9.3.2) to our framework of cluster-robust inference.

For subsampling to be valid, $b$ needs to grow with the number $G$ of clusters but at a slower rate. If $b$ is too close to $G$, then all the subsampled t-statistics will be almost identical to the full-sample t-statistic, resulting in a subsampling distribution being too tight and thus in under-coverage by confidence intervals. On the other hand, if $b$ is too small, then the subsampled t-statistics will be noisy and can result in either under-coverage or over-coverage. Thus, intuitively, we wish to select a $b$ that is in a stable range for the test statistic. The following algorithm formalizes such an idea.

**Algorithm 1** (Minimum volatility method for cluster-robust inference).

1. For $b \in \{b_{\text{small}}, b_{\text{small}} + 1, ..., b_{\text{big}}\}$, compute the critical value $\hat{c}_{G,b}(1 - a)$ at a desired significance level $a$.

2. For $b \in \{b_{\text{small}} + k, b_{\text{small}} + k + 1, ..., b_{\text{big}} - k\}$, compute a volatility index $VI_b$ of the critical value, i.e., the standard deviation of the values $\hat{c}_{G,b-k}(1 - a), \hat{c}_{G,b}(1 - a), ..., \hat{c}_{G,b+k}(1 - a)$ for a small positive integer $k$.

3. Pick $b^*$ that has the smallest $VI_{b^*}$ and the corresponding confidence interval.
Remark 6. As pointed out by Romano and Wolf (1999, Section 11.5), empirical bootstrap is not valid in the presence of heavy-tailed distributions. Thus, the common calibration method for the choice of subsampling block size cannot be used in our setup.

B Mathematical Proofs

This section collects all the mathematical proofs. The order in which the proofs appear below differs from the order in which the corresponding statements appear. Namely, the proof of Theorem 2 uses Lemma 1, and hence we present the proof of Lemma 1 before the proof of Theorem 2. Furthermore, the proof of Theorem 1 uses Lemma 1 and Theorem 2, and hence we present the proofs of Lemma 1 and Theorem 2 before the proof of Theorem 1.

B.1 Proof of Lemma 1

Proof of Lemma 1. Without loss of generality, suppose that $X_{gi}$ is a scalar and $r = 1$, and hence $\delta = \theta$. The proof is divided into two steps. In the first step, we derive the asymptotic distribution of the self-normalized sums that consist of the linear component of the influence function of the estimator. In the second step, we derive the validity of the proposed subsampling inference procedure.

Step 1. Recall that

$$\hat{\theta} - \theta = \left( \sum_{g=1}^{G} X'_{g} X_{g} \right)^{-1} \sum_{g=1}^{G} S_{g}.$$  

We shall derive the asymptotic distribution for the following self-normalized sums of the linear component $\sum_{g=1}^{G} S_{g}$:

$$SN_{1G}(\theta) := \frac{\sum_{g=1}^{G} S_{g}}{\sqrt{\sum_{g=1}^{G} S_{g}^2}}, \quad SN_{2G}(\theta) := \frac{\sum_{g=1}^{G} S_{g}}{\sqrt{\sum_{g=1}^{G} S_{g}^2}},$$

(B.1)

where $\hat{S}_{g} = X'_{g} \hat{U}_{g}$. The asymptotic distribution of a properly re-scaled $(\hat{\theta} - \theta)$ will then follow straightforwardly from the multiplication of $Q^{-1}$ on both the numerator and the
denominator. Since $\alpha \in (1, 2)$, Corollary 1 in [LePage et al., 1981] yields

$$SN_{1G}(\theta) \overset{d}{\to} \sqrt{\frac{\sum_{k=1}^{\infty} \{\epsilon_k Z_k - (2p - 1)\mathbb{E}[Z_k \mathbb{1}(Z_k < 1)]\}}{\sum_{k=1}^{\infty} Z_k^2}}$$

(B.2)

as $G \to \infty$, where

$$p = \lim_{t \to \infty} \frac{\mathbb{P}(S_g > t)}{\mathbb{P}(|S_g| > t)}.$$

$Z_k = (E_1 + \ldots + E_k)^{-1/\alpha}$ for each $k$; $\{E_k\}$ are i.i.d. standard exponential random variables, and $\{\epsilon_k\}$ are i.i.d. random variables that take the value of 1 with probability $p$ and $-1$ with probability $(1 - p)$ and are independent of $\{Z_k\}$.

We now claim that $SN_{2G}(\theta)$ converges in distribution to the same limiting distribution as (B.2). By Theorems 1 and 1' in [LePage et al., 1981],

$$\left(\frac{1}{A' G} \sum_{g=1}^{G} S_g, \frac{1}{A' G} \sum_{g=1}^{G} S_g^2\right) \overset{d}{\to} (S, V) := \left(\sum_{k=1}^{\infty} \{\epsilon_k Z_k - (2p - 1)\mathbb{E}[Z_k \mathbb{1}(Z_k < 1)]\}, \sum_{k=1}^{\infty} Z_k^2\right) = O_p(1)$$

(B.3)

holds for $A_G = G^{1/\alpha} L_1(G)$, where $Z_k$, $\epsilon_k$, and $p$ are defined below Equation (B.2), and $L_1(\cdot)$ is slowly varying at $\infty$; and

$$\frac{1}{(A'_G)^2} \sum_{g=1}^{G} (X'_g X_g)^2 \overset{d}{\to} \sum_{k=1}^{\infty} \tilde{Z}_k^2 = O_p(1)$$

(B.4)

holds where $A'_G = G^{1/\alpha} L_2(G)$, $\tilde{Z}_k = (\tilde{E}_1 + \ldots + \tilde{E}_k)^{-1/\alpha}$ for each $k$; $\{\tilde{E}_k\}$ are i.i.d. standard exponential random variables, and $L_2(\cdot)$ is slowly varying at $\infty$. Because $\alpha \in (1, 2)$ and $L_1$ is slowly varying at $\infty$, Equation (B.3) implies the consistency

$$\|\hat{\theta} - \theta\| = \left\|\sum_{g=1}^{G} X'_g S_g\right\|^{-1} \sum_{g=1}^{G} S_g = O_p(L_1(G) G^{-(1-1/\alpha)}) = o_p(1)$$

(B.5)

under Assumption 1. Using $\hat{U}_g = U_g + X_g(\theta - \hat{\theta})$ and $\hat{S}_g = S_g + X'_g X_g(\theta - \hat{\theta})$, where
\( \hat{u}_g = (\hat{u}_{g1}, \ldots, \hat{u}_{gN_g})' \), we can write
\[
\frac{1}{A_G^2} \sum_{g=1}^{G} \tilde{S}_g^2 = \frac{1}{A_G^2} \sum_{g=1}^{G} S_g^2 + \frac{1}{A_G^2} \sum_{g=1}^{G} (\hat{S}_g - S_g) \hat{S}_g + \frac{1}{A_G^2} \sum_{g=1}^{G} S_g (\hat{S}_g - S_g) \\
= \frac{1}{A_G^2} \sum_{g=1}^{G} S_g^2 + (1) + (2).
\]

We are going to show that the terms (1) and (2) are \( o_p(1) \). First,
\[
\| (1) \| = \left\| \frac{1}{A_G^2} \sum_{g=1}^{G} (S_g + X_g'X_g(\theta - \hat{\theta})) (X_g'X_g(\theta - \hat{\theta}))' \right\|
\leq \left\| \frac{1}{A_G^2} \sum_{g=1}^{G} S_gX_gX_g \right\| \| \hat{\theta} - \theta \| + \left\| \frac{1}{A_G^2} \sum_{g=1}^{G} (X_g'X_g)^2 \right\| \| \hat{\theta} - \theta \|^2
\leq \left\| \frac{1}{A_G^2} \sum_{g=1}^{G} S_g^2 \right\| \| \hat{\theta} - \theta \| + \left\| \frac{1}{A_G^2} \sum_{g=1}^{G} (X_g'X_g)^2 \right\| \| \hat{\theta} - \theta \|^2
= o_p(1)
\]
holds, where the second inequality follows from the Cauchy-Schwarz inequality and the stochastic orders are due to Equations \( (B.3) \), \( (B.4) \), and \( (B.5) \). Second, similar lines of calculations yield
\[
\| (2) \| = \left\| \frac{1}{A_G^2} \sum_{g=1}^{G} S_g(X_g'X_g(\theta - \hat{\theta}))' \right\| = o_p(1).
\]
We have now established that
\[
\frac{1}{A_G^2} \sum_{g=1}^{G} \tilde{S}_g^2 = \frac{1}{A_G^2} \sum_{g=1}^{G} S_g^2 + o_p(1),
\]
and consequently, \( SN_{1G}(\theta) \) is asymptotically equivalent to \( SN_{2G}(\theta) \).

**Step 2.** We next show the validity of cluster robust score subsampling procedure. Define
the conventional subsampling estimator

\[ \tilde{\theta}_{b,j} = \left( \sum_{g \in S_{j}} X'_g X_g \right)^{-1} \sum_{g \in S_{j}} X'_g Y_g. \]

Since \( B^{-1} - A^{-1} = A^{-1}(A - B)B^{-1} \), we have

\[ \tilde{\theta}_{b,j} - \hat{\theta}_{b,j} = \left( \sum_{g \in S_{j}} X'_g X_g \right)^{-1} \sum_{g \in S_{j}} X'_g Y_g - \left( \frac{G}{b} \right) \left( \sum_{g=1}^{G} X'_g X_g \right)^{-1} \sum_{g \in S_{j}} X'_g Y_g \]

\[ = \left( \frac{1}{G} \sum_{g=1}^{G} X'_g X_g \right)^{-1} \left( \frac{1}{G} \sum_{g=1}^{G} X'_g X_g - \frac{1}{b} \sum_{g \in S_{j}} X'_g X_g \right) \left( \frac{1}{b} \sum_{g \in S_{j}} X'_g X_g \right)^{-1} \frac{1}{b} \sum_{g \in S_{j}} X'_g Y_g \]

\[ = o_p(1). \]

This implies \( \tilde{\theta}_{b,j} = \hat{\theta}_{b,j}(1 + o_p(1)) \). Therefore, in the subsampling process, \( \tilde{\theta}_{b,j} \) can be replaced by \( \hat{\theta}_{b,j} \) without changing the asymptotic behavior. Thus, it suffices to establish validity of subsampling procedure based on the conventional subsampling estimator \( \tilde{\theta}_{b,j} \).

Now, since the stable distributions \( S \) and \( V \) defined in the previous step are both continuous and \( V > 0 \) with probability 1, \( S/V^{1/2} \) is continuously distributed and \( J^*(\cdot) \) is continuous. Hence, by invoking Theorem 11.3.1 in Politis et al. (1999), we have

\[ \sup_{t \in \mathbb{R}} |\hat{L}^*_{G,b}(t) - J^*(t)| = o_p(1) \]

as \( G \to \infty \), \( b \to \infty \), and \( b/G = o(1) \). Next, note that \( \hat{L}_{G,b} \) is an empirical CDF consisting of \( M \) i.i.d. summands as we randomly sample the subsample clusters with replacement. By Dvoretzky-Kiefer-Wolfowitz inequality, therefore, we have the uniform convergence of the empirical CDF:

\[ \sup_{t \in \mathbb{R}} |\hat{L}_{G,b}(t) - J^*(t)| = o_p(1) \]

as \( M \to \infty \) and \( G \to \infty \). This concludes the proof.
B.2 Proof of Theorem 2

Proof of Theorem 2. The proof is similar to the proof of Lemma 1 with the following minor modifications. First, when $\alpha = 2$, $S_g$ is in the domain of attraction of the normal distribution and hence Theorem 3.4 in Giné et al. (1997) yields

$$SN_{1G}(\theta) \xrightarrow{d} N(0, 1).$$

Second, to show the asymptotic equivalence of $SN_1(\theta)$ and $SN_2(\theta)$, note that both $S_g$ and $(X'_gX_g)$ belong to the domain of attraction of the normal law when $\alpha = 2$. We branch into two cases. In case that both $S_g$ and $(X'_gX_g)$ have finite variances, we have

$$\frac{1}{G} \sum_{g=1}^{G} \tilde{S}_g^2 = \frac{1}{G} \sum_{g=1}^{G} S_g^2 + o_p(1) \xrightarrow{p} \text{Var}(S_g)$$

by following the standard argument for consistency of the CR variance estimator. In case their variances do not exist, Lemma 3.1 in Giné et al. (1997) yields

$$\frac{1}{A_G^2} \sum_{g=1}^{G} S_g^2 \xrightarrow{p} 1$$

for $A_G$ such that

$$\frac{1}{A_G} \sum_{g=1}^{G} (S_g - \mathbb{E}[S_g]) \xrightarrow{d} N(0, 1).$$

A similar argument holds when $S_g$ is replaced by $(X'_gX_g)$. Then, the arguments for bounding $\|(1)\|$ and $\|(2)\|$ in the proof of Lemma 1 still go through, and thus for the self-normalized sums defined in Equation (B.1), it holds that $SN_2(\theta) = SN_1(\theta) + o_p(1)$. Finally, for the validity of the subsampling procedure, we now invoke Theorem 2.2.1 in Politis et al. (1999) and note that the limiting distribution is normal and hence continuous. □
B.3 Proof of Theorem 1

Proof of Theorem 1. The if part of the statement follows from the proof of Theorem 2. The only if part is a direct implication of Theorem 3.4 in Giné et al. (1997) and the fact that for any $\alpha \in (1,2]$, the self-normalized sums defined in Equation (B.1) satisfy $SN_2(\theta) = SN_1(\theta) + o_p(1)$, as shown in the proofs for Lemma 1 and Theorem 2.

B.4 Proof of Theorem 4

Proof of Theorem 4. Write

\[ T_G = \frac{S_G}{\sqrt{V_G}} := \frac{A_G^{-1} \sum_{g=1}^{G} \left( \sum_{i=1}^{N_g} Y_{gi} \right)}{\sqrt{A_G^{-2} \sum_{g=1}^{G} \left( \sum_{i=1}^{N_g} (Y_{gi} - \hat{\theta}) \right)^2}} \]

and

\[ T_G^* = \frac{S_G^*}{\sqrt{V_G^*}} := \frac{A_G^{-1} \sum_{g=1}^{G} v_g^* \left( \sum_{i=1}^{N_g} Y_{gi}^* \right)}{\sqrt{A_G^{-2} \sum_{g=1}^{G} \left( v_g^* \sum_{i=1}^{N_g} (Y_{gi} - \hat{\theta}^*) \right)^2}}. \]

Let $P$ denote the probability measure for the data and $P^*$ denote the probability measure of Rademacher auxiliary random variables. Define

\[ p = \lim_{t \to \infty} \frac{P \left( \sum_{i=1}^{N_g} Y_{gi} > t \right)}{P \left( \sum_{i=1}^{N_g} Y_{gi}^* > t \right)}. \]

Write $W_g = \left| \sum_{i=1}^{N_g} Y_{gi} \right|$ and the order statistics of $W_1, \ldots, W_G$ as follows:

\[ W_{G1} \geq W_{G2} \geq \ldots \geq W_{GG}. \]

The rescaled counterpart is denoted by $Z_{Gg} = A_G^{-1} W_{Gg}$, for $g = 1, \ldots, G$ – recall that $A_G = G^{1/\alpha} L(G)$ for a slow varying $L(\cdot)$ is defined right before Assumption 1. For each $G$, we can collect them into a countably long vector

\[ Z^G = (Z_{G1}, \ldots, Z_{GG}, 0, 0, \ldots) \in \mathbb{R}^\infty. \]
Similarly defined is the countably long sign vector

$$\epsilon^G = (\epsilon_{G1}, \ldots, \epsilon_{GG}, 1, 1, \ldots) \in \mathbb{R}^\infty,$$

where $\epsilon_{Gg}$ indicates the sign such that $\sum_{i=1}^{N_h} Y_{hi} = \epsilon_{Gg} W_{Gg}$ for the cluster $h$ that corresponds to the $g$-th order statistic $W_{Gg}$ for each $g = 1, \ldots, G$, $G \in \mathbb{N}$. By Lemmas 1 and 2 in [LePage et al.] (1981), we have

$$Z^G \overset{d}{\to} Z = (Z_1, Z_2, \ldots) \quad \text{and} \quad \epsilon^G \overset{d}{\to} \epsilon = (\epsilon_1, \epsilon_2, \ldots),$$

where $\{Z_k\}_k$ and $\{\epsilon_k\}$ are defined in the proof for Lemma [1]. In addition, since $\mathbb{R}^\infty$ is a complete separable metric space under the metric

$$d((x_1, x_2, \ldots), (y_1, y_2, \ldots)) = \sum_{k=1}^\infty \frac{1}{2^k} \cdot \frac{|x_k - y_k|}{1 + |x_k - y_k|},$$

following Skorohod’s representation theorem, on an adequately chosen probability space,

$$d(Z^G, Z) \to 0 \quad \text{and} \quad d(\epsilon^G, \epsilon) \to 0$$

$\mathbb{P}$-almost surely. Denote the countable vector of i.i.d. Rademacher random variables by $v^* = (v^*_1, v^*_2, \ldots) \in \mathbb{R}^\infty$, which is invariant of $G$. We now claim that the weak convergence

$$S^*_G = \sum_{g=1}^G \epsilon_{Gg} Z_{Gg} v^*_g \overset{d^*}{\to} S^* := \sum_{k=1}^\infty \epsilon_k Z_k v^*_k$$

for $(Z, \epsilon)$ with $\mathbb{P}$-probability one, where the convergence in distribution $d^*$ is with respect to $\mathbb{P}^*$. Note that the limiting random variable on the right-hand side is well-defined since

$$\mathbb{E}^* [\epsilon_k Z_k v^*_k] = 0 \quad \text{for all} \quad k \quad \text{and}$$

$$\sum_{k=1}^\infty \mathbb{E}^* [(\epsilon_k Z_k v^*_k)^2] = \sum_{k=1}^\infty Z_k^2 < \infty.$$
\(\mathbb{P}\)-almost surely. The convergence in distribution is shown following the same arguments as in the proof of Theorem 2 in [Knight (1989)] with i.i.d. Rademacher random variables \(v^*_{k}\) in place of their centered i.i.d. Poisson random variables \((M^*_k - 1)\). Specifically, observe that \(Z_k \rightarrow 0\) as \(k \rightarrow \infty\) \(\mathbb{P}\)-almost surely. Following Equation (12) in the proof of Theorem 1 in [LePage et al. (1981)], define \(Z \subset \mathbb{R}^\infty\) be the subspace consists of countable sequences \(z = (z_1, z_2, \ldots)\) such that \(z_1 \geq z_2 \geq \ldots \geq 0\) (note that \(Z\) is also a complete separable space with the inherited topology). Subsequently, for a fixed \(\varepsilon > 0\), define \(\phi: Z \times \{-1, 1\}^\infty \times \{-1, 1\}^\infty\) by

\[
\phi(z, \varepsilon, v^*) = \begin{cases} 
\sum_{k=1}^{\infty} \epsilon_k z_k \mathbb{1}(z_k > \varepsilon) v^*_{k} & \text{if } z_k \rightarrow 0 \text{ as } k \rightarrow \infty, \\
0 & \text{otherwise.}
\end{cases}
\]

Then \(\phi\) is a continuous mapping with respect to the product topology. Thus by the continuous mapping theorem as well as the convergences of \(d(Z^G, Z) \rightarrow 0\) and \(d(e^G, \varepsilon) \rightarrow 0\) with \(\mathbb{P}\)-probability one established earlier, for any \(\varepsilon > 0\),

\[
\sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} \mathbb{1}(Z_{Gg} > \varepsilon) v^*_{g} \xrightarrow{d^*} \sum_{k=1}^{\infty} \epsilon_k Z_k \mathbb{1}(Z_k > \varepsilon) v^*_{k}
\]

for \((Z, \varepsilon)\) with \(\mathbb{P}\)-probability one. In addition, note that

\[
\mathbb{E}^*[\left(\sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} \mathbb{1}(Z_{Gg} \leq \varepsilon) v^*_{g}\right)^2] = \sum_{g=1}^{G} Z_{Gg}^2 \mathbb{1}(Z_{Gg} \leq \varepsilon) \text{Var}^*(v^*_{g}) \leq \sum_{k=1}^{\infty} Z_k^2 \mathbb{1}(Z_k \leq \varepsilon)
\]

holds almost surely in \(\mathbb{P}\) and the right-hand side converges to zero as \(\varepsilon \rightarrow 0\), which implies via Markov’s inequality that, for any \(\delta > 0\),

\[
\lim_{\varepsilon \rightarrow 0} \limsup_{G \rightarrow \infty} \mathbb{P}^*[\left|\sum_{k=1}^{\infty} \epsilon_k Z_{Gk} \mathbb{1}(Z_{Gk} \leq \varepsilon) v^*_{k}\right| > \delta] = 0
\]

\(\mathbb{P}\)-almost surely. Finally, for any \(\delta > 0\),

\[
\lim_{\varepsilon \rightarrow 0} \mathbb{P}^*[\left|\sum_{k=1}^{\infty} \epsilon_k Z_k \mathbb{1}(Z_k \leq \varepsilon) v^*_{k}\right| > \delta] = 0
\]
\( \mathbb{P} \)-almost surely, which follows immediately from the fact that

\[
\mathbb{E}^* \left[ \left( \sum_{k=1}^{\infty} \epsilon_k Z_k 1(\varepsilon_k \leq \varepsilon) u_k^* \right)^2 \right] = \sum_{k=1}^{\infty} Z_k^2 1(\varepsilon_k \leq \varepsilon) \to 0
\]

\( \mathbb{P} \)-almost surely as \( \varepsilon \to 0 \). Combining these results yields that

\[
S^*_G \overset{d^*}{\rightarrow} S^* = \sum_{k=1}^{\infty} \epsilon_k Z_k v_k^*
\]

for \((Z, \epsilon)\) with \( \mathbb{P} \)-probability one. On the other hand, recall from Step 1 in the proof of Lemma 1 that

\[
S_G = \sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} \overset{d}{\rightarrow} S := \sum_{k=1}^{\infty} \{ \epsilon_k Z_k - (2p-1) \mathbb{E}[Z_k 1(Z_k \leq 1)] \},
\]

by Theorem 1 in LePage et al. \(1981\). Note that \( Z_k, \epsilon_k, \) and \( v_g^* \) are all mutually independent from each other. Therefore, the limiting distribution of \( S^*_G \) given \( Y_{1:G} \), i.e. \( S^* \) conditionally on \((Z, \epsilon)\), differs from, \( S \), the limiting \( \alpha \)-stable distribution of \( S_G \) with positive \( \mathbb{P} \)-probability.

Next, to cope with the denominator term of \( S^*_G \), note that, combined with the law of large numbers, the above weak convergence of \( S^*_G \) also implies

\[
\hat{\theta}^* = \frac{1}{N} \sum_{g=1}^{G} \epsilon_{Gg} W_{Gg} v_g^*
\]

\[
= \frac{1}{c + o_p(1)} \cdot \frac{1}{G} \sum_{g=1}^{G} \epsilon_{Gg} W_{Gg} v_g^*
\]

\[
= \frac{1}{c + o_p(1)} \cdot \frac{A_G}{G} \cdot \sum_{g=1}^{G} \epsilon_{Gg} Z_{Gg} v_g^* = o_p(1).
\]

Thus, the denominator term, \((V^*_G)^{1/2}\), of \( S^*_G \) turns out to be asymptotically independent of the auxiliary Rademacher random variables \( v_g^* \).

\[
V^*_G = \frac{1}{A^2_G} \sum_{g=1}^{G} \left( v_g^* \sum_{i=1}^{N_g} (Y_{gi} + o_p(1)) \right)^2 = \sum_{g=1}^{G} Z_{Gg}^2 + o_p(1).
\]
Given $Y_{1:G}$, the denominator is asymptotically constant. Following Step 1 in the proof of Lemma 1 we have

$$V_G = \sum_{g=1}^{G} Z_{Gg}^2 + o_p(1) \xrightarrow{d} \sum_{k=1}^{\infty} Z_k^2 = O_p(1).$$

Thus, given $Y_{1:G}$, the denominator term $(V_G^*)^{1/2}$ is a fixed value, while the original limit of the denominator is an $(\alpha/2)$-stable, non-degenerate continuous distribution. Hence, the limiting distribution of $V_G^*$ given $Y_{1:G}$ and the unconditional limiting distribution of $V_G$ differs with non-zero $\mathbb{P}$-probability.

Finally, note that $V_G^* > 0 \mathbb{P}$-almost surely. Thus, the fact that

$$(S_G^*, V_G^*) \xrightarrow{d} \left( \sum_{k=1}^{\infty} \epsilon_k Z_k v_k^*, \sum_{k=1}^{\infty} Z_k^2 \right)$$

for almost every $(Z, \epsilon)$ and the continuous mapping theorem yield that

$$T_G^* \xrightarrow{d} \frac{\sum_{k=1}^{\infty} \epsilon_k Z_k v_k^*}{\sqrt{\sum_{k=1}^{\infty} Z_k^2}}$$

for $(Z, \epsilon)$ with $\mathbb{P}$-probability one. This, together with the unconditional limiting distribution of $T_G$ implies the conclusion that the unconditional limiting distribution of $T_G$ and the conditional limiting distribution of $T_G^*$ differs with positive $\mathbb{P}$-probability. The inconsistency then follows. \qed
References


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