

Identification and Estimation of Average Marginal Effects in Fixed Effects Logit Models*

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Abstract

This article considers average marginal effects (AME) in a panel data fixed effects logit model. Relating the identified set of the AME to an extremal moment problem, we first show how to obtain sharp bounds on the AME straightforwardly, without any optimization. Then, we consider two strategies to build confidence intervals on the AME. In the first, we estimate the sharp bounds with a semiparametric two-step estimator. The second, very simple strategy estimates instead a quantity known to be at a bounded distance from the AME. It does not require any nonparametric estimation but may result in larger confidence intervals. Monte Carlo simulations suggest that both approaches work well in practice, the second being often very competitive. Finally, we show that our results also apply to average treatment effects, the average structural functions and ordered, fixed effects logit models.

Keywords: Fixed effects logit models, panel data, partial identification.

JEL Codes: C14, C23, C25.

*We gratefully acknowledge financial support from the research grant Otelo (ANR-17-CE26-0015-041). We would like to thank Benjamin Walter for his work at an early stage of this project and Christophe Gaillac for outstanding research assistance. We are also grateful to Iván Fernández-Val, Ulrich Müller and participants of various conferences and seminars for their feedback.

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1 Introduction

In this paper, we consider the identification and estimation of average marginal effects (AME) and average treatment effects (ATE) in the fixed effects (FE) binary logit model, with short panels. Estimation of the common slope parameters dates back to Rasch (1961) and Andersen (1970) (see also Chamberlain, 1980) but up to now, there has been no study of the identification and estimation of the AME and ATE in this model. These parameters are yet of more direct interest than the slope parameter, which only provides information on relative marginal effects. For this reason, and following the influential work of Angrist (2001) (see also Angrist and Pischke, 2008), many applied economists have turned to using FE linear probability models. Besides their simplicity, they allow one to identify the best linear approximation of the true model, if it is nonlinear. Then, one could argue that their slope parameter, which corresponds to the AME if the model is linear, is close to the true AME in practice.

However, FE linear models can be problematic in panel data if “stayers”, namely units with constant covariates over the period, differ from “movers” in their unobserved characteristics. The reason is that the AME in FE linear models is identified using “movers” only. But the true AME may depend on the unobserved heterogeneity and thus be very different for the population of “stayers”. Then, approximating the AME of the whole population using the linear model may be far from the truth. This is especially the case when the proportion of stayers is large, something we illustrate numerically in Appendix A. Also, it seems unfortunate that FE linear models only rely on movers given that in fact, the AME is nonparametrically identified for the population of stayers only (Hoderlein and White, 2012; Chernozhukov et al., 2019).¹

Unlike the FE linear model, nonlinear models such as the FE logit models do allow for heterogeneity of treatment effects, in particular between stayers and movers. Moreover, we demonstrate in this paper that estimation and inference on the AME and the ATE in this model can be performed almost as simply as in the FE linear model. To this end, we first study in Section 2 the identification of the AME (the study of the ATE, which is very similar, is postponed to Section 5). If this parameter is generally not point identified, sharp bounds can be obtained by solving an extremal moment problem, that is, maximizing a moment over

¹Nonlinearities may also cause trouble when using the FE linear model. In Appendix A, we give a simple example, with a difference-in-differences flavor, where the FE linear model identifies a negative ATE, even though the true ATE is positive.

probability distributions given the knowledge of some other moments. Using existing results on such problems, the bounds can then be obtained very simply, without any optimization. Other results on moment problems also highlight that that the bounds are very informative in practice, even if the panel is very short.

Next, we consider in Section 3 an estimator based on the theoretical expressions of the sharp bounds. This involves in particular the nonparametric estimation of a vector-valued function which, after a suitable transformation, corresponds at each point to a vector of raw moments $(1, E(U), \dots, E(U^T))$ for some random variable $U \in [0, 1]$. One difficulty is that standard (e.g., local polynomial) nonparametric estimators, once transformed, may not be vectors of raw moments themselves. We show how to modify any initial estimator so as to satisfy this constraint. We then establish root- n consistency of the estimators of the bounds under regularity conditions. The estimators of the bounds are asymptotically normal except if the corresponding slope coefficient is zero. We build confidence intervals of the true AME that are valid whether this is the case or not.

The previous estimator has the drawback of relying on a nonparametric first-step estimator. We then suggest in Section 4 an even simpler approach that avoids this issue. The idea is to estimate a simple approximation of the true AME and then to make bias aware inference, following the ideas of Donoho (1994) and Armstrong and Kolesár (2018). We can do so because the structure of the model allows us to consistently estimate an upper bound on the distance between the simple approximation and the true AME. The corresponding confidence intervals are asymptotically valid under mild conditions and, if slightly enlarged, even control the asymptotic size uniformly over a large set of data generating processes.

Section 5 shows that the same identification and estimation analysis can be applied to other parameters and models. Specifically, we study the ATE, the average structural function and FE ordered logit models. We also show that our method also applies when the number of observations varies per individual. Thus, our method easily accommodates unbalanced panels and hierarchical data. The R package `MarginalFElogit` and Stata command `mfelogit`, developed with Christophe Gaillac, perform inference on the AME and ATE (depending on whether X is continuous or binary) with the two methods considered here, and accommodates the case of an individual-specific number of observations.²

Next, we study in Section 6 the finite sample properties of our two estimation and inference methods. In line with the theory, they show that the estimated bounds are very informative

²The Stata command is available on the SSC repository and the R package can be found [here](#).

in practice. Also, the two confidence intervals have coverage close to their nominal level already for moderate sample sizes. Interestingly, we also find that the second inference method leads to confidence intervals often of the same size as, and sometimes even shorter than those obtained with the first method. This may seem surprising because as the sample size grows, such confidence intervals tend to an interval that strictly includes the true identified set. But for typical sample sizes and number of periods, it turns out that the distance between the simple approximation and the true AME is very small, leading to a tiny bias correction.

Our work is related to the literature on the identification and estimation of average marginal and treatment effects in panel data. Bias correcting approaches have been developed for panels with large T for both the logit and probit models (Fernández-Val, 2009; Fernández-Val and Weidner, 2016). With fixed T , Aguirregabiria and Carro (2020) shows point identification of a class of average marginal effects in dynamic FE logit models with four or more periods, exploiting the dynamic structure of the model. Another approach consists in using correlated random effects, following Mundlak (1978) and Chamberlain (1982); see e.g., for recent contributions Wooldridge (2019) for nonlinear models and Liu et al. (2021) for semiparametric binary response models. Compared to this approach, we do not impose any restriction on individual effects, which implies that average effects are only partially identified, though the bounds appear to be very informative in practice. An important part of this literature has also studied nonparametric identification (see in particular Altonji and Matzkin, 2005; Hoderlein and White, 2012; Chernozhukov et al., 2013, 2015; Botosaru and Muris, 2017). Our goal is different, as we consider a more constrained model, with the aim of providing a simple characterization and estimation of the bounds in this set-up.

Finally, our work is also related to moment problems, which have been studied extensively since Chebyshev and Markov. We refer to Karlin and Shapley (1953) and Krein and Nudelman (1977) for mathematical expositions and to Dette and Studden (1997) for applications to various statistical problems. D’Haultfœuille and Rathelot (2017) use similar results on moment problems as here to obtain bounds on segregation measures with small units. Finally, Dobronyi et al. (2021) use other results on moment problems to characterize the identified set of common parameters in dynamic logit models, generalizing the work of Honoré and Weidner (2020).

2 Identification

2.1 The set-up and identification of β_0

We consider a panel with T periods and observe binary outcomes Y_1, \dots, Y_T and for each period t , a vector of covariates $X_t := (X_{t1}, \dots, X_{tp})'$. We let $Y := (Y_1, \dots, Y_T)'$, $X := (X_1', \dots, X_T)'$ and make the following assumption.

Assumption 1 *We have $Y_t = \mathbb{1}\{X_t'\beta_0 + \alpha + \varepsilon_t \geq 0\}$, where the $(\varepsilon_t)_{t=1, \dots, T}$ are i.i.d., independent of (α, X) and follow a logistic distribution.*

Importantly, the individual effect α is allowed to be correlated in an unspecified way with X . In this model, $S := \sum_{t=1}^T Y_t$ is a sufficient statistic for α . As a result, identification of β_0 can be achieved by maximizing the expected conditional log-likelihood, where one conditions not only on X but also on S . For any $y = (y_1, \dots, y_T) \in \{0, 1\}^T$, let us define

$$C_k(x, \beta) := \sum_{(d_1, \dots, d_T) \in \{0, 1\}^T: \sum_{t=1}^T d_t = k} \exp\left(\sum_{t=1}^T d_t x_t' \beta\right),$$

$$\ell_c(y|x; \beta) := \sum_{t=1}^T y_t x_t' \beta - \ln \left[C_{\sum_{t=1}^T y_t}(x, \beta) \right],$$

which are respectively the ratio of the probability that $S = k$ given x with the probability that $S = 0$ given x if $\beta_0 = \beta$, and the conditional log-likelihood. To ensure that β_0 is identified as the unique maximizer of the expected conditional log-likelihood, we impose the following condition.

Assumption 2 *$E[\sum_{t, t'} (X_t - X_{t'})(X_t - X_{t'})']$ is nonsingular.*

Assumption 2 is necessary and sufficient for the identification of the slope parameter in fixed effects linear models. The following proposition ensures that this is also the case in FE logit models. It must be well-known, but we have not been able to find it in the literature. Its proof, as the other proofs of identification results, is presented in Appendix B.

Proposition 1 *Suppose that Assumption 1 holds and for all $t \neq t'$ and $k \in \{1, \dots, p\}$, $E[(X_{tk} - X_{t'k})^2] < \infty$. Then β_0 is identified if and only if Assumption 2 holds. In this case, $\beta_0 = \arg \max_{\beta} E(\ell_c(Y|X, \beta))$ and $\mathcal{I}_0 = -E(\partial^2 \ell_c / \partial \beta \partial \beta' (Y|X; \beta_0))$ is nonsingular.*

The second part of Proposition 1 shows that β_0 can be identified as the unique maximizer of the average expected log-likelihood. Under mild regularity conditions, \mathcal{I}_0^{-1} is the asymptotic variance of the conditional maximum likelihood estimator (CMLE) but also the semiparametric efficiency bound for β_0 (Hahn, 1997).

2.2 Identification of the average marginal effects

We now turn to the average marginal effect of a continuous covariate (X_{tk} , say); the average effect of a binary variable is deferred to Section 5.1. Without loss of generality, we focus on the effect in the last period. The average marginal effect is then defined as

$$\Delta := E \left[\frac{\partial P(Y_T = 1|X, \alpha)}{\partial X_{Tk}} \right].$$

By Assumption 1, $P(Y_T = 1|X, \alpha) = \Lambda(X'_T \beta_0 + \alpha)$ with $\Lambda(x) := 1/(1 + \exp(-x))$. Thus,

$$\Delta = \beta_{0k} E[\Lambda'(X'_T \beta_0 + \alpha)]. \quad (1)$$

The identification of Δ is rendered difficult by the fact that α is unobserved and Assumption 1 imposes no restriction on $F_{\alpha|X}$, the cumulative distribution function (cdf) of α given X . The only restrictions on this cdf come from the data, namely from the distribution of $Y|X$. Actually, because S is a sufficient statistic for α , its (conditional) distribution exhausts all the information available on α . Then, the restrictions on $F_{\alpha|X}$ reduce to

$$P(S = k|X = x) = C_k(x, \beta_0) \int \frac{\exp(ka)}{\prod_{t=1}^T [1 + \exp(x'_t \beta_0 + a)]} dF_{\alpha|X}(a|x), \quad (2)$$

which holds for $k \in \{0, \dots, T\}$. Intuitively, because we only have T constraints (one constraint is redundant, since $\sum_{k=0}^T P(S = k|X = x) = 1$), we expect $F_{\alpha|X}$ and in turn Δ not to be point identified. We shall see that this intuition is correct, under qualifications. Before that, we reformulate Δ and its sharp bounds in a more convenient way. This requires additional notation. Let us define

$$\sum_{t=0}^{T+1} \lambda_t(x, \beta_0) u^t := u(1 - u) \prod_{t=1}^{T-1} (u(\exp((x_t - x_T)' \beta_0) - 1) + 1), \quad (3)$$

$$c_t(x) := E \left[\frac{\mathbb{1}\{S \geq t\} \binom{T-t}{S-t} \exp(Sx'_T \beta_0)}{C_S(x, \beta_0)} | X = x \right], \quad t \in \{0, \dots, T\}, \quad (4)$$

$$m_t(x) := \frac{c_t(x)}{c_0(x)}, \quad t \in \{0, \dots, T\},$$

$$r(x, s, \beta_0) := \beta_k \sum_{t=0}^s \binom{T-t}{s-t} \frac{\lambda_t(x; \beta_0) \exp(sx'_T \beta_0)}{C_s(x, \beta_0)}.$$

We then let $m(x) := (m_0(x), \dots, m_T(x))'$. Because β_0 and $F_{S|X}$ are identified, $\lambda_t(x, \beta_0)$, $c_t(x)$, $m_t(x)$ and $r(x, s, \beta_0)$ are identified for each (x, s) .

For any $m \in [0, 1]^{T+1}$, we denote by $\mathcal{D}(m)$ the set of positive measures μ on $[0, 1]$ whose vector of first $T+1$ raw moments $(\int u^0 d\mu(u), \dots, \int u^T d\mu(u))'$ is equal to m . We finally define

$$\underline{q}_T(m) := \inf_{\mu \in \mathcal{D}(m)} \int_0^1 u^{T+1} d\mu(u), \quad \bar{q}_T(m) := \sup_{\mu \in \mathcal{D}(m)} \int_0^1 u^{T+1} d\mu(u). \quad (5)$$

Lemma 1 *Suppose that Assumptions 1-2 hold. Then, there exists a collection of probability measures $(\mu_x)_{x \in \text{Supp}(X)}$, with $\mu_x \in \mathcal{D}(m(x))$, such that*

$$\Delta = E \left[r(X, S, \beta_0) + \beta_{0k} c_0(X) \lambda_{T+1}(X, \beta_0) \int_0^1 u^{T+1} d\mu_X(u) \right]. \quad (6)$$

Moreover, the sharp identified set of Δ is $[\underline{\Delta}, \overline{\Delta}]$, with³

$$\begin{aligned} \underline{\Delta} &= E \left[r(X, S, \beta_0) + \beta_{0k} c_0(X) \lambda_{T+1}(X; \beta_0) \left(\underline{q}_T(m(X)) \right. \right. \\ &\quad \left. \left. \mathbb{1} \{ \beta_{0k} \lambda_{T+1}(X, \beta_0) \geq 0 \} + \overline{q}_T(m(X)) \mathbb{1} \{ \beta_{0k} \lambda_{T+1}(X, \beta_0) < 0 \} \right) \right], \\ \overline{\Delta} &= E \left[r(X, S, \beta_0) + \beta_{0k} c_0(X) \lambda_{T+1}(X; \beta_0) \left(\overline{q}_T(m(X)) \right. \right. \\ &\quad \left. \left. \mathbb{1} \{ \beta_{0k} \lambda_{T+1}(X, \beta_0) \geq 0 \} + \underline{q}_T(m(X)) \mathbb{1} \{ \beta_{0k} \lambda_{T+1}(X, \beta_0) < 0 \} \right) \right]. \end{aligned} \quad (7)$$

Equation (6) essentially follows by showing that Δ can be expressed as a simple function of the $(T + 1)$ -th moment of a variable related to $U := \Lambda(\alpha + X'_T \beta_0) \in [0, 1]$, whose first $T + 1$ moments conditional on $X = x$ are given by $m(x)$.⁴ The constraints on the last T raw moments (ignoring here $m_0(x) = 1$) simply correspond to the T constraints in (2). Thus by Equation (6), Δ is the sum of a point identified term and a term that is only partially identified in general. This second term is

$$\beta_{0k} E \left[c_0(X) \lambda_{T+1}(X, \beta_0) \int_0^1 u^{T+1} d\mu_X(u) \right]. \quad (8)$$

One could consider other decompositions than Equation (6), but this precise expression is helpful for two reasons. First, it leads to the sharp bounds (7), which take simple expressions because $\underline{q}_T(m)$ and $\overline{q}_T(m)$ themselves have a simple closed form, as shown below. When turning to estimation, we present in Section 3 an estimator based on these closed-form expressions. However this first approach requires to estimate non-parametrically, for all x observed in the data, $m(x)$, under the constraint that $m(x)$ is an admissible vector of moments. This constraint is necessary since otherwise $\underline{q}_T(m(x))$ and $\overline{q}_T(m(x))$ are undefined. Then, Equation (6) is also useful to avoid such a complication, because a very simple yet

³Technically, we define here the sharp identified set as the closure of the set of parameters values that can be rationalized by the data and the model. In some cases, the bounds $\underline{\Delta}$ and $\overline{\Delta}$ do not correspond to a valid probability distribution on $\alpha|X = x$, as they amount to put mass at plus or minus infinity. Nevertheless, these bounds can be approached arbitrarily by sequences of probability distributions rationalizing the model and the data.

⁴In the proof, we actually obtain a result on $\Delta(x) := E[\partial P(Y_T = 1|X, \alpha)/\partial X_{Tk}|X = x]$, which leads to Equation (6) by integrating over X . A result akin to Proposition 3 below also holds for $\Delta(x)$.

precise approximation of (8) can actually be obtained. This idea is at the basis of our second estimation method developed in Section 4.

The expression of the bounds shows that the computation of the bounds essentially reduces to that of the functions $\underline{q}_T(\cdot)$ and $\bar{q}_T(\cdot)$ defined by (5). This may seem a difficult task, since the programs are infinite-dimensional. It turns out, however, that they can be computed easily and without any numerical optimization, using results on moment problems (see Karlin and Shapley, 1953; Krein and Nudelman, 1977; Dette and Studden, 1997).

To get intuition on this problem, consider the case $T = 1$. Then, we seek the extremal values of $E(U^2)$, with U a random variable supported on $[0, 1]$ such that $E(U) = m_1$ (since U is a random variable, we also have the constraint $E(U^0) = 1$). The reasoning can be illustrated in Figure 1 below. The problem admits a solution only if $0 \leq m_1 \leq 1$ or equivalently only if a certain polynomial of m is non-negative ($m_1(1 - m_1) \geq 0$). In the boundary cases where $m_1(1 - m_1) = 0$, there is a unique probability distribution with such first moment, which is the Dirac distribution at $m_1 \in \{0, 1\}$. Then, $\underline{q}_T(m) = \bar{q}_T(m) = m_1^2 = m_1$ and $E(U^2)$ is identified. In the interior case $m_1(1 - m_1) > 0$, it follows from Jensen's inequality that $m_1^2 \leq E(U^2)$, with equality only if the distribution of U is a Dirac at $m_1 \in]0; 1[$. This implies that $\underline{q}_T(m) = m_1^2$. And because $U \leq 1$, we also have $E(U^2) \leq E(U) = m_1$, with equality only if U follows a Bernoulli(m_1) distribution. Hence, $\bar{q}_T(m) = m_1 > \underline{q}_T(m) = m_1^2$ and $E(U^2)$ is partly identified. Note that both bounds are rational functions of m .

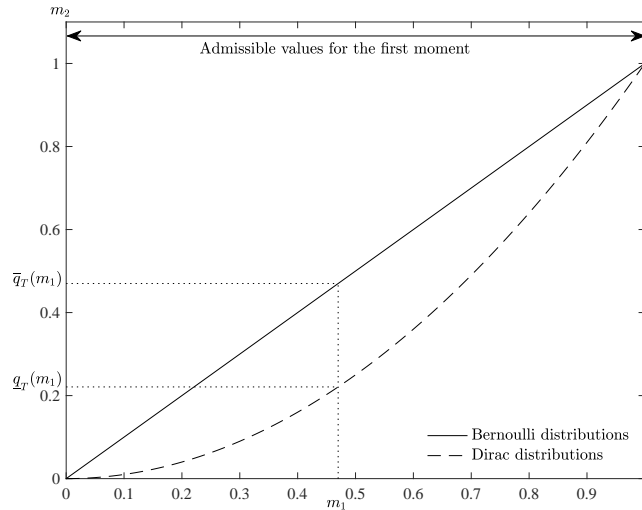


Figure 1: Moment space and bounds $\underline{q}_T(m), \bar{q}_T(m)$ when $T = 1$.

It turns out that the conclusions above for $T = 1$ remain true when $T > 1$. We have point

identification or partial identification of the $(T + 1)$ -th raw moment depending on the nullity of a nonnegative polynomial of m . Also, the sharp bounds on this moment are some rational functions of m .

To describe these general results, we introduce additional notation. Let \mathcal{D} denote the set of probability measures on $[0, 1]$ and for any $t \geq 1$, let

$$\mathcal{M}_t := \left\{ \left(\int_0^1 u^0 d\mu(u), \int_0^1 u^1 d\mu(u), \dots, \int_0^1 u^t d\mu(u) \right)' : \mu \in \mathcal{D} \right\}.$$

This is the set of all possible vectors of the first $t + 1$ raw moments. For instance, in Figure 1, the second and third components of \mathcal{M}_2 lie in between $x \mapsto x^2$ and the 45° line. Now, for any $t \geq 1$, $s > t$ and $m = (m_0, \dots, m_s) \in \mathbb{R}^s$, we define the Hankel matrices $\underline{\mathbb{H}}_t(m)$ and $\overline{\mathbb{H}}_t(m)$ as

$$\begin{aligned} \underline{\mathbb{H}}_t(m) &= (m_{i+j-2})_{1 \leq i, j \leq t/2+1}, & \overline{\mathbb{H}}_t(m) &= (m_{i+j-1} - m_{i+j})_{1 \leq i, j \leq t/2} && \text{if } t \text{ is even,} \\ \underline{\mathbb{H}}_t(m) &= (m_{i+j-1})_{1 \leq i, j \leq (t+1)/2}, & \overline{\mathbb{H}}_t(m) &= (m_{i+j-2} - m_{i+j-1})_{1 \leq i, j \leq (t+1)/2} && \text{if } t \text{ is odd.} \end{aligned}$$

Then, we define $\underline{H}_t(m) = \det(\underline{\mathbb{H}}_t(m))$ and $\overline{H}_t(m) = \det(\overline{\mathbb{H}}_t(m))$.

Proposition 2 *For any $T \geq 1$, we have*

$$\mathcal{M}_T = \left\{ m \in \{1\} \times [0, 1]^T : \forall t = 1, \dots, T, \underline{H}_t(m) \geq 0 \text{ and } \overline{H}_t(m) \geq 0 \right\}. \quad (9)$$

Moreover,

1. If $\underline{H}_T(m) \times \overline{H}_T(m) > 0$, then $\underline{q}_T(m) < \overline{q}_T(m)$. Moreover, $q \mapsto \underline{H}_{T+1}(m, q)$ is strictly increasing, linear and $\underline{H}_{T+1}(m, \underline{q}_T(m)) = 0$. Similarly, $q \mapsto \overline{H}_{T+1}(m, q)$ is strictly decreasing, linear and $\overline{H}_{T+1}(m, \overline{q}_T(m)) = 0$.
2. If $\underline{H}_T(m) \times \overline{H}_T(m) = 0$, then $\underline{q}_T(m) = \overline{q}_T(m)$. Moreover, letting $T' = \min\{t \leq T : \underline{H}_t(m) \times \overline{H}_t(m) = 0\}$, $\underline{q}_T(m) = \overline{q}_T(m)$ is the unique solution of

$$\begin{aligned} \underline{H}_{T'}(m_{T-T'+1}, \dots, m_T, \underline{q}_T(m)) &= 0 && \text{if } \underline{H}_{T'}(m) = 0, \\ \overline{H}_{T'}(m_{T-T'+1}, \dots, m_T, \underline{q}_T(m)) &= 0 && \text{if } \overline{H}_{T'}(m) = 0. \end{aligned}$$

The first point follows by classical results in moment theory, see e.g. Theorems 1.2.7 and 1.4.3 in Dette and Studden (1997). The first part of the second point is also well-known. On the other hand, to the best of our knowledge, the second part is new. The characterization of \mathcal{M}_T through $\underline{H}_t(m) \times \overline{H}_t(m) \geq 0$ for all $t \leq T$ is the extension, for any T , of the constraint

$m_1(1 - m_1) \geq 0$ we mentioned for $T = 1$. Then, the first case corresponds to the interior case described above ($m_1 \in (0, 1)$) for which $\underline{q}_T(m) < \bar{q}_T(m)$. In this case, the two bounds are functions of polynomials of m appearing in $\underline{H}_{T+1}(m, q)$ and $\bar{H}_{T+1}(m, q)$. The second case corresponds to the boundary case described above ($m_1 \in \{0, 1\}$), for which $\mathcal{D}(1, m_1, \dots, m_T)$ is reduced to a single distribution and thus $\underline{q}_T(m) = \bar{q}_T(m)$.

Coming back to the issue of computing the sharp bounds on Δ , the take-away from Proposition 2 is that the functions $\underline{q}_T(\cdot)$ and $\bar{q}_T(\cdot)$ are very easy to compute. For instance in the first case where $\underline{H}_T(m) \times \bar{H}_T(m) > 0$, expansion of the determinant $\underline{H}_{T+1}(m, \underline{q}_T(m)) = 0$ along the last column of $\underline{H}_{T+1}(m, \underline{q}_T(m))$ leads to the linear equation $a(m) + \underline{q}_T(m)b(m) = 0$ for two polynomials $a(m)$ and $b(m)$ of m .

Using Lemma 1 and results related to Proposition 2, we obtain the following further properties of the identified set:

Proposition 3 *Suppose that Assumptions 1-2 hold. Then:*

1. *The length of the identified set satisfies:*

$$\bar{\Delta} - \underline{\Delta} \leq \frac{1}{4^T} E [c_0(X) | \lambda_{T+1}(X, \beta_0)].$$

2. *Δ is point identified if and only if $\beta_{0k} = 0$ or*

$$\Pr \left(\min_{t < T} |(X_t - X_T)' \beta_0| = 0 \cup |Supp(\alpha|X)| \leq T/2 \right) = 1.$$

The first point exploits in particular a result of the theory of moments, namely that $\bar{q}_T(m) - \underline{q}_T(m) \leq 1/4^T$ for any $m \in \mathcal{M}_T$. For some distributions of X , this inequality yields an upper bound on the rate of decrease of the identified set as T increases. Specifically, assume that for all $t < T$, $\Pr(|(X_t - X_T)' \beta_0| \leq \ln(2)) = 1$. Then, additional algebra shows that $E [c_0(X) | \lambda_{T+1}(X, \beta_0)] \leq 1$, which in turn implies

$$\bar{\Delta} - \underline{\Delta} \leq \frac{1}{4^T}.$$

Similarly if for all $t < T$, $\Pr(|(X_t - X_T)' \beta_0| \leq c) = 1$ for some $c \in [\ln(2), \ln(5))$ then $\bar{\Delta} - \underline{\Delta} \leq (e^c - 1)^{T-1}/4^T \leq K^T$ for $K = (e^c - 1)/4 < 1$.

Chernozhukov et al. (2013) also obtain exponential rate of decrease on the length of the identified set of the average structural function $x \mapsto E[\Lambda(x' \beta_0 + \alpha)]$ under some conditions

(see their Theorem 4).⁵ Actually, their result imposed substantially weaker conditions on the distribution of $Y_t|X_t, \alpha$. On the other hand, it only holds for discrete X and imposes additional restrictions on the distribution of (X, α) .

The second result of Proposition 3 characterizes the point identification of Δ . The cases $\beta_{0k} = 0$ and $\min_{t < T} |(X_t - X_T)' \beta_0| = 0$ almost surely can be directly deduced from the first result, since $\min_{t < T} |(X_t - X_T)' \beta_0| = 0$ implies $\lambda_{T+1}(X) = 0$. That Δ is point identified if $\min_{t < T} |(X_t - X_T)' \beta_0| = 0$ could also be expected from Hoderlein and White (2012). They show that average marginal effects are nonparametrically identified on “stayers”, namely individuals for whom X_{it} remains constant between two periods. Finally, point identification can also be achieved if $|\text{Supp}(\alpha|X)| \leq T/2$.⁶ This corresponds to the second, boundary case described in Proposition 2. Intuitively, if $\alpha|X$ has few points of supports, its full distribution is characterized by its first moments. Then, given these moments, the higher moments are fully determined. As an illustration, assume that $T = 2$ and $\alpha|X$ is degenerate and equal to α_0 . Then, some algebra shows that $m(X) = (1, \Lambda(\alpha_0 + X_T' \beta_0), \Lambda(\alpha_0 + X_T' \beta_0)^2)'$. In such a case, the variance of any distribution in $\mathcal{D}(m(X))$ is zero. Thus, $\mathcal{D}(m(X))$ reduces to the Dirac distribution at $\Lambda(\alpha_0 + X_T' \beta_0)$, which implies $\underline{q}_2(m(X)) = \bar{q}_2(m(X)) = \Lambda(\alpha_0 + X_T' \beta_0)^3$.

3 A first estimation and inference method

In this section, we estimate the sharp bounds on Δ and develop inference on this parameter based on these bounds, using a sample $(Y_i, X_i)_{i=1, \dots, n}$.

3.1 Definition of the estimators

We estimate the sharp bounds $(\underline{\Delta}, \overline{\Delta})$ in three steps. Whereas Step 1 and 3 are straightforward, Step 2 is more involved, and further details are given in Online Appendix A.1.

1. Estimation of β_0 by the conditional maximum likelihood estimator $\hat{\beta}$.

⁵This parameter is also different from the average marginal effect. However, our analysis also applies to the average structural function (see Section 5.2 below), so we can also obtain an exponential rate of decrease on this parameter.

⁶Such identification is achieved using the logit structure: as a complement to Hoderlein and White (2012), Chernozhukov et al. (2019) show that average marginal effects are not identified nonparametrically for non-stayers.

2. Estimation of the functions c_0, \dots, c_T and m :

(a) Nonparametric estimation of c_0, \dots, c_T .

The functions c_0, \dots, c_T depend on β_0 and $\gamma_0 = (\gamma_{00}, \dots, \gamma_{0T})$, with $\gamma_{0j}(x) = P(S = j|X = x)$, according to

$$c_t(x) = \sum_{j=t}^T \binom{T-t}{j-t} \frac{\gamma_{0j}(x) \exp(jx'_T \beta_0)}{C_j(x, \beta_0)}. \quad (10)$$

To estimate γ_{0j} ($j = 0, \dots, T$), we consider a local polynomial estimator $\hat{\gamma}_j$, of order ℓ and with bandwidth h_n . Then, the estimators $\hat{c}_0, \dots, \hat{c}_T$ are plug-in estimators replacing β_0 and γ_{0j} with $\hat{\beta}$ and $\hat{\gamma}_j$ in (10).

(b) Nonparametric estimation of m .

The main difficulty of this step is to build an estimator \hat{m} satisfying $\hat{m}(X_i) \in \mathcal{M}_T$ for all i , which is necessary for $q_T(\hat{m}(X_i))$ and $\bar{q}_T(\hat{m}(X_i))$ to be well-defined (the plug-in estimator of $m(x)$ may not lie in \mathcal{M}_T because of sampling errors). To ensure that $\hat{m}(X_i) \in \mathcal{M}_T$, we exploit Proposition 2. We introduce a smoothing parameter c_n to estimate I , the largest t such that $m_{\rightarrow t}(X_i) := (1, m_1(X_i), \dots, m_t(X_i))$ belongs to the relative interior of \mathcal{M}_t . Next for $t \leq I$, $\hat{m}_t(X_i)$ is a function of $\hat{c}_0, \dots, \hat{c}_t$ and for $I < t \leq T$, $\hat{m}_t(X_i)$ is a function of $\hat{m}_{\rightarrow I}(X_i)$.

3. Estimation of the bounds by a plug-in estimator based on the formulas above:

$$\begin{aligned} \hat{\Delta} &= \frac{1}{n} \sum_{i=1}^n r(X_i, S_i, \hat{\beta}) + \hat{\beta}_k \hat{c}_0(X_i) \lambda_{T+1}(X_i, \hat{\beta}) \left[\bar{q}_T(\hat{m}(X_i)) \mathbb{1} \left\{ \hat{\beta}_k \lambda_{T+1}(X_i, \hat{\beta}) \geq 0 \right\} \right. \\ &\quad \left. + q_T(\hat{m}(X_i)) \mathbb{1} \left\{ \hat{\beta}_k \lambda_{T+1}(X_i, \hat{\beta}) < 0 \right\} \right], \\ \hat{\underline{\Delta}} &= \frac{1}{n} \sum_{i=1}^n r(X_i, S_i, \hat{\beta}) + \hat{\beta}_k \hat{c}_0(X_i) \lambda_{T+1}(X_i, \hat{\beta}) \left[q_T(\hat{m}(X_i)) \mathbb{1} \left\{ \hat{\beta}_k \lambda_{T+1}(X_i, \hat{\beta}) \geq 0 \right\} \right. \\ &\quad \left. + \bar{q}_T(\hat{m}(X_i)) \mathbb{1} \left\{ \hat{\beta}_k \lambda_{T+1}(X_i, \hat{\beta}) < 0 \right\} \right], \end{aligned} \quad (11)$$

where $q_T(\hat{m}(X_i))$ and $\bar{q}_T(\hat{m}(X_i))$ are computed according to Proposition 2.

3.2 Asymptotic properties

We show in Section A.2 of the Online Appendix that $(\hat{\Delta}, \hat{\underline{\Delta}})$ is consistent under standard regularity conditions. The key step therein is to show that \hat{m} is uniformly consistent, which is not straightforward because \hat{m} is a complicated function of $\hat{\gamma}$.

To obtain the asymptotic distribution of $(\widehat{\underline{\Delta}}, \widehat{\overline{\Delta}})$, in addition to standard conditions (see Assumptions 5 and 6 in the Online Appendix), we rely on an assumption specific to our context. To introduce it, let $\mathcal{B}(u, \epsilon)$ denote the closed ball centered at $u \in \mathbb{R}^d$ and with radius $\epsilon > 0$. Note that for any $J \geq 1$, \mathcal{M}_J is a convex subset of \mathbb{R}^{J+1} included in an affine subspace of dimension J . Then, let $\text{Int } \mathcal{M}_J$ and $\partial\mathcal{M}_J$ denote the relative interior and the relative boundary of \mathcal{M}_J .

Assumption 3 *There exists $\epsilon > 0$ and $I \in \{1, \dots, T\}$ such that for all $x \in \text{Supp}(X)$; (i) $\mathcal{B}(m_{\rightarrow I}(x), \epsilon) \subset \text{Int } \mathcal{M}_I$; (ii) if $I < T$, $m_{\rightarrow I+1}(x) \in \partial\mathcal{M}_{I+1}$.*

The index I therein corresponds to the index I introduced in Step 2b of the estimation. It does not need to be known by the researcher. I is related to $|\text{Supp}(\alpha|X = x)|$. Specifically, one can prove that if $I < T$, then I is odd and $|\text{Supp}(\alpha|X = x)| = (I + 1)/2$ for all $x \in \text{Supp}(X)$. Also, if $I = T$, then $|\text{Supp}(\alpha|X = x)| > T/2$ for all $x \in \text{Supp}(X)$, in which case $|\text{Supp}(\alpha|X = x)|$ may vary with x . Hence, Assumption 3 is violated if there exists $(x, x') \in \text{Supp}(X)^2$ such that

$$\begin{aligned} & |\text{Supp}(\alpha|X = x)| \neq |\text{Supp}(\alpha|X = x')|, \\ & \min\left(|\text{Supp}(\alpha|X = x)|, |\text{Supp}(\alpha|X = x')|\right) \leq T/2. \end{aligned}$$

We impose this restriction because \underline{q}_T and \overline{q}_T are not regular everywhere for $T \geq 3$. Specifically, whereas these functions are continuous on \mathcal{M}_T and infinitely differentiable on $\text{Int } \mathcal{M}_T$, they may not be even directionally differentiable at $m \in \partial\mathcal{M}_T$.⁷

Theorem 1 presents the asymptotic distribution of $(\widehat{\underline{\Delta}}, \widehat{\overline{\Delta}})$ as a function of $(\underline{\psi}_i, \overline{\psi}_i)$, which are defined in Equations (42)-(43) of the Online Appendix. Σ denotes the variance matrix of $(\underline{\psi}_i, \overline{\psi}_i)$ and $\widehat{\Sigma} = \frac{1}{n} \sum_{i=1}^n (\widehat{\underline{\psi}}_i, \widehat{\overline{\psi}}_i)' (\widehat{\underline{\psi}}_i, \widehat{\overline{\psi}}_i)$, where $(\widehat{\underline{\psi}}_i, \widehat{\overline{\psi}}_i)$ are estimators of $(\underline{\psi}_i, \overline{\psi}_i)$, defined in Equations (44)-(45) of the Online Appendix.

Theorem 1 *Suppose that Assumptions 1-3 and 5,6 hold. Then:*

1. *If $\beta_{0k} > 0$,*

$$\sqrt{n} \left(\widehat{\underline{\Delta}} - \underline{\Delta}, \widehat{\overline{\Delta}} - \overline{\Delta} \right) = \frac{1}{\sqrt{n}} \sum_{i=1}^n (\underline{\psi}_i, \overline{\psi}_i) + o_P(1) \xrightarrow{d} (\underline{Z}, \overline{Z}),$$

with $(\underline{Z}, \overline{Z}) \sim \mathcal{N}(0, \Sigma)$. If $\beta_{0k} < 0$, the same result holds by just exchanging the roles of $\underline{\psi}_i$ (resp. \underline{Z}) and $\overline{\psi}_i$ (resp. \overline{Z}).

⁷See D'Haultfoeuille and Rathelot (2017) for proofs of the first two statements. Regarding the third, one can show that, e.g., $m_1 \mapsto \underline{q}_3(m_0, m_1, m_2, m_3)$ is not differentiable at $m = (1, m_1, m_1^2, m_1^3)$.

2. If $\beta_{0k} = 0$,

$$\sqrt{n} \left(\widehat{\underline{\Delta}} - \underline{\Delta}, \widehat{\overline{\Delta}} - \overline{\Delta} \right) \xrightarrow{d} \left(\min(\underline{Z}, \overline{Z}), \max(\underline{Z}, \overline{Z}) \right).$$

3. We have $\widehat{\Sigma} \xrightarrow{P} \Sigma$.

The key step to prove this theorem is to show how the influence functions of $\widehat{\underline{\Delta}}$ and $\widehat{\overline{\Delta}}$ depend on the first-step estimators. Assumption 3 is key for this purpose.

The estimated bounds are asymptotically normal if $\beta_{0k} \neq 0$, but not in general if $\beta_{0k} = 0$. An exception is when the whole vector β_0 is equal to 0. Then $\underline{\psi} = \overline{\psi}$, which implies that $\underline{Z} = \overline{Z}$. In this case $\sqrt{n}(\widehat{\overline{\Delta}} - \widehat{\underline{\Delta}}) = o_P(1)$, and both bounds are asymptotically normal.

With Theorem 1 at hand, we can construct confidence intervals on Δ that are asymptotically valid whether or not $\beta_{0k} = 0$, at least in a pointwise sense (namely, for a fixed joint distribution of (X, Y)). To this end, let φ_α denote a consistent test with asymptotic level α of $\beta_{0k} = 0$, e.g., a t -test. Following Imbens and Manski (2004), let c_α denote the unique solution to

$$\Phi \left(c_\alpha + \frac{n^{1/2} \left(\widehat{\overline{\Delta}} - \widehat{\underline{\Delta}} \right)}{\max \left(\widehat{\Sigma}_{11}^{1/2}, \widehat{\Sigma}_{22}^{1/2} \right)} \right) - \Phi(-c_\alpha) = 1 - \alpha,$$

with Φ the cdf of a standard normal distribution and Σ_{ij} the (i, j) term of Σ . Then, we define $CI_{1-\alpha}^1$ as

$$CI_{1-\alpha}^1 := \begin{cases} \left[\widehat{\underline{\Delta}} - c_\alpha (\widehat{\Sigma}_{11}/n)^{1/2}, \widehat{\overline{\Delta}} + c_\alpha (\widehat{\Sigma}_{22}/n)^{1/2} \right] & \text{if } \varphi_\alpha = 1, \\ \left[\min \left(0, \widehat{\underline{\Delta}} - c_\alpha (\widehat{\Sigma}_{11}/n)^{1/2} \right), \max \left(0, \widehat{\overline{\Delta}} + c_\alpha (\widehat{\Sigma}_{22}/n)^{1/2} \right) \right] & \text{if } \varphi_\alpha = 0. \end{cases}$$

The following proposition shows that $CI_{1-\alpha}^1$ is pointwise valid as $n \rightarrow \infty$.

Proposition 4 *Suppose that Assumptions 1-3 and 5,6 hold and $\min(\Sigma_{11}, \Sigma_{22}) > 0$. Then $\liminf_n \inf_{\Delta \in [\underline{\Delta}, \overline{\Delta}]} P(\Delta \in CI_{1-\alpha}^1) \geq 1 - \alpha$, with equality when $\beta_{0k} \neq 0$.*

Intuitively, $CI_{1-\alpha}^1$ asymptotically reaches its nominal level when $\beta_{0k} \neq 0$ because it includes $[\widehat{\underline{\Delta}} - c_\alpha (\widehat{\Sigma}_{11}/n)^{1/2}, \widehat{\overline{\Delta}} + c_\alpha (\widehat{\Sigma}_{22}/n)^{1/2}]$, and the latter interval has asymptotic coverage $1 - \alpha$, by Theorem 1. When $\beta_{0k} = 0$, the asymptotic coverage of $CI_{1-\alpha}^1$ is also at least $1 - \alpha$, because $\Delta = 0 \in CI_{1-\alpha}^1$ as soon as $\varphi_\alpha = 0$.

The interval $CI_{1-\alpha}^1$ may have a uniform coverage over an appropriate set of data generating processes (DGPs), even if β_0 varies over Θ . Establishing this formally would however require to establish the uniform convergence in distribution of $(\widehat{\underline{\Delta}}, \widehat{\overline{\Delta}})$, a multistep estimator with a nonparametric first step. We leave this issue for future research. Note, on the other hand, that we consider below other confidence intervals that are uniformly conservative.

4 An alternative, simple estimator and inference method

4.1 The estimator

As shown in Lemma 1, the only reason why Δ is not identified is the integral term $\int u^{T+1} d\mu_X(u)$. In particular, because $\mu_x \in \mathcal{D}(m(x))$, Δ would become point identified if we replaced u^{T+1} by any polynomial of degree T . An idea, then, is to use a good approximation of u^{T+1} by such a lower degree polynomial. Following this strategy, we construct a very simple estimator that, in particular, does not require any first-step nonparametric estimator.

Specifically, note that among polynomials of degree $T + 1$ with leading coefficient equal to 1, the (renormalized) Chebyshev polynomial \mathbb{T}_{T+1}^c has the lowest supremum norm over $[-1, 1]$. Thus, the same holds on $[0, 1]$ for $\mathbb{T}_{T+1}(u) := 2^{-T-1}\mathbb{T}_{T+1}^c(2u - 1)$. Then, the best approximation of $u \mapsto u^{T+1}$ by a polynomial of degree T for the supremum norm is $P_T^*(u) := u^{T+1} - \mathbb{T}_{T+1}(u)$. Figure 2 displays $u \mapsto u^{T+1}$ and P_T^* for $T = 2, 3$ and 4. As we can see, the approximation is already good for $T = 2$, and the two functions become indistinguishable for $T = 4$.

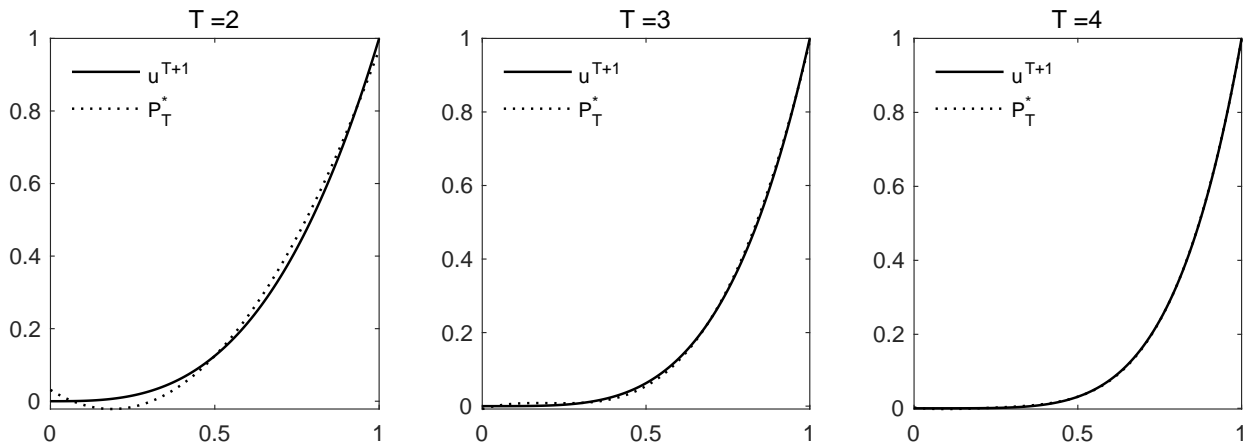


Figure 2: Approximation of $u \mapsto u^{T+1}$ by P_T^* .

Then, using (6), we simply approximate Δ by

$$\tilde{\Delta} = E \left[r(X, S, \beta_0) + \beta_{0k} c_0(X) \lambda_{T+1}(X, \beta_0) \int_0^1 P_T^*(u) d\mu_X(u) \right].$$

Let b_t denote the coefficient of u^t in P_T^* . Since $\int u^t d\mu_X(u) = m_t(x) = c_t(x)/c_0(x)$, we obtain

$$\begin{aligned}\tilde{\Delta} &= E \left[r(X, S, \beta_0) + \beta_{0k} \lambda_{T+1}(X, \beta_0) \sum_{t=0}^T b_t c_t(X) \right] \\ &= \beta_{0k} E \left[p(X, S, \beta_0) \right],\end{aligned}$$

where we define

$$p(X, S, \beta_0) := r(X, S, \beta_0) + \sum_{t=0}^S \binom{T-t}{S-t} \frac{(\lambda_t(x, \beta_0) + b_t \lambda_{T+1}(x, \beta_0)) \exp(SX'_T \beta_0)}{C_S(X, \beta_0)}.$$

We then estimate Δ by the simple plug-in estimator of $\tilde{\Delta}$:

$$\hat{\Delta} = \frac{\hat{\beta}_k}{n} \sum_{i=1}^n p(X_i, S_i, \hat{\beta}). \quad (12)$$

4.2 Inference on Δ

Even if $\hat{\Delta}$ is not a consistent estimator of Δ when T is fixed, we now show that we can build asymptotically valid confidence intervals for Δ using $\hat{\Delta}$. By a slight adjustment, we can even control their asymptotic size over a large class of DGPs. The confidence intervals shrink to $\{\Delta\}$ if $T \rightarrow \infty$ and remain of positive length otherwise. To construct these confidence intervals, we rely on two results. The first is the root- n asymptotic normality of $\hat{\Delta}$. Before displaying this result, we introduce some notation. Let $\phi_i = (\phi_{i1}, \dots, \phi_{iK})' := \mathcal{I}_0^{-1} \partial \ell_c / \partial \beta (Y_i | X_i; \beta_0)$ be the influence function of $\hat{\beta}$ and $\hat{\phi}_i$ be its plug-in estimator. Then, let

$$\begin{aligned}\psi_i &= E [p(X, S, \beta_0)] \phi_{ik} + \beta_{0k} \left\{ p(X, S, \beta_0) - E [p(X, S, \beta_0)] + E \left[\frac{\partial p}{\partial \beta} (X, S, \beta_0) \right]' \phi_i \right\}, \\ \hat{\psi}_i &= \left(\frac{1}{n} \sum_{i=1}^n p(X_i, S_i, \hat{\beta}) \right) \hat{\phi}_{ik} + \hat{\beta}_k \left\{ p(X_i, S_i, \hat{\beta}) - \frac{1}{n} \sum_{i=1}^n p(X_i, S_i, \hat{\beta}) \right. \\ &\quad \left. + \left(\frac{1}{n} \sum_{i=1}^n \frac{\partial p}{\partial \beta} (X_i, S_i, \hat{\beta}) \right)' \hat{\phi}_i \right\}.\end{aligned}$$

Finally, we define $\sigma^2 = V(\psi)$ and $\hat{\sigma}^2 = \sum_{i=1}^n \hat{\psi}_i^2 / n$. The following result follows from standard arguments. As all results of this section (except Lemma 3 below, proved in Appendix B), it is proved in Online Appendix A.

Lemma 2 *Suppose that Assumptions 1, 2 and 5 hold. Then*

$$n^{1/2} (\hat{\Delta} - \tilde{\Delta}) \xrightarrow{d} \mathcal{N}(0, \sigma^2). \quad (13)$$

Moreover, $\hat{\sigma} \xrightarrow{P} \sigma$.

The second result is a bound on $\tilde{\Delta} - \Delta$, which essentially follows from the fact that the Chebyshev polynomial \mathbb{T}_{T+1} satisfies $\sup_{u \in [0,1]} |\mathbb{T}_{T+1}(u)| \leq 1/[2 \times 4^T]$. Below, we let $\mathcal{M}^+ = \arg \max_{u \in [0,1]} \mathbb{T}_{T+1}(u)$, $\mathcal{M}^- = \arg \min_{u \in [0,1]} \mathbb{T}_{T+1}(u)$ and

$$R := E \left[c_0(X) \lambda_{T+1}(X, \beta_0) \int_0^1 (u^{T+1} - P_T^*(u)) d\mu_X(u) \right],$$

$$\bar{R} := E \left[\binom{T}{S} \frac{|\lambda_{T+1}(X, \beta_0)| \exp(SX'_T \beta_0)}{2 \times 4^T \times C_S(X, \beta_0)} \right].$$

Lemma 3 *Suppose that Assumption 1 holds. Then*

$$|\tilde{\Delta} - \Delta| = |\beta_{0k}| R \leq \bar{b} := |\beta_{0k}| \bar{R}.$$

Moreover, $|\tilde{\Delta} - \Delta| = \bar{b}$ if and only if:

1. $\beta_{0k} = 0$;
2. Or, conditional on $X = x$, $\Lambda(x'_T \beta_0 + \alpha)$ is supported on \mathcal{M}^+ for almost all x such that $\lambda_{T+1}(x, \beta_0) > 0$ and on \mathcal{M}^- for almost all x such that $\lambda_{T+1}(x, \beta_0) < 0$;
3. Or, conditional on $X = x$, $\Lambda(x'_T \beta_0 + \alpha)$ is supported on \mathcal{M}^- for almost all x such that $\lambda_{T+1}(x, \beta_0) > 0$ and on \mathcal{M}^+ for almost all x such that $\lambda_{T+1}(x, \beta_0) < 0$.

To build a confidence interval on Δ , we first estimate \bar{b} by $\hat{\bar{b}} = |\hat{\beta}_k| \hat{\bar{R}}$, with

$$\hat{\bar{R}} = \frac{1}{2 \times 4^T} \frac{1}{n} \sum_{i=1}^n \binom{T}{S_i} \frac{|\lambda_{T+1}(X_i, \hat{\beta})| \exp(S_i X'_{iT} \hat{\beta})}{C_{S_i}(X_i, \hat{\beta})}.$$

Let $Z_n := n^{1/2} (\hat{\Delta} - \Delta) / \hat{\sigma}$. To motivate the construction of the confidence intervals, let us first assume that $\hat{\sigma} = \sigma$, $\hat{\bar{b}} = \bar{b}$ and the asymptotic approximation (13) is exact. Then

$$Z_n \sim \mathcal{N} \left(n^{1/2} \frac{\tilde{\Delta} - \Delta}{\hat{\sigma}}, 1 \right). \quad (14)$$

Let $q_\alpha(b)$ denote the quantile of order $1 - \alpha$ of a $|\mathcal{N}(b, 1)|$. It is not difficult to show that $b \mapsto q_\alpha(b)$ is symmetric and increasing on $[0, \infty)$. Then, by Lemma 3, if $\hat{\bar{b}} = \bar{b}$ and (14) holds,

$$P \left(|Z_n| \leq q_\alpha \left(\frac{n^{1/2} \hat{\bar{b}}}{\hat{\sigma}} \right) \right) \geq 1 - \alpha. \quad (15)$$

We then define

$$\text{CI}_{1-\alpha}^2 = \left[\hat{\Delta} \pm q_\alpha \left(\frac{n^{1/2} \hat{\bar{b}}}{\hat{\sigma}} \right) \frac{\hat{\sigma}}{n^{1/2}} \right].$$

Inequality (15) implies that if (14) holds and $(\hat{\sigma}, \hat{b}) = (\sigma, \bar{b})$, $CI_{1-\alpha}^2$ has a level greater than $1-\alpha$. Theorem 2 below shows that actually, the same property holds asymptotically without these conditions, as long as $R < \bar{R}$. The only difference between $CI_{1-\alpha}^2$ and a standard confidence interval is that because of the possible bias, we consider $q_\alpha\left(n^{1/2}\hat{b}/\hat{\sigma}\right)$ instead of the usual normal quantile $q_\alpha(0)$. This difference is important, however: one can show that $CI_{1-\alpha}^2$ converges to $[\tilde{\Delta} \pm \bar{b}] \ni \Delta$ if $\bar{b} > 0$, rather than to $\{\tilde{\Delta}\}$ (but if $\bar{b} = 0$, $CI_{1-\alpha}^2$ converges to $\{\tilde{\Delta}\} = \{\Delta\}$).

We consider a second class of slightly wider confidence intervals, which has the advantage of being uniformly valid among a large class of DGP. To this end, we now partially take into account the randomness of \hat{b} . Specifically, consider $\gamma, \delta > 0$ such that $\gamma + \delta = \alpha$ and let $z_{1-\gamma}$ be the quantile of order $1-\gamma$ of a $\mathcal{N}(0, 1)$. Define $\hat{b}_\gamma = \left(|\hat{\beta}_k| + z_{1-\gamma}n^{-1/2}\hat{\tau}_k\right)\hat{R}$, where $\hat{\tau}_k$ denotes the estimator of the asymptotic variance τ_k of $\hat{\beta}_k$. We consider the following modification of $CI_{1-\alpha}^2$,

$$CI_{1-\alpha}^3 = \left[\hat{\Delta} \pm \frac{\hat{\sigma}}{n^{1/2}} q_\delta \left(\frac{n^{1/2}\hat{b}_\gamma}{\hat{\sigma}} \right) \right].$$

To define the class of DGPs for which the validity of $CI_{1-\alpha}^3$ is uniform, fix Θ a compact subset of \mathbb{R}^p , $\bar{M}, \underline{\sigma} \geq 0, \zeta \geq 0$ and let \underline{A} be a symmetric positive definite matrix. Define the following subset of probability distributions:

$$\mathcal{P} := \left\{ P : \text{Assumption 1 holds, } \beta_0 \in \Theta, P(\|X\| \leq \bar{M}) = 1, \mathcal{I}_{0P} \gg \underline{A}, \sigma_P^2 \geq \underline{\sigma}^2 \right. \\ \left. \text{and } \bar{R}_P > R_P(1 + \zeta) \right\}, \quad (16)$$

where $B \gg A$ means that $B - A$ is symmetric positive definite and we index $\mathcal{I}_0, \sigma^2, R$ and \bar{R} by P to underline their dependence in P .

Theorem 2 *1. Suppose that Assumptions 1, 2 and 5 hold, $\sigma^2 > 0$ and either $\bar{b} = 0$ or $R < \bar{R}$. Then:*

$$\liminf_{n \rightarrow \infty} P\left(\Delta \in CI_{1-\alpha}^2\right) \geq 1 - \alpha.$$

2. Suppose that Assumption 5.1 holds, $\gamma + \delta = \alpha$ in the definition of $CI_{1-\alpha}^3$ and $\underline{\sigma} > 0$ and $\zeta > 0$ in the definition of \mathcal{P} . Then:

$$\liminf_{n \rightarrow \infty} \inf_{P \in \mathcal{P}} P\left(\Delta \in CI_{1-\alpha}^3\right) \geq 1 - \alpha.$$

Given Lemma 3, the condition that either $\bar{b} = 0$ or $R < \bar{R}$ is very weak: it is violated only for very peculiar $F_{\alpha|X}$. The first point shows that under this condition, $CI_{1-\alpha}^2$ is pointwise

asymptotically conservative. One potential issue, however, is that because $CI_{1-\alpha}^2$ does not account for the variability of \widehat{b} , it may not be uniformly asymptotically conservative, in particular if we consider sequences of β_{0k} of the kind $n^{-1/2}c_k$. By (partially) accounting for the variability of \widehat{b} , $CI_{1-\alpha}^3$ is uniformly valid on \mathcal{P} , which allows for such sequences of β_{0k} .

One limitation of $CI_{1-\alpha}^2$ and $CI_{1-\alpha}^3$ is that they are not guaranteed to work in the very specific cases for which $R = \overline{R}$. To include such cases, one would have to account for the variability of \widehat{R} . However, its asymptotic distribution is non-normal and complicated when $P(\lambda_{T+1}(X, \beta_0) = 0) > 0$,⁸ so we leave this issue for future research. Also, simulations discussed below suggest that $CI_{1-\alpha}^2$ and $CI_{1-\alpha}^3$ may still have good coverage even if $R = \overline{R}$.

5 Extensions

We mention in this section other parameters and models for which very similar identification and estimation strategies apply. Specifically, we study the ATE, the average structural function and FE ordered logit models. We also consider the case where T varies per individual. We mostly focus on identification below, but also discuss in some cases how the inference methods above adapt to these set-ups.

5.1 Average treatment effects

When the regressor X_k is a binary treatment, we usually consider other parameters than the average marginal effect. Let X_T^0 (resp. X_T^1) be as X_T but with a 0 (resp. 1) in its k -th component. One usual parameter is the average treatment on the treated at period T :

$$\Delta^{ATT} = E \left[\Lambda \left(X_T^1 \beta_0 + \alpha \right) - \Lambda \left(X_T^0 \beta_0 + \alpha \right) \mid X_{kT} = 1 \right].$$

This is the average effect of a ceteris paribus change of X_{Tk} from 0 to 1, for all individuals satisfying $X_{Tk} = 1$. Because

$$E \left(\Lambda \left(X_T^1 \beta_0 + \alpha \right) \mid X_{kT} = 1 \right) = E \left(\Lambda \left(X_T^1 \beta_0 + \alpha \right) \mid X_{kT} = 1 \right) = E(Y \mid X_{kT} = 1)$$

is identified, we just have to focus on the bounds of

$$\Delta^{(1)} = E \left[\Lambda \left(X_T^0 \beta_0 + \alpha \right) \mid X_{kT} = 1 \right].$$

⁸For this reason also, we cannot use the results of Imbens and Manski (2004) and Stoye (2009) to construct uniformly valid confidence intervals.

The functions c_0, \dots, c_T , m_0, \dots, m_T , $\lambda_0, \dots, \lambda_{T+1}$ and r used for the average marginal effect have to be slightly adapted to $\Delta^{(1)}$. Specifically, for all $t = 0, \dots, T$, let

$$\begin{aligned} \sum_{t=1}^{T+1} \lambda_t^{(1)}(x, \beta_0) u^t &:= u \prod_{t=1}^T \left[1 + u(\exp((x_t - x_T^0)' \beta_0) - 1) \right], \\ c_t^{(1)}(x) &:= E \left[\mathbb{1} \{S \geq t\} \binom{T-t}{S-t} \exp(Sx_T^{0'} \beta_0) / C_S(x, \beta_0) \mid X = x, X_{kT} = 1 \right], \\ m_t^{(1)}(x) &:= c_t^{(1)}(x) / c_0^{(1)}(x), \\ r^{(1)}(x, s, \beta_0) &:= \sum_{t=0}^s \binom{T-t}{s-t} \lambda_t^{(1)}(x, \beta_0) \exp(sx_T^{0'} \beta_0) / C_s(x, \beta_0) \end{aligned}$$

The following result mimics Lemma 1 for $\Delta^{(1)}$. Its proof is very similar and is thus omitted.

Lemma 4 *Suppose that Assumptions 1-2 hold, $X_{kT} \in \{0; 1\}$. The sharp identified set of $\Delta^{(1)}$ is $[\underline{\Delta}^{(1)}, \overline{\Delta}^{(1)}]$, with*

$$\begin{aligned} \underline{\Delta}^{(1)} &= E \left[r^{(1)}(X, S, \beta_0) + c_0^{(1)}(X) \lambda_{T+1}^{(1)}(X, \beta_0) \left(\underline{q}_T(m^{(1)}(X)) \right. \right. \\ &\quad \left. \left. \mathbb{1} \{ \lambda_{T+1}^{(1)}(X, \beta_0) \geq 0 \} + \overline{q}_T(m^{(1)}(X)) \mathbb{1} \{ \lambda_{T+1}^{(1)}(X, \beta_0) < 0 \} \right) \mid X_{kT} = 1 \right] \\ \overline{\Delta}^{(1)} &= E \left[r^{(1)}(X, S, \beta_0) + c_0^{(1)}(X) \lambda_{T+1}^{(1)}(X, \beta_0) \left(\overline{q}_T(m^{(1)}(X)) \right. \right. \\ &\quad \left. \left. \mathbb{1} \{ \lambda_{T+1}^{(1)}(X, \beta_0) \geq 0 \} + \underline{q}_T(m^{(1)}(X)) \mathbb{1} \{ \lambda_{T+1}^{(1)}(X, \beta_0) < 0 \} \right) \mid X_{kT} = 1 \right] \end{aligned}$$

A similar result holds for the average treatment on the untreated, defined as

$$\Delta^{ATU} = E \left[\Lambda \left(X_T^{1'} \beta_0 + \alpha \right) - \Lambda \left(X_T^{0'} \beta_0 + \alpha \right) \mid X_{kT} = 0 \right].$$

Finally, consider the average treatment effect

$$\Delta^{ATE} = E \left[\Lambda \left(X_T^{1'} \beta_0 + \alpha \right) - \Lambda \left(X_T^{0'} \beta_0 + \alpha \right) \right].$$

To compute the bounds on Δ^{ATE} , remark that

$$\Delta^{ATE} = P(X_{kT} = 1) \Delta^{ATT} + P(X_{kT} = 0) \Delta^{ATU}.$$

Moreover, the distribution of $\alpha \mid X, X_{kT} = 1$ does not restrict the distributions of $\alpha \mid X, X_{kT} = 0$. As a result, the sharp lower bound $\underline{\Delta}^{ATE}$ on Δ^{ATE} simply satisfies

$$\underline{\Delta}^{ATE} = P(X_{kT} = 1) \underline{\Delta}^{ATT} + P(X_{kT} = 0) \underline{\Delta}^{ATU}.$$

The same holds for the sharp upper bound.

We can also simply estimate Δ^{ATE} using the second method. Following the same logic as in Section 4, we obtain the following approximation for Δ^{ATE} :

$$\tilde{\Delta}^{ATE} = E[p^{ATE}(X, Y, S, \beta_0)], \quad (17)$$

where we define

$$p^{ATE}(X, Y, S, \beta_0) = Y_T(2X_{kT} - 1) + \sum_{t=0}^S \binom{T-t}{S-t} \frac{(d_t(x, s, \beta_0) + b_t^* d_{T+1}(x, s, \beta_0)) \exp(SX'_T \beta_0)}{C_S(X, \beta_0)},$$

$$d_t(x, s, \beta_0) = -\lambda_t^{(1)}(x, \beta_0) \exp(-s\beta_{0k})x_{kT} + \lambda_t^{(0)}(x, \beta_0) \exp(s\beta_{0k})(1 - x_{kT}),$$

$$\sum_{t=1}^{T+1} \lambda_t^{(0)}(x, \beta_0) u^t = u \prod_{t=1}^T [1 + u(\exp((x_t - x_T^1)' \beta_0) - 1)],$$

and $-(b_0^*, \dots, b_T^*)$ are the first T coefficients of \mathbb{T}_{T+1} . The estimator $\hat{\Delta}^{ATE}$ of Δ^{ATE} is then a plug-in estimator based on (17). Also, with the same reasoning as for deriving the upper bound on $\tilde{\Delta} - \Delta$, we obtain

$$|\Delta^{ATE} - \tilde{\Delta}^{ATE}| \leq \bar{b}^{ATE} := E \left[\frac{\binom{T}{S} \exp(SX'_T \beta_0)}{2 \times 4^T \times C_S(X, \beta_0)} \left(|\lambda_{T+1}^{(1)}(X, \beta_0)| \exp(-S\beta_{0k}) X_T \right. \right. \\ \left. \left. + |\lambda_{T+1}^{(0)}(X, \beta_0)| \exp(S\beta_{0k})(1 - X_T) \right) \right].$$

Then, we can build confidence intervals on Δ^{ATE} using $\hat{\Delta}^{ATE}$, a plug-in estimator of \bar{b}^{ATE} and an estimator of the asymptotic variance of $\hat{\Delta}^{ATE}$, which is similar to $\hat{\sigma}$.

5.2 Average structural function

We now turn to the average structural function defined, for any $x_0 \in \mathbb{R}^p$, by:

$$\Delta_{x_0} := E(\Lambda(x'_0 \beta + \alpha)).$$

In a similar way as above, we define

$$\sum_{t=1}^{T+1} \lambda_t^{(2)}(x, \beta_0) u^t := u \prod_{t=1}^T [1 + u(\exp((x_t - x_0)' \beta_0) - 1)],$$

$$c_t^{(2)}(x) := E \left[\mathbb{1}\{S \geq t\} \binom{T-t}{S-t} \exp(Sx'_0 \beta_0) / C_S(x, \beta_0) | X = x \right],$$

$$m_t^{(2)} := c_t^{(2)}(x) / c_0^{(2)}(x),$$

$$r^{(2)}(x, s, \beta_0) := \sum_{t=0}^s \binom{T-t}{s-t} \lambda_t(x, \beta_0) \exp(sx'_0 \beta_0) / C_s(x, \beta_0)$$

Again, we obtain a similar result as Lemma 1 on the sharp bounds of $\Delta_{x_0}^{(2)}$:

Lemma 5 *Suppose that Assumptions 1-2 hold. Then the sharp identified set of $\Delta_{x_0}^{(2)}$ is $[\underline{\Delta}_{x_0}^{(2)}, \overline{\Delta}_{x_0}^{(2)}]$, with*

$$\begin{aligned}\underline{\Delta}_{x_0}^{(2)} &= E \left[r_t^{(2)}(X, S, \beta_0) + c_0^{(2)}(X) \lambda_{T+1}^{(2)}(X, \beta_0) \left(\underline{q}_T(m^{(2)}(X)) \right. \right. \\ &\quad \left. \left. \mathbb{1}\{\lambda_{T+1}^{(2)}(X, \beta_0) \geq 0\} + \bar{q}_T(m^{(2)}(X)) \mathbb{1}\{\lambda_{T+1}^{(2)}(X, \beta_0) < 0\} \right) \right] \\ \overline{\Delta}_{x_0}^{(2)} &= E \left[r_t^{(2)}(X, S, \beta_0) + c_0^{(2)}(X) \lambda_{T+1}^{(2)}(X, \beta_0) \left(\bar{q}_T(m^{(2)}(X)) \right. \right. \\ &\quad \left. \left. \mathbb{1}\{\lambda_{T+1}^{(2)}(X, \beta_0) \geq 0\} + \underline{q}_T(m^{(2)}(X)) \mathbb{1}\{\lambda_{T+1}^{(2)}(X, \beta_0) < 0\} \right) \right]\end{aligned}$$

5.3 Average marginal effect in ordered logit models

We now consider a model where the outcome is ordered and takes $J \geq 2$ values.

Assumption 4 *We have $Y_t = \sum_{k=1}^{J-1} k \mathbb{1}\{\gamma_k \leq X_t' \beta_0 + \alpha + \varepsilon_t < \gamma_{k+1}\}$ with $\gamma_1 = 0 < \dots < \gamma_J = +\infty$ and $(\varepsilon_t)_{t=1, \dots, T}$ are iid, independent of (α, X) and follow a logistic distribution.*

The condition $\gamma_1 = 0$ is a mere normalization: only the differences $\gamma_j - \gamma_{j'}$ are identified since the location of the distribution of α is left unrestricted. In this model, we consider the following AME, for any $j_0 \in \{1, \dots, J-1\}$:

$$\Delta^{(3)} = E \left[\frac{\partial P(Y_T \geq j_0 | X, \alpha)}{\partial X_{Tk}} \right].$$

To identify $(\beta_0, \gamma_2, \dots, \gamma_{J-1})$, we follow Muris (2017). Let Π be the set of functions from $\{1, \dots, T\}$ into $\{1, \dots, J-1\}$ and for $\pi \in \Pi$, let $Y_t^\pi = \mathbb{1}\{Y_t \geq \pi(t)\}$. By conditioning on $S^\pi = \sum_t Y_t^\pi$, we get the conditional log-likelihood

$$\begin{aligned}\ell_c^\pi(y|x; \beta, \gamma_2, \dots, \gamma_{J-1}) &:= \sum_{t=1}^T y_t (x_t' \beta - \gamma_{\pi(t)}) - \ln \left[C_{\sum_{t=1}^T y_t}^\pi(x, \beta, \gamma) \right], \\ \text{with } C_k^\pi(x, \beta, \gamma) &:= \sum_{(d_1, \dots, d_T) \in \{0,1\}^T: \sum_{t=1}^T d_t = k} \exp \left(\sum_{t=1}^T d_t (x_t' \beta - \gamma_{\pi(t)}) \right).\end{aligned}$$

The parameters $\theta_0 = (\beta_0, \gamma_2, \dots, \gamma_{J-1})$ are then identified by stacking, over all $\pi \in \Pi$, the first-order conditions $E[\partial \ell_c^\pi / \partial \theta(Y|X; \theta_0)] = 0$ of the conditional log-likelihood maximization.

For any $(j, t) \in \{1, \dots, J-1\} \times \{1, \dots, T\}$, let $\rho(j, t, x) = \exp((x_t - x_T)' \beta_0 - \gamma_j + \gamma_{j_0}) - 1$ and

$$w(u) = \frac{1}{\prod_{\substack{1 \leq j \leq J-1 \\ 1 \leq t \leq T}} (1 + u \rho(j, t, x))}.$$

Note that $w(u)$ is well-defined and positive on $[0, 1]$ since $\rho(j, t, x) > -1$. Finally, let $U = \Lambda(x'_T \beta_0 - \gamma_{j_0} + \alpha)$. We show in the proof of Lemma 6 below that:

$$\begin{aligned} & \text{span} \left\{ u \mapsto P((Y_1, \dots, Y_T) = y | X = x, U = u), y \in \{1, \dots, J\}^T \right\} \\ &= \text{span} \left\{ u \mapsto u^t w(u), t \in \{0, \dots, (J-1)T\} \right\}. \end{aligned}$$

This means that there exist identified, non-negative functions $(c_0^{(3)}(x), \dots, c_{(J-1)T}^{(3)}(x))$ such that the $(J-1)T + 1$ equations $\int_0^1 u^t w(u) dF_{U|X=x}(u) = c_t^{(3)}(x)$ exhaust the information provided by the knowledge of $(P((Y_1, \dots, Y_T) = y | X = x))_{y \in \{1, \dots, J\}^T}$. Next, let

$$m^{(3)}(x) := (c_0^{(3)}(x)/c_0^{(3)}(x), \dots, c_{(J-1)T}^{(3)}(x)/c_0^{(3)}(x))$$

and as previously, define $\lambda_0(x, \beta_0), \dots, \lambda_{(J-1)T+1}(x, \beta_0)$ as the coefficient of the polynomial $\frac{u(1-u)}{w(u)}$ of degree $(J-1)T + 1$:

$$\sum_{t=0}^{(J-1)T+1} \lambda_t^{(3)}(x, \beta_0) u^t := \frac{u(1-u)}{w(u)}.$$

Finally, let us define

$$r^{(3)}(x, \beta_0) := \beta_{0k} \sum_{t=0}^{(J-1)T} \lambda_t^{(3)}(x, \beta_0) c_t^{(3)}(x).$$

Again, sharp bounds on $\Delta^{(3)}$ can be obtained as in Lemma 1.

Lemma 6 *Suppose that Assumptions 2 and 4 hold. Then the sharp identified set of $\Delta^{(3)}$ is $[\underline{\Delta}^{(3)}, \overline{\Delta}^{(3)}]$, with*

$$\begin{aligned} \underline{\Delta}^{(3)} &= E \left[r^{(3)}(X, \beta_0) + \beta_{0k} c_0^{(3)}(X) \lambda_{(J-1)T+1}^{(3)}(X, \beta_0) \left(\underline{q}_{(J-1)T}(m^{(3)}(X)) \right. \right. \\ &\quad \left. \left. \mathbf{1}\{\beta_{0k} \lambda_{(J-1)T+1}(X, \beta_0) \geq 0\} + \bar{q}_{(J-1)T}(m^{(3)}(X)) \mathbf{1}\{\beta_{0k} \lambda_{(J-1)T+1}(X, \beta_0) < 0\} \right) \right] \\ \overline{\Delta}^{(3)} &= E \left[r^{(3)}(X, \beta_0) + \beta_{0k} c_0^{(3)}(X) \lambda_{(J-1)T+1}^{(3)}(X, \beta_0) \left(\bar{q}_{(J-1)T}(m^{(3)}(X)) \right. \right. \\ &\quad \left. \left. \mathbf{1}\{\beta_{0k} \lambda_{(J-1)T+1}(X, \beta_0) \geq 0\} + \underline{q}_{(J-1)T}(m^{(3)}(X)) \mathbf{1}\{\beta_{0k} \lambda_{(J-1)T+1}(X, \beta_0) < 0\} \right) \right]. \end{aligned}$$

The main difference between this result and Lemma 1 above is that the bounds are related to moments of order $(J-1)T + 1$ of distributions for which the first $(J-1)T$ raw moments are known. Hence, the bounds are tighter than in the binary case, and substantially more so given Proposition 3 above.

5.4 Varying number of periods

Missing data or attrition are common in panel data. A “panel” may also correspond to hierarchical data where (i, t) corresponds to a unit t belonging to a group i (e.g. individuals within a household). In both cases, T is a random variable varying from one individual (or group) to another. Our method still applies in this case, provided that T is conditionally exogenous. Specifically, we assume that $(\varepsilon_1, \dots, \varepsilon_T)$ is independent of (T, X, α) . Note, on the other hand, that we remain agnostic on the dependence between T and (X, α) . Then, we consider the average marginal effects at period $\underline{T} = \min \text{Supp}(T)$. Other choices are of course possible but this parameter has the advantage of being easily interpretable in the panel case.⁹

Under the independence condition above, the identification and estimation of β_0 remains unchanged. Also, Lemma 1 holds conditional on $T = \bar{t}$. The only changes therein are that (i) the polynomial in (3) is now $u(1-u) \prod_{t \neq \underline{T}} (u(\exp((x_t - x_{\underline{T}})' \beta_0) - 1))$; (ii) one should replace x_T by $x_{\underline{T}}$ in (4). Next, we obtain the sharp bounds on Δ by integrating over T . Similarly, the first estimation method applies for each subpopulation satisfying $T = \bar{t}$, and then one can just sum over all $\bar{t} \in \text{Supp}(T)$.

The second method can also be easily adapted. An inspection of $\tilde{\Delta}$ reveals that the formula remains similar, with the following changes: (i) Equations (3) and (4) should be modified as above; (ii) the Chebyshev polynomials used for the approximation $P_T^*(u, x)$ now vary with T . The estimator $\hat{\Delta}$ and the formulas of σ^2 and \bar{b} should be adjusted in a similar way. These features are

6 Monte Carlo simulations

We now study the finite sample performances of our two methods and compare them with the popular linear probability model estimator.¹⁰ In all the DGPs we consider, we assume that (X_1, \dots, X_T) are i.i.d., with $X_t \in \mathbb{R}$, uniformly distributed on $[-1/2, 1/2]$ and $\beta_0 = 1$. We also suppose that $\alpha = -X_T' \beta_0 + \eta$. Then, we first consider three correctly specified DGPs that differ by their distribution of $\eta|X$:

⁹With hierarchical data, the choice of the “period” does not matter anyway.

¹⁰For an application of our methodology to a real dataset with both continuous and discrete regressors, see the `documentation` of our R package `MarginalFELogit`.

1. DGP1: $\eta = 0$. Since $|\text{Supp}(\alpha|X)| = 1$ a.s., $\Delta = 0.25$ is point identified for all $T \geq 2$.
2. DGP2: $\eta|X \sim \mathcal{N}(0, 1)$. Because $|\text{Supp}(\alpha|X)| = \infty$, $\Delta \simeq 0.2067$ is partially identified for all T . The true bounds are $(\underline{\Delta}, \overline{\Delta}) \simeq (0.2006, 0.2124)$ if $T = 2$ and $(\underline{\Delta}, \overline{\Delta}) \simeq (0.2059, 0.2069)$ if $T = 3$.
3. DGP3: this corresponds to Point 2 of Lemma 3, for which $|\tilde{\Delta} - \Delta| = \bar{b}$. Specifically, $\eta|X$ is uniformly distributed over $\Lambda^{-1}(\mathcal{M}^+)$ (resp. over $\Lambda^{-1}(\mathcal{M}^-)$) if $\lambda_{T+1}(X, \beta_0) \geq 0$ (resp. if $\lambda_{T+1}(X, \beta_0) < 0$). Then, $\Delta = \underline{\Delta} = \overline{\Delta} \simeq 0.1875$ if $T = 2$ and $\Delta \simeq 0.1667$ with $(\underline{\Delta}, \overline{\Delta}) \simeq (0.1652, 0.1667)$ if $T = 3$.

For each of the three DGPs above, we consider $T \in \{2, 3\}$, $n \in \{250; 500; 1,000\}$ and perform 500 simulations for each such (T, n) . We then compute the estimators of the first and second methods, and $\text{CI}_{0.95}^1$, $\text{CI}_{0.95}^2$ and $\text{CI}_{0.95}^3$. To estimate nonparametrically γ_0 , we use local linear estimators with a Gaussian product kernel. We use data-driven bandwidths h_n and thresholds c_n , on which further details are given in Section C of the Online Appendix. In $\text{CI}_{0.95}^3$, we use $\gamma = 0.01$ and $\delta = 0.04$.

Table 1 displays the properties of the estimators underlying the two methods. The estimators of the bounds appear to have a small bias in all cases, except perhaps with DGP3 and $T = 2$. Note that in this case, the distribution of $\eta|X = x$ does not vary in a smooth way with x : $\eta = \Lambda^{-1}(1/4)$ when $x_1 \leq x_2$ while $\eta = \Lambda^{-1}(3/4)$ otherwise. As a result, the regularity condition we impose on γ_0 (see Assumption 6.2) is actually violated, which could explain the larger bias in this case. The bias of the estimator $\hat{\Delta}$ is very small compared to its standard deviation; except with DGP3, $T = 2$ and $n = 1,000$, it is always more than ten times smaller. The exceptional case could be expected, as in this case $|\tilde{\Delta} - \Delta|$ reaches its upper bound \bar{b} . Even in this case, the bias of $\hat{\Delta}$ is more than four times smaller than its standard deviation. Note that the bias is very small with $T = 3$, even under DGP3. This illustrates the fact that the bound on $|\tilde{\Delta} - \Delta|$ decreases quickly with T .

Table 2 presents the coverage rate and length of both confidence intervals. The coverage rates of the third confidence interval are always greater than 95%. This is the case even with DGP3, for which Theorem 2 does not provide any guarantee. Hence, neglecting the variability of \bar{R} does not seem to lead to undercoverage here. The second confidence interval also shows a very good coverage, always greater than 94%. The first method leads to somewhat smaller coverage. It is still close to 95% for DGP1, DGP2 and DGP3 with $T = 3$. The lower coverage for DGP3 and $T = 2$ is probably due to the bias of the estimators of the bounds, which could be due to the aforementioned irregularity of γ_0 .

In terms of length of the confidence intervals, the second method actually performs better than the first method when $T = 2$, and also when $T = 3$ with DGP3; otherwise the two methods are very comparable. This was not obvious: when $n \rightarrow \infty$, the length of $CI_{0.95}^1$ becomes smaller than that of $CI_{0.95}^2$ because $\bar{\Delta} - \underline{\Delta} < \bar{R}$. Hence, at least with our three DGPs, $\bar{R} - (\bar{\Delta} - \underline{\Delta})$ is small compared to the standard errors of $\hat{\underline{\Delta}}$, $\hat{\bar{\Delta}}$ and $\hat{\Delta}$, even with $n = 1,000$. Finally, the third confidence interval is of course larger than $CI_{0.95}^2$, but the increase in length is modest, around 6% on average and never more than 8%.

DGP	T	n	First method				Second method	
			$\sigma(\hat{\Delta})$	Bias($\hat{\Delta}$)	$\sigma(\hat{\bar{\Delta}})$	Bias($\hat{\bar{\Delta}}$)	$\sigma(\hat{\Delta})$	Bias($\hat{\Delta}$)
1	2	250	0.145	0.024	0.15	0.03	0.118	0.006
		500	0.095	0.018	0.098	0.024	0.077	0.002
		1,000	0.066	0.016	0.069	0.021	0.056	0*
	3	250	0.069	-0.009	0.069	-0.009	0.078	0.002
		500	0.051	-0.004	0.051	-0.004	0.057	0.004
		1,000	0.036	-0.003	0.036	-0.003	0.040	0.004
2	2	250	0.148	0.039	0.156	0.039	0.109	0.011
		500	0.096	0.026	0.102	0.026	0.076	0.006
		1,000	0.058	0.014	0.062	0.013	0.050	-0.001
	3	250	0.066	-0.010	0.066	-0.011	0.072	0.002
		500	0.05	-0.006	0.050	-0.006	0.052	0.001
		1,000	0.034	-0.006	0.034	-0.006	0.037	0*
3	2	250	0.176	0.064	0.184	0.077	0.110	-0.002
		500	0.102	0.041	0.108	0.053	0.071	-0.007
		1,000	0.071	0.030	0.075	0.041	0.052	-0.011
	3	250	0.086	0.011	0.087	0.010	0.064	0.004
		500	0.050	0.001	0.050	0.001	0.045	-0.004
		1,000	0.035	0.004	0.035	0.004	0.032	0*

Notes: in the three DGPs, $\alpha = -X_T\beta_0 + \eta$ with $\eta = 0$ in DGP1, $\eta|X \sim \mathcal{N}(0, 1)$ in DGP2 and $\eta|X$ uniformly distributed over $\Lambda^{-1}(\mathcal{M}^+)$ (resp. over $\Lambda^{-1}(\mathcal{M}^-)$) if $\lambda_{T+1}(X, \beta_0) > 0$ (resp. if $\lambda_{T+1}(X, \beta_0) < 0$) in DGP3. The results are obtained with 500 simulations. *: in absolute values, smaller than 0.0005.

Table 1: Properties of the estimators

DGP	T	n	CI _{0.95} ¹		CI _{0.95} ²		CI _{0.95} ³		
			Coverage	Avg. length	Coverage	Avg. length	Coverage	Avg. length	
1	2	250	0.94	0.492	0.95	0.461	0.97	0.492	
		500	0.93	0.344	0.96	0.325	0.97	0.347	
		1,000	0.93	0.243	0.96	0.231	0.97	0.248	
	3	250	0.97	0.309	0.96	0.317	0.97	0.332	
		500	0.98	0.218	0.96	0.223	0.96	0.234	
		1,000	0.97	0.155	0.96	0.158	0.96	0.166	
	2	2	250	0.93	0.463	0.96	0.420	0.97	0.454
			500	0.92	0.318	0.96	0.296	0.98	0.319
			1,000	0.94	0.221	0.97	0.210	0.98	0.226
3		250	0.96	0.279	0.95	0.282	0.95	0.296	
		500	0.95	0.197	0.95	0.201	0.96	0.210	
		1,000	0.96	0.139	0.94	0.141	0.95	0.148	
3		2	250	0.90	0.540	0.96	0.422	0.97	0.453
			500	0.92	0.365	0.96	0.296	0.97	0.318
			1,000	0.88	0.250	0.94	0.209	0.95	0.224
	3	250	0.95	0.279	0.95	0.249	0.96	0.261	
		500	0.95	0.187	0.95	0.175	0.96	0.184	
		1,000	0.96	0.130	0.95	0.124	0.96	0.130	

Notes: in the three DGPs, $\alpha = -X_T\beta_0 + \eta$ with $\eta = 0$ in DGP1, $\eta|X \sim \mathcal{N}(0, 1)$ in DGP2 and $\eta|X$ uniformly distributed over $\Lambda^{-1}(\mathcal{M}^+)$ (resp. over $\Lambda^{-1}(\mathcal{M}^-)$) if $\lambda_{T+1}(X, \beta_0) > 0$ (resp. if $\lambda_{T+1}(X, \beta_0) < 0$).

Table 2: Coverage and average length of $CI_{0.95}^k$ for $k \in \{1, 2, 3\}$.

Finally, we compare $\hat{\Delta}$ and $CI_{0.95}^2$ with the linear probability model (LPM) estimator and the corresponding confidence interval (CI^{LPM}) based on asymptotic normality and the usual standard error accounting for clustering at the individual level. We consider DGP1 but also two incorrectly specified models. In DGP4, the $(\varepsilon_t)_{t=1, \dots, T}$ still marginally follow a logistic distribution (so that Δ is the same as in DGP1), but they are autocorrelated: the copula of $(\varepsilon_s, \varepsilon_t)$ is gaussian with correlation coefficient $1/2^{|s-t|}$. In DGP5, we assume instead that the ε_t are independent over time but $\varepsilon_t \sim \mathcal{N}(0, 8/\pi)$. We chose this variance so that again, Δ is the same as in DGP1. We consider $T \in \{2, 3, 4\}$, $n = 1,000$ and two possible values of

β_0 , namely $\beta_0 = 1$ and $\beta_0 = 2$.

DGP	β_0	T	Bias		Coverage	
			$\hat{\Delta}$	$\hat{\Delta}_{\text{LPM}}$	$\text{CI}_{0.95}^2$	$\text{CI}_{0.95}^{\text{LPM}}$
1	1	2	0.003	-0.006	0.95	0.94
		3	0.001	-0.008	0.96	0.96
		4	-0.002	-0.01	0.96	0.95
	2	2	0.002	-0.056	0.99	0.78
		3	0.005	-0.055	0.96	0.67
		4	-0.002	-0.057	0.95	0.50
4	1	2	0.003	-0.006	0.97	0.93
		3	0.001	-0.008	0.96	0.95
		4	0*	-0.008	0.95	0.93
	2	2	0.003	-0.055	1	0.73
		3	0*	-0.056	0.97	0.58
		4	0*	-0.054	0.95	0.44
5	1	2	0.003	-0.005	0.95	0.93
		3	0.003	-0.006	0.95	0.94
		4	0.002	-0.006	0.94	0.95
	2	2	0.013	-0.047	0.97	0.85
		3	0.018	-0.047	0.96	0.78
		4	0.013	-0.046	0.95	0.63

Notes: DGP1 is as above. DGP4 as DGP1, but with autocorrelated $(\varepsilon_t)_{t=1,\dots,T}$. DGP5 as DGP1, but with $\varepsilon_t \sim \mathcal{N}(0, 8/\pi)$. In the three DGPs, $\Delta = 0.25\beta_0$. Results based on 500 simulations.

*: in absolute values, smaller than 0.0005.

Table 3: Comparison with the linear probability model

Table 3 displays the results. It first shows that in the two misspecified DGPs we consider, our estimator $\hat{\Delta}$ still performs very well: its bias remains small and the corresponding confidence interval $\text{CI}_{0.95}^2$ still exhibits a coverage very close to or above 95%. The linear probability model estimator also has good performances when $\beta_0 = 1$, with low bias and a coverage always larger than 93%. However, when $\beta_0 = 2$, its performance deteriorates, especially for larger T . Note that this sensitivity on β_0 may be more exacerbated with fixed effects. To

see this, we consider the same DGP as DGP1 but with $\alpha = 0$. Then, the coverage rate of $CI_{0.95}^{\text{LPM}}$ only decreases from 95% when $\beta_0 = 1$ to 89% when $\beta_0 = 2$ with $T = 4$, as opposed to the decrease from 95% to 50% with DGP1.

7 Conclusion

In the FE logit model, the AME can be written as a function of the $(T + 1)$ -th raw moment of an unknown distribution for which the first T moments are known. By results in the theory of moments, this implies simple expressions for the sharp bounds of the AME. These bounds can be estimated consistently under weak conditions. Using instead the best uniform approximation of u^{T+1} by a polynomial of degree T yields an even simpler approach for inference on the AME. We expect both ideas to apply to other set-up involving latent variables, such as α in our context.¹¹

The theory is simple here because only raw moments are involved; but similar results hold with other moments, provided that the corresponding functions form a so-called Chebyshev system (See, e.g., Krein and Nudelman, 1977, for a mathematical exposition). Results on these systems have already been applied to the optimal design of experiments (see Dette and Studden, 1997) and the measure of segregation with small units (D’Haultfœuille and Rathelot, 2017). By drawing attention on these tools, we hope that this paper will contribute to their use in econometrics.

¹¹Noteworthy, Dobronyi et al. (2021) apply related results on moment problems to obtain a simple characterization of the identified set of slope parameters in a dynamic FE logit model.

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A Potential pitfalls of using FE linear models

We illustrate here two points made in the introduction on the use of FE linear models for binary outcomes. First, the FE linear model will only approximate the effect on the “movers” (in terms of covariates), and this effect may be very different from the effect on the “stayers”, and thus also different from the average effect on the whole population. Second, the linear approximation of a true, nonlinear model may be so poor that the approximation of a true treatment effect is of the wrong sign.

To illustrate the first point, suppose the true model is the FE logit model and assume that $T = 2$. Suppose $X_t \in \mathbb{R}$, $\beta_0 = 1$ and we have a dummy variable M , with $M = 1$ if the individual is a mover, $M = 0$ otherwise. In the first case, X_1 and X_2 are i.i.d. and continuous (so that $P(X_1 = X_2|M = 1) = 0$) whereas in the second case, $X_1 = X_2$ a.s. Assume that the individual effect is such that $|\alpha|$ is very large when $M = 0$, whereas $\alpha = 0$ if $M = 1$. Then, the true AME is

$$\begin{aligned}\Delta &= P(M = 1)E[\Lambda'(X_2)|M = 1] + P(M = 0)E[\Lambda'(X_2 + \alpha)|M = 0] \\ &\simeq P(M = 1)E[\Lambda'(X_2)|M = 1].\end{aligned}$$

On the other hand, the linear approximation Δ_{lin} of Δ , equal to the slope parameter of the FE linear model, satisfies

$$\begin{aligned}\Delta_{\text{lin}} &= \frac{E[(Y_2 - Y_1)(X_2 - X_1)]}{E[(X_2 - X_1)^2]} \\ &= \frac{E[(Y_2 - Y_1)(X_2 - X_1)|M = 1]}{E[(X_2 - X_1)^2|M = 1]} \\ &= \frac{E[(\Lambda(X_2) - \Lambda(X_1))(X_2 - X_1)|M = 1]}{E[(X_2 - X_1)^2|M = 1]} \\ &\simeq E[\Lambda'(X_2)|M = 1],\end{aligned}$$

where the last approximation follows by a Taylor expansion, if $X_2 - X_1$ is small. Thus, in this example, Δ_{lin} will overestimate Δ by the factor $1/P(M = 1)$, which can be arbitrarily large. Note that the reverse holds true if, instead, $|\alpha|$ is very large when $M = 1$ and $\alpha = 0$ when $M = 0$.

To illustrate the second point, suppose that potential outcomes $Y_t(d)$ satisfy

$$Y_t(d) = \mathbb{1}\{\alpha + \mathbb{1}\{t = 2\} + d + \varepsilon_t \geq 0\}, \quad t \in \{1, 2\},$$

where $\varepsilon_1, \varepsilon_2$ are i.i.d. and follow a logistic distribution. We observe $Y_t := Y_t(D_t)$, where the binary treatment satisfies $D_1 = 0$ a.s., whereas $P(D_2 = 1) = 0.5$. Assume further that $\alpha = -0.5 + 1.5D_2$. Then, the true ATE at the second period satisfies

$$\begin{aligned}\Delta^{ATE} &= E[Y_2(1) - Y_2(0)] \\ &= 0.5 [E\Lambda(1 + 1 + 1) - \Lambda(1 + 1)] + 0.5 [\Lambda(-0.5 + 1 + 1) - \Lambda(-0.5 + 1)] \\ &\simeq 0.13.\end{aligned}$$

On the other hand, the FE linear model yields the simple difference-in-difference:

$$\begin{aligned}\Delta_{\text{lin}}^{ATE} &= E[Y_2 - Y_1 | D_2 = 1] - E[Y_2 - Y_1 | D_2 = 0] \\ &= \Lambda(1 + 1 + 1) - \Lambda(1) - [\Lambda(-0.5 + 1) - \Lambda(-0.5)] \\ &\simeq -0.02.\end{aligned}$$

This problem arises because, basically, the common trends condition is violated in this nonlinear model. One could argue that the difference-in-difference estimand identifies the ATT, not the ATE, under common trends. But note that the ATT is equal to 0.07 and thus also of opposite sign to $\Delta_{\text{lin}}^{ATE}$.

B Proofs of the identification results

B.1 Proposition 1

First, suppose that Assumption 2 does not hold. Then, there exists $\lambda \neq 0$ such that $X_1' \lambda = \dots = X_T' \lambda$ almost surely (a.s.). For any $v \in \mathbb{R}$, let $\alpha' = \alpha - v X_t' \lambda$ and $\beta = \beta_0 + v \lambda$. Then

$$Y_t = \mathbf{1} \{X_t' \beta + \alpha' + \varepsilon \geq 0\}.$$

This model satisfies Assumption 1. Thus, β_0 is not identified.

Now, assume that Assumption 2 holds. By the concavity of the logarithm and Jensen's inequality,

$$E(\ell_c(Y|X; \beta)) \leq E(\ell_c(Y|X; \beta_0))$$

with equality if and only if $\ell_c(Y|X; \beta) = \ell_c(Y|X; \beta_0)$ a.s. Assume that the latter holds.

Then, a.s.,

$$\exp[\ell_c(Y|X; \beta)] \mathbf{1}\{S = 1\} = \exp[\ell_c(Y|X; \beta_0)] \mathbf{1}\{S = 1\}. \quad (18)$$

Let us define

$$P_t(\beta) := \frac{\exp(X_t'\beta)}{\sum_{s=1}^T \exp(X_s'\beta)}.$$

Equality (18) is equivalent to

$$\sum_{t=1}^T (P_t(\beta) - P_t(\beta_0)) Y_t \prod_{s \neq t} (1 - Y_s) = 0 \quad \text{a.s.}$$

Because the variables $(Y_t \prod_{s \neq t} (1 - Y_s))_t$ are mutually exclusive, we have, for all t ,

$$(P_t(\beta) - P_t(\beta_0)) Y_t \prod_{s \neq t} (1 - Y_s) = 0 \quad \text{a.s.}$$

By taking the expectation with respect to X and noting that $P(Y_t \prod_{s \neq t} (1 - Y_s) | X) > 0$ a.s., we get, a.s., $P_t(\beta) = P_t(\beta_0)$. This in turn, implies that $X_t'(\beta - \beta_0)$ does not depend on t .

Hence, a.s.,

$$\sum_{t,s} [(X_t - X_s)'(\beta - \beta_0)]^2 = 0.$$

Taking the expectation, this implies that

$$(\beta - \beta_0)' E \left[\sum_{t,s} (X_t - X_s)(X_t - X_s)' \right] (\beta - \beta_0) = 0.$$

By Assumption 2, $\beta = \beta_0$. Hence, β_0 is identified and $\beta_0 = \arg \max_{\beta} E(\ell_c(Y|X, \beta))$.

We finally turn to the last result. If $S = 1$, we have $\partial \ell_c / \partial \beta(Y|X; \beta_0) = \sum_{t=1}^T X_t (Y_t - P_t(\beta_0))$.

Then, conditional on $S = 1$,

$$\begin{aligned} \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y|X; \beta_0) &= - \sum_{t=1}^T X_t P_t(\beta_0) \sum_{s=1}^T (X_t - X_s)' P_s(\beta_0) \\ &= - \frac{1}{2} \sum_{s,t} P_s(\beta_0) P_t(\beta_0) (X_t - X_s) (X_t - X_s)'. \end{aligned}$$

Let λ be such that $\lambda' \mathcal{I}_0 \lambda = 0$. Because $-\partial^2 \ell_c / \partial \beta \partial \beta'$ is positive semidefinite, we have

$$\begin{aligned} \lambda' \mathcal{I}_0 \lambda &\geq \lambda' E \left[\frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y|X; \beta_0) \mathbf{1}\{S = 1\} \right] \lambda \\ &= \frac{1}{2} \sum_{s,t} E \left[P_s(\beta_0) P_t(\beta_0) \mathbf{1}\{S = 1\} \lambda' (X_t - X_s) (X_t - X_s)' \lambda \right] \\ &= \frac{1}{2} \sum_{s,t} E \left[P_s(\beta_0) P_t(\beta_0) P(S = 1|X) [(X_t - X_s)' \lambda]^2 \right]. \end{aligned}$$

Hence, for all (s, t) , $P_s(\beta_0) P_t(\beta_0) P(S = 1|X) [(X_t - X_s)' \lambda]^2 = 0$ almost surely. Since $P(S = 1|X) > 0$, we have $(X_t - X_s)' \lambda = 0$ almost surely. In turn, this implies that

$$\lambda' E \left[\sum_{s,t} (X_t - X_s)(X_t - X_s)' \right] \lambda = 0.$$

Thus, by Assumption 2, $\lambda = 0$, proving that \mathcal{I}_0 is nonsingular.

B.2 Lemma 1

Remark that $\Delta = E(\Delta(X))$, where $\Delta(x)$ is defined by

$$\begin{aligned}\Delta(x) &:= \beta_{0k} E[\Lambda'(x'_T \beta_0 + \alpha) | X = x] \\ &= \beta_{0k} \int \Lambda'(x'_T \beta_0 + a) dF_{\alpha|X}(a|x).\end{aligned}\quad (19)$$

Let us first show that the identified set I_x of $\Delta(x)$ satisfies $I_x = E_x$, with

$$E_x := \left\{ E(r(X, S, \beta_0) | X = x) + \beta_{0k} c_0(x) \lambda_{T+1}(x, \beta_0) \int_0^1 u^{T+1} d\mu_x(u) : \mu_x \in \mathcal{D}(m_x) \right\}. \quad (20)$$

To this end, let us define $U = \Lambda(\alpha + x'_T \beta_0)$. Then, a change of variable in (19) and (2) shows that

$$\Delta(x) = \beta_{0k} \int_0^1 u(1-u) dF_{U|X}(u|x), \quad (21)$$

$$\begin{aligned}P(S = k | X = x) &= C_k(x, \beta_0) \exp(-kx'_T \beta_0) \\ &\times \int_0^1 \frac{u^k (1-u)^{T-k}}{\prod_{t=1}^{T-1} [1 + u(\exp((x_t - x_T)' \beta_0) - 1)]} dF_{U|X}(u|x).\end{aligned}\quad (22)$$

Remark that for all $t \in \{0, \dots, T\}$, $\sum_{k=t}^T \binom{T-t}{k-t} u^{k-t} (1-u)^{T-k} = 1$. Then, for such t ,

$$\begin{aligned}c_t(x) &= E \left[\mathbb{1}\{S \geq t\} \binom{T-t}{S-t} \exp(Sx'_T \beta_0) / C_S(x, \beta_0) | X = x \right] \\ &= \int_0^1 \frac{\sum_{k=t}^T \binom{T-t}{k-t} u^k (1-u)^{T-k}}{\prod_{t=1}^{T-1} [1 + u(\exp((x_t - x_T)' \beta_0) - 1)]} dF_{U|X}(u|x) \\ &= \int_0^1 \frac{u^t}{\prod_{t=1}^{T-1} [1 + u(\exp((x_t - x_T)' \beta_0) - 1)]} dF_{U|X}(u|x).\end{aligned}$$

Let μ_x be the measure having a density with respect to $F_{U|X}(\cdot|x)$ equal to

$$f(u) = \frac{\prod_{t=1}^{T-1} 1/[1 + u(\exp((x_t - x_T)' \beta_0) - 1)]}{\int_0^1 \prod_{t=1}^{T-1} 1/[1 + v(\exp((x_t - x_T)' \beta_0) - 1)] dF_{U|X}(v|x)}.$$

Then, by definition of $m_t(x)$, we obtain, for all $t \in \{0, \dots, T\}$

$$m_t(x) = \int_0^1 u^t d\mu_x(u), \quad (23)$$

so that $\mu_x \in \mathcal{D}(m(x))$. By a change of measure in (21) and by definition of $(\lambda_t(x, \beta_0))_{t=0, \dots, T+1}$, we also get

$$\Delta(x) = \beta_{0k} c_0(x) \int_0^1 \sum_{t=0}^{T+1} \lambda_t(x, \beta_0) u^t d\mu_x(u). \quad (24)$$

Then, using (23), $c_t(x) = c_0(x)m_t(x)$ and $E(r(X, S, \beta_0)|X = x) = \beta_{0k} \sum_{t=0}^T c_t(x)\lambda_t(x; \beta_0)$, we obtain $I_x \subset E_x$.

Conversely, for all $x \in \text{Supp}(X)$, define the probability measure G_x through its density with respect to μ_x :

$$\frac{dG_x}{d\mu_x}(u) = \frac{\prod_{t=1}^{T-1} [1 + u(\exp((x_t - x_T)' \beta_0) - 1)]}{\int_0^1 \prod_{t=1}^{T-1} [1 + u(\exp((x_t - x_T)' \beta_0) - 1)] d\mu_x(u)}.$$

Then, let $U|X = x \sim G_x$ and define the distribution of $\alpha|X = x$ as the distribution of $\Lambda^{-1}(U) - x'_T \beta_0$. Using again the definitions of $c_t(x)$ and $m_t(x)$, we obtain that (19) and (2) hold. This implies that $E_x \subset I_x$. Hence, $I_x = E_x$.

Now, remark that the distribution of $\alpha|X = x$ is not constrained by the distribution of $(\alpha|X = x')_{x' \neq x}$. As a result, the identified set of Δ is

$$\left\{ E \left[r(X, S, \beta_0) + \beta_{0k} c_0(X) \lambda_{T+1}(X, \beta_0) \int_0^1 u^{T+1} d\mu_X(u) \right] : \mu_X \in \mathcal{D}(m_X) \text{ a.s.} \right\}.$$

Hence, (6) holds. Finally, because $\mathcal{D}(m(x))$, and thus $\{\int u^{T+1} d\mu(u) : \mu \in \mathcal{D}(m(x))\}$, are convex, we get that the closure of $\{\int_0^1 u^{T+1} d\mu_x(u) : \mu_x \in \mathcal{D}(m_x)\}$ is $[\underline{q}_T(m(x)); \bar{q}_T(m(x))]$. The two equalities in (7) follow.

B.3 Proposition 2

Equation (9) follows from Point 1 of Theorem 1.4.3 in Dette and Studden (1997).

1. If $\underline{H}_T(m) \bar{H}_T(m) > 0$, by Point of Theorem 1.4.3 in Dette and Studden (1997), $m \in \text{Int } \mathcal{M}_T$. Then, by Theorem 1.2.7 in Dette and Studden (1997), $\underline{q}_T(m) < \bar{q}_T(m)$. Moreover, $(m_0, \dots, m_{T-1}) \in \text{Int } \mathcal{M}_{T-1}$. Thus, again by the second part of Theorem 1.4.3 in Dette and Studden (1997),

$$\underline{H}_{T-1}(m_0, \dots, m_{T-1}) > 0.$$

By expanding the determinant $\underline{H}_{T+1}(m, q)$ along its last column, we get that $q \mapsto \underline{H}_{T+1}(m, q)$ is linear and strictly increasing. The same reasoning applies to $\bar{q}_T(m)$.

2. If $\underline{H}_T(m) \bar{H}_T(m) = 0$, Theorem 1.4.3 in Dette and Studden (1997) implies that $m \in \partial \mathcal{M}_T$. Then, by Theorem 1.2.5 in Dette and Studden (1997), there is a unique distribution corresponding to m . Let U a random variable with this unique distribution. Note that $\underline{q}_T(m) = \bar{q}_T(m) = E(U^{T+1})$. Suppose first that T' is even and $\underline{H}_{T'}(m) = 0$. Then, there

exists a vector $\lambda = (\lambda_1, \dots, \lambda_{T'/2+1})'$ such that $\mathbb{H}_{T'}(m)\lambda = 0$. Hence, for all $i \in \{1, \dots, T'/2+1\}$, $\sum_{j=1}^{T'/2+1} \lambda_j m_{i+j-2} = 0$. Thus, for all $i \in \{0, \dots, T'/2\}$,

$$E \left[U^i \sum_{j=0}^{T'/2} \lambda_{j+1} U^j \right] = 0.$$

Hence, $E \left[\left(\sum_{j=0}^{T'/2} \lambda_{j+1} U^j \right)^2 \right] = 0$, which implies that almost surely, $\sum_{j=0}^{T'/2} \lambda_{j+1} U^j = 0$. In particular, for all $k \geq 1$ and letting $m_k := E(U^k)$ for $k > T$, we have

$$\sum_{j=1}^{T'/2+1} \lambda_j m_{j+k-2} = 0.$$

Since this holds for $k \in \{T+2-T', \dots, T+2-T'/2\}$, we have $\mathbb{H}_{T'}(m_{T+1-T'}, \dots, m_{T+1})\lambda = 0$.

Therefore, $\underline{H}_{T'}(m_{T+1-T'}, \dots, m_{T+1}) = 0$, with $\underline{q}_T(m) = \bar{q}_T(m) = m_{T+1}$.

The reasoning is the same if T' is odd and still $\underline{H}_{T'}(m) = 0$, with just one difference. Instead of having $E \left[\left(\sum_{j=0}^{T'/2-1} \lambda_{j+1} U^j \right)^2 \right] = 0$, we have

$$E \left[U \left(\sum_{j=0}^{(T'-1)/2-1} \lambda_{j+1} U^j \right)^2 \right] = 0.$$

But since $U \geq 0$, this still implies $U \left(\sum_{j=0}^{(T'-1)/2-1} \lambda_{j+1} U^j \right)^2 = 0$, and the rest of the proof is similar as above. When instead $\bar{H}_{T'}(m) = 0$ and T' is even, we have instead

$$E \left[U(1-U) \left(\sum_{j=0}^{T'/2-1} \lambda_{j+1} U^j \right)^2 \right] = 0,$$

implying again $U(1-U) \left(\sum_{j=0}^{T'/2-1} \lambda_{j+1} U^j \right)^2 = 0$. Finally, when $\bar{H}_{T'}(m) = 0$ and T' is odd, we have

$$E \left[(1-U) \left(\sum_{j=0}^{(T'-1)/2-1} \lambda_{j+1} U^j \right)^2 \right] = 0,$$

implying again $(1-U) \left(\sum_{j=0}^{(T'-1)/2-1} \lambda_{j+1} U^j \right)^2 = 0$.

B.4 Proposition 3

1. From Lemma 1, we have

$$\bar{\Delta} - \underline{\Delta} = |\beta_{0k}| \times E \left(c_0(X) \times |\lambda_{T+1}(X, \beta_0)| \times [\bar{q}_T(m(X)) - \underline{q}_T(m(X))] \right). \quad (25)$$

Now, by a result of Karlin and Shapley (1953), we have, for all $m \in \mathcal{M}_T$,

$$\bar{q}_T(m) - \underline{q}_T(m) \leq \frac{1}{4^T}. \quad (26)$$

The result follows.

2. Δ is point identified if and only if $\underline{\Delta} = \bar{\Delta}$ or equivalently if and only if

$$\beta_{0k} c_0(X) |\lambda_{T+1}(X, \beta_0)| \left(\bar{q}_T(m(X)) - \underline{q}_T(m(X)) \right) = 0 \text{ a.s.} \quad (27)$$

We have $c_0(X) > 0$ almost surely. Then, (27) holds if and only if $\beta_{0k} = 0$ or

$$|\lambda_{T+1}(X, \beta_0)| \left(\bar{q}_T(m(X)) - \underline{q}_T(m(X)) \right) = 0 \text{ a.s.}$$

We have $\lambda_{T+1}(x, \beta_0) = 0$ if and only if $(x_t - x_T)' \beta_0 = 0$ for some $t < T$. By Proposition 2, $\underline{q}_T(m) = \bar{q}_T(m)$ is equivalent to $\underline{H}_T(m) \times \bar{H}_T(m) = 0$. By the proof of Lemma 1 and, e.g., Theorem 1.2.5 in Dette and Studden (1997), this holds if and only if the index of the conditional distribution of $U = \Lambda(\alpha + x_T' \beta_0)$ is smaller than or equal to $T/2$. The index denotes here the number of support points, except that 0 and 1 are counted only as one-half. Now, because $1 < U < 0$ almost surely, the number of support points of $U|X = x$ (or equivalently $\alpha|X = x$), is equal to its index. The result follows.

B.5 Lemma 3

Define $\tilde{\Delta}(x) = \beta_{0k} \sum_{t=0}^T \lambda_t(x, \beta_0) c_t(x) + \beta_{0k} \lambda_{T+1}(x, \beta_0) \sum_{t=0}^T b_t c_t(x)$. Then

$$\tilde{\Delta}(x) = \beta_{0k} \sum_{t=0}^T \lambda_t(x, \beta_0) c_t(x) + \beta_{0k} c_0(x) \lambda_{T+1}(x, \beta_0) \int_0^1 P_T^*(u) d\mu_x(u),$$

which implies that $\tilde{\Delta} = E(\tilde{\Delta}(X))$. Then by (24), we obtain

$$\begin{aligned} \left| \tilde{\Delta}(x) - \Delta(x) \right| &\leq |\beta_{0k} \lambda_{T+1}(x, \beta_0)| c_0(x) \sup_{u \in [0,1]} |\mathbb{T}_{T+1}(u)| \\ &= \frac{|\beta_{0k} \lambda_{T+1}(x, \beta_0)| c_0(x)}{2^{T+1}} \sup_{u \in [-1,1]} |\mathbb{T}_{T+1}^c(u)| \\ &= \frac{|\beta_{0k} \lambda_{T+1}(x, \beta_0)| c_0(x)}{2 \times 4^T}. \end{aligned} \quad (28)$$

The last equality follows by standard properties of Chebyshev polynomials, see, e.g., Mason and Handscomb (2002). The first result follows by integration, using

$$|\tilde{\Delta} - \Delta| \leq E \left[\left| \tilde{\Delta}(X) - \Delta(X) \right| \right]. \quad (29)$$

By what precedes, $|\tilde{\Delta} - \Delta| = \bar{b}$ if and only if we have an equality in (28) for almost all x , and an equality in (29). The latter holds if and only if $\beta_{0k} = 0$, or the sign of $\lambda_{T+1}(X) \int_0^1 \mathbb{T}_{T+1}(u) dG(u|X)$ is constant. The former holds if and only if $\beta_{0k} = 0$ or the support of $G(\cdot|x)$ is either \mathcal{M}^+ or \mathcal{M}^- . The characterization of the equality $|\tilde{\Delta} - \Delta| = \bar{b}$ follows.

B.6 Lemma 6

We have $\Delta^{(3)} = E(\Delta^{(3)}(X))$, for $\Delta^{(3)}(x) = E[\partial P(Y_T \geq j_0|X, \alpha) / \partial X_{Tk} | X = x]$. Because the conditional distribution of $\alpha|X$ is not constrained, we have to find the sharp bounds of $\Delta^{(3)}(x)$ for each $x \in \text{Supp}(X)$.

Let Π_0 be the set of functions from $\{1, \dots, T\}$ into $\{0, 1, \dots, J-1\}$. First, we prove that the set of conditional probabilities $(P(S^\pi = s|X, U))_{s=0, \dots, T, \pi \in \Pi_0}$ is in one-to-one linear mapping with $P(Y = y|X, U)_{y \in \{0, 1, \dots, J-1\}^T}$. First,

$$P(Y = (J-1, \dots, J-1)|X, U) = P(S^{\bar{\pi}} = T|X, U)$$

with $\bar{\pi}$ such that $\bar{\pi}(t) = J-1$ for all t . Next, for any $y = (y_1, \dots, y_T) \in \{0, 1, \dots, J-1\}^T$, let $\pi \in \Pi_0$ be such that $\pi(t) = y_t$. Then:

$$P(Y = y|X, U) = P(S^\pi = T|X, U) - \sum_{\substack{y': y' \neq y \\ \forall t, y'_t \geq y_t}} P(Y = y'|X, U).$$

Hence, by a decreasing induction on y , using the lexicographic order, $P(Y = y|X, U)$ is a linear combination of the $(P(S^\pi = T|X, U))_{\pi \in \Pi_0}$. Conversely, $P(S^\pi = s|X, U) = \sum_{y \in \mathcal{Y}_s^\pi} P(Y = y|X, U)$ with $\mathcal{Y}_s^\pi = \{y \in \{0, 1, \dots, J-1\}^T : \sum_t \mathbf{1}\{y_t \geq \pi(t)\} = s\}$. This ensures that $(S^\pi)_{\pi \in \Pi_0}$ is exhaustive for U and

$$\begin{aligned} & \text{span} \{u \mapsto P(Y = y|X, U = u), y \in \{0, 1, \dots, J-1\}^T\} \\ &= \text{span} \{u \mapsto P(S^\pi = s|X, U = u), s = 0, \dots, T, \pi \in \Pi_0\}. \end{aligned}$$

Then the sharp lower bound (say) $\underline{\Delta}^{(3)}(x)$ satisfies:

$$\begin{aligned} \underline{\Delta}^{(3)}(x) &= \arg \min_{F_{U|X}(\cdot|x)} \int \frac{\partial P(Y_T \geq j_0|X = x, U = u)}{\partial X_{Tk}} dF_{U|X}(u|x) \\ & \text{s.t.} \int P(S^\pi = s|X = x, U = u) dF_{U|X}(u|x) = P(S^\pi = s|X = x), \\ & \pi \in \Pi_0, s \in \{0, 1, \dots, T\}. \end{aligned}$$

Thus, to conclude the proof, it suffices to show

$$\begin{aligned} & \text{span} \{u \mapsto P(S^\pi = s | X = x, U = u), \pi \in \Pi_0, s = 0, \dots, T\} \\ &= \text{span} \left\{ u \mapsto u^t w(u), t = 0, \dots, (J-1)T \right\}. \end{aligned} \quad (30)$$

For $\pi \in \Pi_0$, let $\mathcal{T}_+^\pi = \{t : \pi(t) > 0\}$ and for $k \leq |\mathcal{T}_+^\pi|$, let $\mathcal{D}_k^\pi = \{d \in \{0, 1\}^{\mathcal{T}_+^\pi} : \sum_{t \in \mathcal{T}_+^\pi} d_t = k\}$ and

$$C_k^\pi(x, \beta, \gamma) := \sum_{d \in \mathcal{D}_k^\pi} \exp \left(\sum_{t \in \mathcal{T}_+^\pi} d_t (x'_t \beta - \gamma_{\pi(t)}) \right).$$

For any $\pi \in \Pi_0$, let $s_0^\pi = T - |\mathcal{T}_+^\pi|$. We have

$$\begin{aligned} & P(S^\pi = s | X = x, U = u) \\ &= \frac{C_{s-s_0^\pi}^\pi(x, \beta_0, \gamma) \exp(-(s - s_0^\pi)(x'_T \beta_0 - \gamma_{j_0})) u^{s-s_0^\pi} (1-u)^{T-s}}{\prod_{t \in \mathcal{T}_+^\pi} [1 + u\rho(\pi(t), t, x)]} \mathbb{1} \{s_0^\pi \leq s \leq T\}. \end{aligned}$$

The Bernstein polynomials $\{u \mapsto u^{s-s_0^\pi} (1-u)^{T-s}, s = s_0^\pi, \dots, T\}$ are a basis of polynomials of degree lower than $|\mathcal{T}_+^\pi|$. Thus,

$$\begin{aligned} & \text{span} \{u \mapsto P(S^\pi = s | X = x, U = u), \pi \in \Pi_0, s = 0, \dots, T\} \\ &= \text{span} \left\{ u \mapsto \frac{u^t}{\prod_{t \in \mathcal{T}_+^\pi} [1 + u\rho(\pi(t), t, x)]}, \pi \in \Pi_0, t = 0, \dots, |\mathcal{T}_+^\pi| \right\} \\ &\subset \text{span} \left\{ u \mapsto u^t w(u), t = 0, \dots, (J-1)T \right\}. \end{aligned} \quad (31)$$

Conversely, let \sim be the equivalence relation on $\{1, \dots, J-1\} \times \{1, \dots, T\}$ defined by:

$$(j, t) \sim (j', t') \Leftrightarrow \rho(j, t, x) = \rho(j', t', x).$$

Then let $[(j, t)]$ denote the equivalence class of (j, t) and let \mathcal{E} be a set of representatives of all the equivalence classes, except $[(j_0, T)]$. Let also $n(j, t) = |[(j, t)]|$. Using $\rho(j_0, T, x) = 0$ and partial fraction decompositions, we obtain

$$\begin{aligned} & \text{span} \left\{ u \mapsto u^t w(u), t = 0, \dots, (J-1)T \right\} \\ &\subset \text{span} \left\{ u \mapsto u^d, d = 0, \dots, n(j_0, T), u \mapsto (1 + u\rho(j, t', x))^{-d}, \right. \\ &\quad \left. (j, t') \in \mathcal{E}, d = 1, \dots, n(j, t') \right\}. \end{aligned} \quad (32)$$

Fix $(j, t') \in \mathcal{E}$, $d \in \{1, \dots, n(j, t')\}$ and let $(j_1, t_1), \dots, (j_d, t_d)$ denote d distinct elements of $[(j, t')]$. By definition of ρ , t_1, \dots, t_d are all distinct. Then, define $\pi \in \Pi_0$ as $\pi(t_i) = j_i$ for $i = 1, \dots, d$ and $\pi(t) = 0$ for $t \notin \{t_1, \dots, t_d\}$. Then:

$$\frac{1}{(1 + u\rho(j, t', x))^d} = \frac{1}{\prod_{t \in \mathcal{T}_+^\pi} [1 + u\rho(\pi(t), t, x)]}. \quad (33)$$

Next, fix $d \in \{0, \dots, n(j_0, T)\}$, and let $(j_1, t_1), \dots, (j_d, t_d)$ denote d distinct elements of $[(j_0, T)]$. Define $\pi \in \Pi_0$ exactly as above if $d > 0$ and $\pi(t) = 0$ for all t if $d = 0$. Using $\rho(j_i, t_i, x) = 0$ for $i = 1, \dots, d$ and the definition of \mathcal{T}_+^π , we obtain $d = |\mathcal{T}_+^\pi|$ and

$$u^d = \frac{u^d}{\prod_{t \in \mathcal{T}_+^\pi} [1 + u\rho(\pi(t), t, x)]}. \quad (34)$$

Using (32) (33) and (34) and then (31), we finally obtain

$$\begin{aligned} & \text{span} \left\{ u \mapsto u^t w(u), t = 0, \dots, (J-1)T \right\} \\ & \subset \text{span} \left\{ u \mapsto \frac{u^t}{\prod_{t \in \mathcal{T}_+^\pi} [1 + u\rho(\pi(t), t, x)]} \pi \in \Pi_0, t = 0, \dots, |\mathcal{T}_+^\pi| \right\} \\ & = \text{span} \left\{ u \mapsto P(S^\pi = s | X = x, U = u), \pi \in \Pi_0, s = 0, \dots, T \right\}. \end{aligned}$$

Equation (30) follows, and this concludes the proof.

Online Appendix

A Proofs of the asymptotic results

A.1 Estimation of (c_0, \dots, c_T) and m

A.1.1 Estimation of (c_0, \dots, c_T)

Let $\gamma_{0j}(x) = P(S = j|X = x)$ for $j = 0, \dots, T$. The functions $(c_t)_{t=0\dots T}$ and $(\gamma_{0j})_{j=0\dots T}$ are related through

$$(c_0(x), \dots, c_T(x))' = \Gamma \left(\frac{\gamma_{00}(x) \exp(0 \times x'_T \beta_0)}{C_0(x, \beta_0)}, \dots, \frac{\gamma_{0T}(x) \exp(T \times x'_T \beta_0)}{C_T(x, \beta_0)} \right)', \quad (35)$$

where Γ is a square matrix of size $T + 1$ with coefficients $\Gamma_{ij} = \binom{T-i}{j-i} \mathbb{1}\{i \leq j\}$ for $i, j = 1, \dots, T + 1$. We first estimate $\gamma_0 := (\gamma_{00}, \dots, \gamma_{0T})$ nonparametrically. We use local polynomial estimators of order ℓ to avoid boundary effects. Let K denote a kernel function and for a given $0 \leq j \leq T$, define

$$\hat{a}^j(x) := \operatorname{argmin}_a \sum_{i=1}^n K \left(\frac{X_i - x}{h_n} \right) \left(\mathbb{1}\{S_i = j\} - \sum_{|b| \leq \ell} a_b (X_i - x)^b \right)^2, \quad (36)$$

where, in this definition, for $b \in \mathbb{N}^{pT}$, $|b| = \sum_{j=1}^{pT} b_j$ and $x^b = x_1^{b_1} \dots x_{pT}^{b_{pT}}$. The estimator of $\gamma_{0j}(x)$ is then $\hat{\gamma}_j(x) = \hat{a}_0^j(x)$. Our estimator for $c_t(x)$, $\hat{c}_t(x)$, uses (35), replacing γ_0 and β_0 with their estimators.

A.1.2 Estimation of m

Given its definition, a natural estimator of m is

$$\tilde{m}(x) = \left(1, \frac{\hat{c}_1(x)}{\hat{c}_0(x)}, \dots, \frac{\hat{c}_T(x)}{\hat{c}_0(x)} \right).$$

However, this estimator may not satisfy $\tilde{m}(x) \in \mathcal{M}_T$. This is especially the case if $m(x)$ is at the boundary of \mathcal{M}_T , or for a “large” T , because the volume of \mathcal{M}_T decreases very quickly with T (Karlin and Shapley, 1953). In our simulations, this already occurs with $T = 3$ and $n = 1,000$, even if $m(x)$ is in the interior of \mathcal{M}_T . That $\tilde{m}(x) \notin \mathcal{M}_T$ is an issue because then $\underline{q}_T(\tilde{m}(x))$ and $\bar{q}_T(\tilde{m}(x))$ are undefined. We thus consider another estimator \hat{m} such that $\hat{m}(x) \in \mathcal{M}_T$.

To this end, we rely on Proposition 2. For any $(m_t)_{t \geq 0}$ and $t \in \{0, \dots, T\}$, let $m_{\rightarrow t} = (m_0, \dots, m_t)$. The idea of the estimator is to use the first elements of $\widetilde{m}(x)$, until $\widetilde{m}_t(x)$ falls too close to $\underline{q}_{t-1}(\widetilde{m}_{\rightarrow t-1}(x))$ or $\overline{q}_{t-1}(\widetilde{m}_{\rightarrow t-1}(x))$. In such a case, we simply replace $\widetilde{m}_t(x)$ by $\underline{q}_{t-1}(\widetilde{m}_{\rightarrow t-1}(x))$ or $\overline{q}_{t-1}(\widetilde{m}_{\rightarrow t-1}(x))$. We finally complete the vector using the second part of Proposition 2.

Specifically, let c_n be a sequence tending to 0 at a rate specified later and define

$$\widehat{I}(x) := \max \left\{ t \in \{1, \dots, T\} : \underline{H}_t(\widetilde{m}_{\rightarrow t}(x)) \times \overline{H}_t(\widetilde{m}_{\rightarrow t}(x)) > c_n \right\}.$$

with the convention that $\max \emptyset = 0$. We then let

$$\widehat{m}_{\rightarrow \widehat{I}(x)}(x) := \widetilde{m}_{\rightarrow \widehat{I}(x)}(x).$$

If $\widehat{I}(x) = T$, $\widehat{m}(x)$ is fully defined. Otherwise, we complete $\widehat{m}(x)$ by first letting

$$\widehat{m}_{\widehat{I}(x)+1}(x) := \begin{cases} \underline{q}_{\widehat{I}(x)}(\widetilde{m}_{\rightarrow \widehat{I}(x)}(x)) & \text{if } \underline{H}_{\widehat{I}(x)+1}(\widetilde{m}_{\rightarrow \widehat{I}(x)+1}(x)) < c_n^{1/2}, \\ \overline{q}_{\widehat{I}(x)}(\widetilde{m}_{\rightarrow \widehat{I}(x)}(x)) & \text{otherwise.} \end{cases}$$

Next, if $\widehat{I}(x) + 1 < T$, by construction, we have

$$\underline{H}_{\widehat{I}(x)+1}(\widehat{m}_{\rightarrow \widehat{I}(x)+1}(x)) \times \overline{H}_{\widehat{I}(x)+1}(\widehat{m}_{\rightarrow \widehat{I}(x)+1}(x)) = 0.$$

Then, applying Part 2 of Proposition 2, we construct by induction the unique possible moments $\widehat{m}_{\widehat{I}(x)+2}, \dots, \widehat{m}_T$ that are compatible with $\widehat{m}_{\rightarrow \widehat{I}(x)+1}(x)$. By construction, the corresponding vector $\widehat{m}(x)$ belongs to \mathcal{M}_T .

A.2 Consistency of $(\widehat{\Delta}, \widehat{\Delta})$

We first establish consistency of the estimated bounds under the following conditions.

Assumption 5

1. The variables $(X_i, \alpha_i, \varepsilon_{i1}, \dots, \varepsilon_{iT})$ are i.i.d across i .
2. $\text{Supp}(X)$ is a compact set and $\beta_0 \in \Theta$, where Θ is a compact set.

Assumption 6

1. X admits a density f_X with respect to the Lebesgue measure on \mathbb{R}^{pT} . f_X is C^1 and bounded away from 0 on $\text{Supp}(X)$,

2. (a) γ_0 is $C^{\ell+2}$ on $\text{Supp}(X)$,
(b) $\ell \geq pT/2$,
3. K is a Lipschitz density on \mathbb{R}^{pT} with compact support including a neighborhood of 0,
4. (a) $h_n \rightarrow 0$ and $nh_n^{pT}/\ln n \rightarrow \infty$ as $n \rightarrow \infty$,
(b) $nh_n^{2(\ell+1)} \rightarrow 0$ and $n[h_n^{pT}/\ln n]^3 \rightarrow \infty$ as $n \rightarrow \infty$,
5. $[(\ln n/(nh_n^{pT}))^{1/2} + h_n^{\ell+1}]/c_n \rightarrow 0$ as $n \rightarrow \infty$.

Assumption 5 is sufficient for $\hat{\beta}$ to be consistent. Also, Assumptions 5 and 6.1 to 6.4 guarantee that $\hat{\gamma}$ converges uniformly to γ_0 over the support of X at a rate at least δ_n , with $\delta_n := (\ln n/(nh_n^{pT}))^{1/2} + h_n^{\ell+1}$. Note that Assumption 6.2 is in fact a smoothness condition on the distribution of α given X . For instance, if this distribution is discrete and both support points and weighting probabilities are $C^{\ell+2}$ as functions of X on $\text{Supp}(X)$, then Assumption 6.2 holds. Last, Assumption 6.5 imposes condition on the rate of convergence on the smoothing parameter c_n used to estimate $\widehat{m}(X_i)$.

Theorem 3 *Suppose that Assumptions 1, 5, 6.1, 6.2a, 6.3, 6.4a and 6.5 hold. Then*

$$(\widehat{\Delta}, \widehat{\bar{\Delta}}) \xrightarrow{P} (\underline{\Delta}, \overline{\Delta}).$$

Proof: We focus on $\widehat{\Delta}$ hereafter, as the proof for the upper bound is the same. The proof proceeds in three steps. First, we show the uniform consistency of \widetilde{m} over $\text{Supp}(X)$. Second, we prove that \widehat{m} is also uniformly consistent. Finally, we show the consistency of $\widehat{\Delta}$. For any function f from \mathcal{D} to \mathbb{R}^q , we let $\|f\|_\infty = \sup_{x \in \mathcal{D}} \|f(x)\|$, where $\|\cdot\|$ denotes the Euclidean norm on \mathbb{R}^q . We denote by C , \underline{C} and \overline{C} generic constants subject to change from one line to the next.

Step 1: Uniform consistency of \widetilde{m}

Remark that the Under Assumptions 1-5, $P \in \mathcal{P}'$ as defined in Lemma 8, with $\underline{\sigma} = 0$ and some appropriate \overline{M} and A . Then, by Lemma 8, $\widehat{\beta} \xrightarrow{P} \beta_0$. Moreover, $\text{Supp}(X)$ is compact. Then, for all $(k, x, \beta) \in \{0, \dots, T\} \times \text{Supp}(X) \times \{\beta_0, \widehat{\beta}\}$, with probability approaching one (wpao),

$$\overline{C} > C_k(x, \beta) \geq \underline{C} > 0, \quad \overline{C} > \exp(kx'_T \beta) \geq \underline{C}. \quad (37)$$

Moreover, by definition of $c_k(\gamma, x, \beta)$,

$$\begin{aligned} & \left\| c(\hat{\gamma}, x, \hat{\beta}) - c(\gamma_0, x, \beta_0) \right\| \\ & \leq C \left\| \left(\frac{\hat{\gamma}_0(x) e^{0 \times x'_T \hat{\beta}}}{C_0(x, \hat{\beta})}, \dots, \frac{\hat{\gamma}_T(x) e^{T \times x'_T \hat{\beta}}}{C_T(x, \hat{\beta})} \right)' - \left(\frac{\gamma_{00}(x) e^{0 \times x'_T \beta_0}}{C_0(x, \beta_0)}, \dots, \frac{\gamma_{0T}(x) e^{T \times x'_T \beta_0}}{C_T(x, \beta_0)} \right)' \right\|. \end{aligned}$$

Fix $0 \leq k \leq T$. Then wpao,

$$\left| \frac{\hat{\gamma}_k(x) e^{k \times x'_T \hat{\beta}}}{C_k(x, \hat{\beta})} - \frac{\gamma_{0k}(x) e^{k \times x'_T \beta_0}}{C_k(x, \beta_0)} \right| \leq \frac{|\hat{\gamma}_k(x) - \gamma_{0k}(x)| e^{k \times x'_T \hat{\beta}}}{C_k(x, \hat{\beta})} + \gamma_{0k}(x) \left| \frac{e^{k \times x'_T \beta_0}}{C_k(x, \beta_0)} - \frac{e^{k \times x'_T \hat{\beta}}}{C_k(x, \hat{\beta})} \right|.$$

The derivatives of $\beta \mapsto e^{k \times x'_T \beta} / C_k(x, \beta)$ are uniformly bounded over β and $x \in \text{Supp}(X)$ wpao. Combined with (37), this implies that wpao,

$$\left| \frac{\hat{\gamma}_k(x) e^{k \times x'_T \hat{\beta}}}{C_k(x, \hat{\beta})} - \frac{\gamma_{0k}(x) e^{k \times x'_T \beta_0}}{C_k(x, \beta_0)} \right| \leq C \left(|\hat{\gamma}_k(x) - \gamma_{0k}(x)| + \|\beta_0 - \hat{\beta}\| \right).$$

Therefore, recalling that $\hat{c} = c(\hat{\gamma}, x, \hat{\beta})$,

$$\|\hat{c} - c\|_\infty \leq C \left(\|\hat{\gamma} - \gamma_0\|_\infty + \|\beta_0 - \hat{\beta}\| \right). \quad (38)$$

Next, by (35), (37) and $\sum_{j=0}^T \gamma_{0j}(x) = 1$, for all $(x, \beta) \in \text{Supp}(X) \times \{\beta_0, \hat{\beta}\}$, wpao,

$$c_0(\gamma_0, x, \beta) > \sum_{j=0}^T \gamma_{0j}(x) \underline{C} / \bar{C} = \underline{C} / \bar{C}. \quad (39)$$

The conditions in Theorem 6 of Masry (1996) hold under Assumptions 1-6. Thus, $\hat{\gamma}$ is uniformly consistent. Given (38) and (39), we then have $c_0(\hat{\gamma}, x, \hat{\beta}) > C$ wpao.

By definition of \tilde{m} , we have, for all $(k, x) \in \{0, \dots, T\} \times \text{Supp}(X)$, wpao,

$$\begin{aligned} |\tilde{m}_k(x) - m_k(x)| & \leq \frac{1}{c_0(\gamma_0, x, \beta_0)} |c_k(\hat{\gamma}, x, \hat{\beta}) - c_k(\gamma, x, \beta_0)| \\ & \quad + \frac{1}{\hat{c}_0^2} |c_k(\hat{\gamma}, x, \hat{\beta})| \times |c_0(\hat{\gamma}, x, \hat{\beta}) - c_0(\gamma_0, x, \beta_0)| \end{aligned} \quad (40)$$

where $\hat{c}_0^2 \geq \min(c_0(\gamma_0, x, \beta)^2, c_0(\hat{\gamma}, x, \hat{\beta})^2) > C$ and $|c_k(\hat{\gamma}, x, \hat{\beta})|$ is bounded in probability in view of (38). Therefore, by (40) and, again, (38),

$$\begin{aligned} \|\tilde{m} - m\|_\infty & \leq C \left(\|c - \hat{c}\|_\infty + \|c_0 - \hat{c}_0\|_\infty \right) \\ & \leq C \left(\|\hat{\gamma} - \gamma_0\|_\infty + \|\beta_0 - \hat{\beta}\| \right). \end{aligned}$$

The result follows by uniform consistency of $\hat{\gamma}$ and consistency of $\hat{\beta}$.

Step 2: Uniform consistency of \widehat{m}

We drop the dependence in x and write m, \widehat{m}, \dots instead of $m(x), \widehat{m}(x), \dots$ to simplify notation as all the statements to follow hold uniformly over $x \in \text{Supp}(X)$. We start by showing that for all $\epsilon > 0$ and for n large enough, if $\widehat{I} = t$ then $|m_{t+1} - \widehat{m}_{t+1}| \leq 2\epsilon$. A first step is to notice that for all $\epsilon > 0$, there exists N_0 such that $n \geq N_0$, $m \in \mathcal{M}_T$ and $\underline{H}_{t+1}(m_1, \dots, m_{t+1}) < 2c_n^{1/2}$ implies $|m_{t+1} - \underline{q}_t(m_{\rightarrow t})| = |m_{t+1} - \widehat{m}_{t+1}| \leq \epsilon$. To see this, suppose the contrary. Then there exists $\epsilon > 0$ and a subsequence $(m^{\phi(n)}) \in \mathcal{M}_T^{\mathbb{N}}$ such that for all $n \in \mathbb{N}$,

$$0 < \underline{H}_{t+1}(m_1^{\phi(n)}, \dots, m_{t+1}^{\phi(n)}) < 2c_{\phi(n)}^{1/2} \quad \text{and} \quad |m_{t+1}^{\phi(n)} - \underline{q}_t(m_{\rightarrow t}^{\phi(n)})| > \epsilon.$$

The set \mathcal{M}_T is compact, thus there exists a further subsequence $(m^{\phi'(n)})$ converging to some m^0 . By continuity of the functions \underline{q}_t and \underline{H}_{t+1} , we have $\underline{H}_{t+1}(m_1^0, \dots, m_{t+1}^0) = 0$ and $|m_{t+1}^0 - \underline{q}_t(m_{\rightarrow t}^0)| \geq \epsilon > 0$. But this contradicts Proposition 2. The same result holds for \overline{H}_{t+1} .

Define C' a Lipschitz constant valid for both \overline{H}_t and \underline{H}_t for all $t \leq T$. Take $\epsilon > 0$, N_1 larger than the corresponding N_0 and such that $n > N_1$ implies

$$\begin{aligned} \forall t \leq T, \|m_{\rightarrow t} - m'_{\rightarrow t}\| \leq \delta_n &\Rightarrow \|q_t(m_{\rightarrow t}) - q_t(m'_{\rightarrow t})\| \leq \epsilon, \\ \delta_n \leq \epsilon \quad \text{and} \quad \delta_n &\leq c_n^{1/2}/C'. \end{aligned}$$

Then for $n \geq N_1$, for all $t \leq T$, if $\widetilde{m}_{\rightarrow t} \in \mathcal{M}_t$ and $0 < \underline{H}_{t+1}(\widetilde{m}_1, \dots, \widetilde{m}_t, \widetilde{m}_{t+1}) < c_n^{1/2}$ then wpa0, $0 \leq \underline{H}_{t+1}(m_1, \dots, m_{t+1}) \leq c_n^{1/2} + C' \times \delta_n \leq 2c_n^{1/2}$. Thus if $\widehat{I} = t$ and we are in the case $0 < \underline{H}_{t+1}(\widetilde{m}_1, \dots, \widetilde{m}_t, \widetilde{m}_{t+1}) < c_n^{1/2}$ then wpa0

$$\begin{aligned} |m_{t+1} - \widehat{m}_{t+1}| &= |m_{t+1} - \underline{q}_t(\widetilde{m}_{\rightarrow t})| \leq |m_{t+1} - \underline{q}_t(m_{\rightarrow t})| + |\underline{q}_t(m_{\rightarrow t}) - \underline{q}_t(\widetilde{m}_{\rightarrow t})| \\ &\leq 2\epsilon. \end{aligned}$$

The same result holds for \overline{H}_{t+1} . We can then proceed by induction, as

$$\begin{aligned} |m_{t+2} - \widehat{m}_{t+2}| &= |m_{t+2} - \underline{q}_{t+1}(\widetilde{m}_{\rightarrow t}, \widehat{m}_{t+1})| \\ &\leq |\overline{q}_{t+1}(m_{\rightarrow t+1}) - \underline{q}_{t+1}(m_{\rightarrow t+1})| + |\underline{q}_{t+1}(m_{\rightarrow t+1}) - \underline{q}_{t+1}(\widetilde{m}_{\rightarrow t}, \widehat{m}_{t+1})| \\ &\leq |\overline{q}_{t+1}(m_{\rightarrow t+1}) - \overline{q}_{t+1}(\widetilde{m}_{\rightarrow t}, \widehat{m}_{t+1})| + |\underline{q}_{t+1}(\widetilde{m}_{\rightarrow t}, \widehat{m}_{t+1}) - \underline{q}_{t+1}(m_{\rightarrow t+1})| \\ &\quad + |\underline{q}_{t+1}(m_{\rightarrow t+1}) - \underline{q}_{t+1}(\widetilde{m}_{\rightarrow t}, \widehat{m}_{t+1})| \end{aligned}$$

where the last inequality follows from $\underline{q}_{t+1}(\widetilde{m}_{\rightarrow t}, \widehat{m}_{t+1}) = \overline{q}_{t+1}(\widetilde{m}_{\rightarrow t}, \widehat{m}_{t+1})$. Using recursively the uniform continuity of $\overline{q}_{t'}$ and $\underline{q}_{t'}$ as functions of $m_{\rightarrow t}$ over \mathcal{M}_t and properly adjusting recursive choices of the ϵ 's, we then obtain the uniform convergence of $\widehat{m} - m$ to 0.

Step 3: Consistency of the lower bound

Let $\hat{A}(x) := \hat{\beta}_k \hat{c}_0(x) \lambda_{T+1}(x, \hat{\beta}) \hat{q}_T(\hat{m}(x))$ and $\hat{B}(x) := \hat{\beta}_k \hat{c}_0(x) \lambda_{T+1}(x, \hat{\beta}) \hat{q}_T(\hat{m}(x))$. By Equation (11), $\hat{\Delta}$ satisfies

$$\hat{\Delta} = \frac{1}{n} \sum_{i=1}^n r(X_i, S_i, \hat{\beta}) + \frac{1}{n} \sum_{i=1}^n \min(\hat{A}(X_i), \hat{B}(X_i)). \quad (41)$$

Since λ_t is infinitely differentiable for all $t \leq T$, $\text{Supp}(X)$ is compact and $\hat{\beta}$ is consistent, $x \mapsto \lambda_t(x, \hat{\beta})$ converges uniformly in probability to $x \mapsto \lambda_t(x, \beta_0)$. The same holds for $(x, s) \mapsto C_s(x, \hat{\beta})$ and $(x, s) \mapsto \exp(sX'_T \hat{\beta})$. Because wpa0 $C_s(x, \hat{\beta}) > \underline{C}$ for all $x \in \text{Supp}(X)$ and $s \leq T$, $(x, s) \mapsto r(x, s, \hat{\beta})$ converges uniformly in probability to $(x, s) \mapsto r(x, s, \beta_0)$. Then, by the triangle inequality and the law of large numbers (LLN),

$$\frac{1}{n} \sum_{i=1}^n r(X_i, S_i, \hat{\beta}) \xrightarrow{P} E(r(X, S, \beta_0)).$$

Next, let us show the convergence in probability of the second term in (41). The functions \bar{q}_T and q_T are continuous and thus uniformly continuous over the compact set \mathcal{M}_T . Then, by Step 2 and since by construction $(m(x), \hat{m}(x)) \in \mathcal{M}_T^2$, $x \mapsto q_T(\hat{m}(x))$ and $x \mapsto \bar{q}_T(\hat{m}(x))$ converge uniformly in probability to $x \mapsto q_T(m(x))$ and $x \mapsto \bar{q}_T(m(x))$ respectively. Thus, the functions \hat{A} and \hat{B} converge uniformly in probability to their corresponding limits, which we write A and B . Since $\min(A, B) = (A + B - |A - B|)/2$, $x \mapsto \min(\hat{A}(x), \hat{B}(x))$ also converges uniformly in probability to A and B . Then, by the triangle inequality and the LLN,

$$\frac{1}{n} \sum_{i=1}^n \min(\hat{A}(X_i), \hat{B}(X_i)) \xrightarrow{P} E(\min(A(X), B(X))).$$

The result follows.

A.3 Theorem 1

Before proving Theorem 1, we introduce additional notation. First, for any vector of functions $\gamma = (\gamma_0, \dots, \gamma_T)$, let

$$(c_0(\gamma, x, \beta), \dots, c_T(\gamma, x, \beta))' := \Gamma \left(\frac{\gamma_0(x) \exp(0 \times x'_T \beta)}{C_0(x, \beta)}, \dots, \frac{\gamma_T(x) \exp(T \times x'_T \beta)}{C_T(x, \beta)} \right)',$$

where Γ is a square matrix of size $T + 1$ with coefficients $\Gamma_{ij} = \binom{T-i}{j-i} \mathbb{1}\{i \leq j\}$ for $i, j = 1, \dots, T + 1$. Note that $\hat{c}_t(x) = c_t(\hat{\gamma}, x, \hat{\beta})$. Then, with I defined in Assumption 3, let

$$m(\gamma, x, \beta) := \left(1, \frac{c_1(\gamma, x, \beta)}{c_0(\gamma, x, \beta)}, \dots, \frac{c_I(\gamma, x, \beta)}{c_0(\gamma, x, \beta)} \right),$$

so that $m(\gamma_0, x, \beta_0) = m_{\rightarrow I}(x)$ and $m(\hat{\gamma}, x, \hat{\beta}) = \tilde{m}_{\rightarrow I}(x)$. Now, if $I = T$, we let, with a slight abuse of notation, $\underline{q}_T(\gamma, x, \beta) = \underline{q}_T(m(\gamma, x, \beta))$. If $I < T$, by Assumption 3 and Proposition 2, $m_{I+1}(x) = \underline{q}_I(m_{\rightarrow I}(x))$ or $m_{I+1}(x) = \bar{q}_I(m_{\rightarrow I}(x))$. Then, by Proposition 2 again and a straightforward induction, we can define $m_t(x)$ for $t \in \{I+1, \dots, T\}$ as a function of $m_{\rightarrow I}(x)$. We let $\text{Ext}(\cdot)$ denote the corresponding extension function. Then $m(x) = \text{Ext}(m_{\rightarrow I}(x))$. Finally, we let (with again a slight abuse of notation)

$$\underline{q}_T(\gamma, x, \beta) := \underline{q}_T(\text{Ext}(m(\gamma, x, \beta))).$$

We define similarly $\bar{q}_T(\gamma, x, \beta)$. Note that $\underline{q}_T(\cdot, \cdot, \cdot)$ and $\bar{q}_T(\cdot, \cdot, \cdot)$ depend on the unknown I , and when $I < T$, on the true function m , since the definition of E involves this true function. However, we show in the proof of Theorem 1 below that with probability approaching one, $\underline{q}_T(\hat{m}(x)) = \underline{q}_T(\hat{\gamma}, x, \hat{\beta})$.

Then, we also define

$$\begin{aligned} \underline{h}(x, s, \gamma, \beta) &= r(x, s, \beta) + \beta_k c_0(\gamma, x, \beta) \lambda_{T+1}(x, \beta) \left[\underline{q}_T(\gamma, x, \beta) \mathbb{1} \{ \lambda_{T+1}(x, \beta) > 0 \} \right. \\ &\quad \left. + \bar{q}_T(\gamma, x, \beta) \mathbb{1} \{ \lambda_{T+1}(x, \beta) < 0 \} \right], \\ \bar{h}(x, s, \gamma, \beta) &= r(x, s, \beta) + \beta_k c_0(\gamma, x, \beta) \lambda_{T+1}(x, \beta) \left[\bar{q}_T(\gamma, x, \beta) \mathbb{1} \{ \lambda_{T+1}(x, \beta) > 0 \} \right. \\ &\quad \left. + \underline{q}_T(\gamma, x, \beta) \mathbb{1} \{ \lambda_{T+1}(x, \beta) < 0 \} \right]. \end{aligned}$$

Note that $\underline{h}(x, s, \gamma, \beta)$ (and similarly $\bar{h}(x, s, \gamma, \beta)$) depends on γ only through $\gamma(x)$. Also, \underline{h} is differentiable with respect to β and the vector $\gamma(x)$. We denote its corresponding partial derivatives as $D_\beta \underline{h}(x, s, \gamma, \beta)$ and $D_\gamma \underline{h}(x, s, \gamma, \beta)$.

The influence functions of $\hat{\underline{\Delta}}$ and $\hat{\bar{\Delta}}$ are:

$$\begin{aligned} \underline{\psi}_i &= \underline{h}(X_i, S_i, \gamma_0, \beta_0) - E[\underline{h}(X, S, \gamma_0, \beta_0)] + E[D_\beta \underline{h}(X, S, \gamma_0, \beta_0)]' \phi_i \\ &\quad + D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)' [Z_i - \gamma_0(X_i)], \end{aligned} \tag{42}$$

$$\begin{aligned} \bar{\psi}_i &= \bar{h}(X_i, S_i, \gamma_0, \beta_0) - E[\bar{h}(X, S, \gamma_0, \beta_0)] + E[D_\beta \bar{h}(X, S, \gamma_0, \beta_0)]' \phi_i \\ &\quad + D_\gamma \bar{h}(X_i, S_i, \gamma_0, \beta_0)' [Z_i - \gamma_0(X_i)], \end{aligned} \tag{43}$$

where $Z_i = (\mathbb{1} \{S_i = 0\}, \dots, \mathbb{1} \{S_i = T\})'$ and $\phi_i = \mathcal{I}_0^{-1} \partial \ell_c / \partial \beta(Y_i | X_i; \beta_0)$ is the influence function of $\hat{\beta}$. We let Σ denote the variance-covariance matrix of $(\underline{\psi}, \bar{\psi})$. We introduce $\hat{\phi}_i$ as the sample analog of ϕ_i and similarly, sample analogs of $\underline{\psi}_i$ and $\bar{\psi}_i$ are

$$\hat{\underline{\psi}}_i = \underline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}) - \frac{1}{n} \sum_{j=1}^n \underline{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) + \left(\frac{1}{n} \sum_{j=1}^n D_\beta \underline{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) \right)' \hat{\phi}_i$$

$$+ D_\gamma \underline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta})' [Z_i - \hat{\gamma}(X_i)] \quad (44)$$

$$\begin{aligned} \hat{\psi}_i &= \bar{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}) - \frac{1}{n} \sum_{j=1}^n \bar{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) + \left(\frac{1}{n} \sum_{j=1}^n D_\beta \bar{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) \right)' \hat{\phi}_i \\ &+ D_\gamma \bar{h}(X_i, S_i, \hat{\gamma}, \hat{\beta})' [Z_i - \hat{\gamma}(X_i)] \end{aligned} \quad (45)$$

We finally estimate Σ by $\hat{\Sigma} = \frac{1}{n} \sum_{i=1}^n (\hat{\psi}_i, \hat{\psi}_i)' (\hat{\psi}_i, \hat{\psi}_i)$.

Proof of Theorem 1:

Part 1: asymptotic approximation and normality when $\beta_{0k} \neq 0$.

We show the linear approximation here. The convergence in distribution then follows directly from the central limit theorem (CLT). Also, we focus on $\hat{\Delta}$: the proofs for $\hat{\Delta}$ is similar. We prove the result in three steps. First, we show that wpao, $\hat{I}(X_i) = I$ for all i . Second, we prove that

$$\hat{\Delta} = \frac{1}{n} \sum_{i=1}^n \underline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}) \mathbf{1} \{ \hat{\beta}_k \geq 0 \} + \bar{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}) \mathbf{1} \{ \hat{\beta}_k < 0 \} + o_P(n^{-1/2}). \quad (46)$$

These two steps are valid whatever the sign of β_{0k} . Finally, we show the result in the third step, assuming that $\beta_{0k} > 0$; the proof when $\beta_{0k} < 0$ follows similarly.

Step 1: wpao, $\hat{I}(X_i) = I$ for all i .

First, let $T_t(m) := \underline{H}_t(m) \bar{H}_t(m)$. By definition of $\hat{I}(x)$ and I ,

$$\hat{I}(x) > I \Rightarrow T_{I+1}(m(x)) = 0 \text{ and } T_{I+1}(\tilde{m}(x)) > c_n.$$

Moreover,

$$T_{I+1}(\tilde{m}(x)) > c_n \Rightarrow \underline{H}_{I+1}(\tilde{m}(x)) \bar{H}_{I+1}(\tilde{m}(x)) - \underline{H}_{I+1}(m(x)) \bar{H}_{I+1}(m(x)) > c_n,$$

The functions \underline{H}_{I+1} and \bar{H}_{I+1} are infinitely differentiable on the compact set \mathcal{M}_{I+1} . The product of these functions is thus Lipschitz on this set. By induction, $(\tilde{m}_0(x), \dots, \tilde{m}_{I+1}(x))$ lies in \mathcal{M}_{I+1} . Indeed, otherwise we would not have $\hat{I}(x) > I + 1$. This implies that for any given value $x \in \text{Supp}(X)$, wpao

$$T_{I+1}(\tilde{m}(x)) > c_n \Rightarrow \|\tilde{m}(x) - m(x)\| > Cc_n.$$

Because $\delta_n/c_n \rightarrow 0$, this cannot occur for any $x \in \text{Supp}(X)$, wpao. Hence, wpao, $\hat{I}(X_i) \leq I$ for all i .

Now, assume that $\hat{I}(x) < I$ for some $x \in \text{Supp}(X)$. Then,

$$\exists k = \hat{I}(x) < I, \underline{H}_k(\tilde{m}(x))\overline{H}_k(\tilde{m}(x)) \leq c_n \text{ and } \underline{H}_k(m(x))\overline{H}_k(m(x)) > 0.$$

We know $m_{\rightarrow k}(x) \in \mathcal{M}_k^\epsilon$ for any $k \leq t$. Thus if $\underline{H}_k(m(x))\overline{H}_k(m(x)) > 2c_n$ then

$$\begin{aligned} \underline{H}_k(\tilde{m}(x))\overline{H}_k(\tilde{m}(x)) &> 2c_n - |\underline{H}_k(\tilde{m}(x))\overline{H}_k(\tilde{m}(x)) - \underline{H}_k(m(x))\overline{H}_k(m(x))| \\ &> 2c_n - C\|m - \tilde{m}\|_\infty, \end{aligned}$$

using again the Lipschitz property of the product $\underline{H}_k\overline{H}_k$ on \mathcal{M}_k . By $\delta_n/c_n \rightarrow 0$ and by $\|\tilde{m} - m\|_\infty = O_P(\delta_n)$, wpao and for n large enough we have

$$\exists N_0, \forall x, n \geq N_0, \underline{H}_k(m(x))\overline{H}_k(m(x)) > 2c_n \Rightarrow \underline{H}_k(\tilde{m}(x))\overline{H}_k(\tilde{m}(x)) > c_n,$$

or alternatively

$$\underline{H}_k(\tilde{m}(x))\overline{H}_k(\tilde{m}(x)) \leq c_n \Rightarrow \underline{H}_k(m(x))\overline{H}_k(m(x)) \leq 2c_n$$

while $k < I$. Thus, wpao, for $n \geq N_0$

$$\hat{I}(x) < I \Rightarrow \exists k = \hat{I}(x) < I, \underline{H}_k(m(x))\overline{H}_k(m(x)) \leq 2c_n.$$

But since $m_{\rightarrow k}(x) \in \mathcal{M}_k^\epsilon$ for all $x \in \text{Supp}(X)$, $\underline{H}_k\overline{H}_k$ is a continuous function and \mathcal{M}_k is a compact set, we know that there exists ϵ' such that for all x , $\underline{H}_k(m(x))\overline{H}_k(m(x)) > \epsilon'$ is strictly positive. This makes it impossible to have for $n \geq N_0$, $\underline{H}_k(m(x))\overline{H}_k(m(x)) \leq 2c_n$ for any x . Thus we get $\{i \in \{1, \dots, n\} : \hat{I}(X_i) < I\} = \emptyset$ wpao.

In conclusion, wpao, we have $\hat{I}(X_i) = I$ for all $i \in \{1, \dots, n\}$.

Step 2: (46) holds.

$P(\forall i \in \{1, \dots, n\}, \underline{q}_T(\widehat{m}(X_i)) = \underline{q}_T(\widehat{\gamma}, X_i, \widehat{\beta})) \rightarrow 1$ as $n \rightarrow \infty$. This in turn implies wpao

$$\begin{aligned} \widehat{\Delta} &= \frac{1}{n} \sum_{i=1}^n r(X_i, S_i, \widehat{\beta}) + \widehat{\beta}_k \widehat{c}_0(X_i) \lambda_{T+1}(X_i, \widehat{\beta}) \left[\underline{q}_T(\widehat{\gamma}, X_i, \widehat{\beta}) \mathbb{1} \left\{ \widehat{\beta}_k \lambda_{T+1}(X_i, \widehat{\beta}) \geq 0 \right\} \right. \\ &\quad \left. + \overline{q}_T(\widehat{\gamma}, X_i, \widehat{\beta}) \mathbb{1} \left\{ \widehat{\beta}_k \lambda_{T+1}(X_i, \widehat{\beta}) < 0 \right\} \right]. \end{aligned} \quad (47)$$

To obtain (46), we define the set $\mathcal{V}_0 = \{x \in \text{Supp}(X) \mid \lambda_{T+1}(x, \beta_0) \geq 0\}$ and $J_n := \widehat{\Delta} - \frac{1}{n} \sum_{i=1}^n \underline{h}(X_i, S_i, \widehat{\gamma}, \widehat{\beta}) \mathbb{1} \left\{ \widehat{\beta}_k \geq 0 \right\} + \overline{h}(X_i, S_i, \widehat{\gamma}, \widehat{\beta}) \mathbb{1} \left\{ \widehat{\beta}_k < 0 \right\}$. Note first that wpao

$$J_n = \frac{1}{n} \sum_{i=1}^n \widehat{\beta}_k \widehat{c}_0(X_i) \lambda_{T+1}(X_i, \widehat{\beta}) \left[\overline{q}_T(\widehat{m}(X_i)) \left(\mathbb{1} \left\{ \lambda_{T+1}(X_i, \widehat{\beta}) \geq 0 \right\} - \mathbb{1} \left\{ X_i \in \mathcal{V}_0 \right\} \right) \right]$$

$$\begin{aligned}
& + \underline{q}_T(\widehat{m}(X_i)) \left(\mathbf{1} \left\{ \lambda_{T+1}(X_i, \widehat{\beta}) < 0 \right\} - \mathbf{1} \left\{ X_i \in \mathcal{V}_0^c \right\} \right) \\
= & \frac{1}{n} \sum_{i=1}^n \widehat{\beta}_k \widehat{c}_0(X_i) \lambda_{T+1}(X_i, \widehat{\beta}) \left[\underline{q}_T(\widehat{\gamma}, X_i, \widehat{\beta}) - \bar{q}_T(\widehat{\gamma}, X_i, \widehat{\beta}) \right] \\
& \left[\mathbf{1} \left\{ \lambda_{T+1}(X_i, \widehat{\beta}) < 0 \right\} - \mathbf{1} \left\{ X_i \in \mathcal{V}_0^c \right\} \right],
\end{aligned}$$

and we denote the right-hand side of the second equality as I_n . We prove now that $I_n = o_P(n^{-1/2})$ which will guarantee that (46) holds.

By definition, $\lambda_{T+1}(x, \beta) = -\prod_{t=1}^{T-1} \left(e^{(x_t - x_T)' \beta} - 1 \right) = -e^{(T-1)x_T' \beta} \prod_{t=1}^{T-1} \left(e^{x_t' \beta} - e^{x_T' \beta} \right)$. Define the random variables $V_{it} := e^{X_{it}' \beta_0}$ and $\widehat{V}_{it} = e^{X_{it}' \widehat{\beta}}$ for $i \leq n$ and $t \leq T$. Then $\lambda_{T+1}(X_i, \beta_0) = -V_{it}^{T-1} \prod_{t=1}^{T-1} (V_{it} - V_{iT})$. The same equality holds replacing variables with their estimators. Define $L(X_i, \beta_0) = \prod_{t=1}^{T-1} (V_{it} - V_{iT})$. Using previous results and Proposition 3, wpa0,

$$I_n \leq C \frac{1}{n} \sum_{i=1}^n \left| L(X_i, \widehat{\beta}) \right| \left| \mathbf{1} \left\{ \lambda_{T+1}(X_i, \widehat{\beta}) < 0 \right\} - \mathbf{1} \left\{ X_i \in \mathcal{V}_0^c \right\} \right|.$$

Note that wpa0,

$$\begin{aligned}
& \left| \mathbf{1} \left\{ \lambda_{T+1}(X_i, \widehat{\beta}) < 0 \right\} - \mathbf{1} \left\{ X_i \in \mathcal{V}_0^c \right\} \right| \\
& \leq \mathbf{1} \left\{ \exists t < T : V_{it} - V_{iT} < 0 < \widehat{V}_{it} - \widehat{V}_{iT} \text{ ou } V_{it} - V_{iT} > 0 > \widehat{V}_{it} - \widehat{V}_{iT} \right\}.
\end{aligned}$$

Moreover $\left| \widehat{V}_{it} - \widehat{V}_{iT} - (V_{it} - V_{iT}) \right| \leq \left| \widehat{V}_{it} - V_{it} \right| + \left| \widehat{V}_{iT} - V_{iT} \right|$. We use $|\exp(a) - \exp(b)| \leq \exp(b + |b - a|)|b - a|$, $\|X_{it}\| \leq C$ and the Cauchy-Schwarz inequality to obtain

$$\left| \widehat{V}_{it} - V_{it} \right| \leq C \|\widehat{\beta} - \beta_0\| \exp(C \|\widehat{\beta} - \beta_0\|). \quad (48)$$

Take $(r_n)_n$ a sequence such that $r_n \rightarrow \infty$ and $r_n = o(n^{1/4})$. The previous inequalities give

$$\begin{aligned}
& \left| \mathbf{1} \left\{ \lambda_{T+1}(X_i, \widehat{\beta}) < 0 \right\} - \mathbf{1} \left\{ x \in \mathcal{V}_{0k}^c \right\} \right| \\
& \leq \mathbf{1} \left\{ \sqrt{n} \|\widehat{\beta} - \beta_0\| \leq r_n \right\} \mathbf{1} \left\{ \exists t : |V_{it} - V_{iT}| < 2C(r_n/\sqrt{n}) \exp(C(r_n/\sqrt{n})) \right\} \\
& \quad + \mathbf{1} \left\{ \sqrt{n} \|\widehat{\beta} - \beta_0\| > r_n \right\}.
\end{aligned} \quad (49)$$

Write $u_n = 2C(r_n/\sqrt{n}) \exp(Cr_n/\sqrt{n})$ where only here C is fixed to be the constant in the previous inequality. Assume $\sqrt{n} \|\widehat{\beta} - \beta_0\| \leq r_n$ and $|V_{it^*} - V_{iT}| < u_n$ for some t^* . Then

$$\left| \widehat{V}_{it^*} - \widehat{V}_{iT} \right| \leq 2u_n, \text{ and } \left| \widehat{V}_{it^*} - \widehat{V}_{iT} \right| \leq C + u_n.$$

Thus we have

$$\sqrt{n} I_n \leq \frac{C}{n^{1/2}} \sum_{i=1}^n I_{ni} + C \sqrt{n} \mathbf{1} \left\{ \sqrt{n} \|\widehat{\beta} - \beta_0\| > r_n \right\}$$

with $I_{ni} = |L(X_i, \hat{\beta})| \mathbf{1} \{|\exists t : |V_{it} - V_{iT}| \leq u_n\}$. We imposed $r_n \rightarrow \infty$, the second term is thus $o_P(1)$. Note that $|L(X_i, \hat{\beta})| \leq C$. The second term will thus be an $o_P(1)$ as well if

$$E \left[\frac{u_n}{n^{1/2}} \sum_{i=1}^n I_{ni} \right] \rightarrow 0. \quad (50)$$

Note now that under Assumptions 5 and 6, if $\beta_0 \neq 0$,

$$\exists u_0, u \leq u_0 \Rightarrow P(\exists t : |(X_{it} - X_{iT})' \beta_0| \leq u) \leq C' u.$$

This is a consequence of $\text{Supp}(X)$ being compact and f_X being absolutely continuous with respect to the Lebesgue measure on \mathbb{R}^{pT} . Since $u_n \rightarrow 0$, this implies that $E(I_{ni}) \leq C' u_n$ and

$$E \left[\frac{u_n}{n^{1/2}} \sum_{i=1}^n I_{ni} \right] \lesssim \frac{r_n^2}{\sqrt{n}} = o(1),$$

since $r_n = o(n^{1/4})$. This proves (50) and $I_n = o_P(n^{-1/2})$.

Finally, consider the case $\beta_0 = 0$. Using (48) and $\hat{\beta} \in \Theta$ compact, we get

$$I_{ni} \leq |L(X_i, \hat{\beta})| \leq C \|\hat{\beta}\|^{T-1}.$$

By Lemma 8 (with \mathcal{P}' defined with $\sigma = 0$ and some appropriate \bar{M} and A) and the fact that $\hat{\beta}$ is bounded imply that $n^{1/2} E \left[\|\hat{\beta}\|^{T-1} \right]$ is bounded. Thus, $n^{1/2} E(I_{ni})$ is bounded, which implies that (50) holds. Thus, in all cases, (46) holds.

Step 3: conclusion

Define $H_n(\gamma, \beta) := \frac{1}{n} \sum_{i=1}^n \underline{h}(X_i, S_i, \gamma, \beta)$ and $H(\gamma, \beta) := E(\underline{h}(X, S, \gamma, \beta))$, so that $H(\gamma_0, \beta_0) = \underline{\Delta}$. Moreover, by Lemma 8, $\hat{\beta}_k \xrightarrow{P} \beta_{0k} > 0$. Thus, $\hat{\beta}_k \geq 0$ wpao. Then, by the previous step,

$$\hat{\underline{\Delta}} = H_n(\hat{\gamma}, \hat{\beta}) + o_P(n^{-1/2}).$$

Now, $H_n(\hat{\gamma}, \hat{\beta})$ is a semiparametric estimator with a nonparametric first step. We then show the result by applying Chen et al. (2003). To this end, let $\alpha := \lceil pT/2 \rceil$ and following Chen et al. (2003), let us define, for any function γ from $\text{Supp}(X)$ to \mathbb{R}^{T+1} admitting at least α derivatives,

$$\|\gamma\|_{\mathcal{G}} := \max_{|a| \leq \alpha} \|D^a \gamma\|_{\infty}.$$

For any $c > 0$, we let \mathcal{C}_c^α denote the set of functions γ admitting at least α derivatives and such that $\|\gamma\|_{\mathcal{G}} \leq c$. By Assumptions 5.2 and 6.2a, there exists C such that $\gamma_0 \in \mathcal{C}_C^\alpha$. Hereafter, we let $\mathcal{G} := \mathcal{C}_{C'}^\alpha$ for some $C' > C$. We prove in Lemma 7 (in Section B of the Online Appendix) the five following conditions:

1. Condition 1: for all $(\epsilon_n)_{n \geq 1}$ such that $\epsilon_n \rightarrow 0$,

$$\sup_{\substack{\|\beta - \beta_0\| \leq \epsilon_n, \\ \|\gamma - \gamma_0\|_{\mathcal{G}} \leq \epsilon_n}} | [H_n(\gamma, \beta) - H(\gamma, \beta)] - [H_n(\gamma_0, \beta_0) - H(\gamma_0, \beta_0)] | = o_P(n^{-1/2}).$$

2. Condition 2: The functional pathwise and ordinary derivatives of \underline{h} with respect to γ and β exist. Moreover, there exists $b(\cdot)$ such that $E(b(X_i)) < \infty$ and

$$\begin{aligned} & |\underline{h}(X_i, S_i, \gamma, \beta) - \underline{h}(X_i, S_i, \gamma_0, \beta_0) \\ & - D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)' [\gamma(X_i) - \gamma_0(X_i)] - D_\beta \underline{h}(X_i, S_i, \gamma_0, \beta_0) [\beta - \beta_0]| \\ & \leq b(X_i) (\|\gamma - \gamma_0\|_\infty^2 + |\beta - \beta_0|^2). \end{aligned}$$

3. Condition 3: We have $\sqrt{n}(\hat{\beta} - \beta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \phi_i + o_P(1)$.

4. Condition 4: We have $\hat{\gamma} \in \mathcal{G}$ wpao, $\|\hat{\gamma} - \gamma_0\|_\infty = o_P(n^{-1/4})$ and $\|\hat{\gamma} - \gamma_0\|_{\mathcal{G}} = O_P(\tilde{\epsilon}_n)$ for some $\tilde{\epsilon}_n \rightarrow 0$.

5. Condition 5: Holding fixed the nonparametric estimator $\hat{\gamma}$ in the expectation,

$$\begin{aligned} & \sqrt{n} E^* \left(D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0) [\hat{\gamma}(X_i) - \gamma_0(X_i)] \right) \\ & = \frac{1}{\sqrt{n}} \sum_{i=1}^n D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)' [Z_i - E(Z_i|X_i)] + o_P(1). \end{aligned}$$

Then, we have

$$\begin{aligned} \sqrt{n}[\hat{\Delta} - \Delta] & = \sqrt{n}[H_n(\hat{\gamma}, \hat{\beta}) - H(\gamma_0, \beta_0)] \\ & = \sqrt{n}[H_n(\gamma_0, \beta_0) - H(\gamma_0, \beta_0)] + \sqrt{n}[H(\hat{\gamma}, \hat{\beta}) - H(\gamma_0, \beta_0)] \\ & \quad + \sqrt{n}[H_n(\hat{\gamma}, \hat{\beta}) - H(\hat{\gamma}, \hat{\beta})] - \sqrt{n}[H_n(\gamma_0, \beta_0) - H(\gamma_0, \beta_0)] \\ & = \frac{1}{\sqrt{n}} \sum_{i=1}^n \left[\underline{h}(X_i, S_i, \gamma_0, \beta_0) - E(\underline{h}(X, S, \gamma_0, \beta_0)) + E[D_\beta \underline{h}(X, S, \gamma_0, \beta_0)]' \right. \\ & \quad \left. \times \phi_i + E[D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)|X_i]' [Z_i - E(Z_i|X_i)] \right] + o_P(1). \end{aligned}$$

The result follows using $E[D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)|X_i] = D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)$ and the definition of $\underline{\psi}_k$.

Part 2: case $\beta_{0k} = 0$.

First, let us define

$$\underline{Z}_n = \frac{1}{n^{1/2}} \sum_{i=1}^n \underline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}), \quad \bar{Z}_n = \frac{1}{n^{1/2}} \sum_{i=1}^n \bar{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}).$$

Remark that when $\beta_{0k} = 0$, $\underline{\Delta} = \overline{\Delta} = 0$. Then, by Step 2 above (which holds regardless of the value of β_{0k}) and remarking that $\underline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}) \leq \overline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta})$ if and only if $\hat{\beta}_k \geq 0$, we get

$$\sqrt{n} \left(\widehat{\underline{\Delta}} - \underline{\Delta}, \widehat{\overline{\Delta}} - \overline{\Delta} \right) = \left(\min \left(\underline{Z}_n, \overline{Z}_n \right), \max \left(\underline{Z}_n, \overline{Z}_n \right) \right) + o_P(1).$$

Now, the proof of asymptotic linearity of \underline{Z}_n in Part 1 above also applies when $\beta_{0k} = 0$. Thus, $\left(\underline{Z}_n, \overline{Z}_n \right) \xrightarrow{d} \left(\underline{Z}, \overline{Z} \right)'$. The result follows by the continuous mapping theorem.

Part 3: Consistency of $\widehat{\Sigma}$.

Let us assume wlog that $\beta_{0k} > 0$. The estimator of the variance covariance matrix is $\widehat{\Sigma} = \frac{1}{n} \sum_{i=1}^n (\widehat{\underline{\psi}}_i, \widehat{\overline{\psi}}_i) (\widehat{\underline{\psi}}_i, \widehat{\overline{\psi}}_i)'$ where

$$\begin{aligned} \widehat{\underline{\psi}}_i &= \underline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}) - \frac{1}{n} \sum_{j=1}^n \underline{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) + \left(\frac{1}{n} \sum_{j=1}^n D_\beta \underline{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) \right)' \widehat{\phi}_i \\ &\quad + D_\gamma \underline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta})' [Z_i - \hat{\gamma}(X_i)] \\ \widehat{\overline{\psi}}_i &= \overline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta}) - \frac{1}{n} \sum_{j=1}^n \overline{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) + \left(\frac{1}{n} \sum_{j=1}^n D_\beta \overline{h}(X_j, S_j, \hat{\gamma}, \hat{\beta}) \right)' \widehat{\phi}_i \\ &\quad + D_\gamma \overline{h}(X_i, S_i, \hat{\gamma}, \hat{\beta})' [Z_i - \hat{\gamma}(X_i)] \end{aligned}$$

We show that the functions of (X_i, S_i) appearing in $\widehat{\underline{\psi}}_i$ and $\widehat{\overline{\psi}}_i$ converge uniformly to their pointwise limits. Similarly to what we argued in the proof of Theorem 3, $(x, s) \mapsto \underline{h}(x, s, \hat{\gamma}, \hat{\beta}) - \frac{1}{n} \sum_{j=1}^n \underline{h}(X_j, S_j, \hat{\gamma}, \hat{\beta})$ converges uniformly in probability to $(x, s) \mapsto \underline{h}(x, s, \gamma_0, \beta_0) - E(\underline{h}(X, S, \gamma_0, \beta_0))$.

The smoothness arguments given in Condition 1 (Part 2) implies in particular that the derivatives of \underline{h} with respect to both the vector $\gamma(x)$ and β are Lipschitz continuous on \mathcal{C}_c^α and Θ , with Lipschitz constant uniform over $x \in \text{Supp}(X)$. This implies that $(x, s, z) \mapsto D_\gamma \underline{h}(x, s, \hat{\gamma}, \hat{\beta})' [z - \hat{\gamma}(x)]$ converges uniformly in probability to $(x, s, z) \mapsto D_\gamma \underline{h}(x, s, \gamma_0, \beta_0)' [z - \gamma_0(x)]$. The same results follow for \overline{h} . By \mathcal{I}_0 nonsingular and $C_s(x, \beta)$ bounded away from 0 uniformly over (s, x, β) , the derivatives of $\beta \mapsto \left[\frac{1}{\sqrt{n}} \sum_{j=1}^n \partial^2 \ell_c / \partial \beta^2 (Y_j | X_j; \beta) \right]^{-1} \partial \ell_c / \partial \beta (y | x; \beta)$ are uniformly bounded over (y, x, β) wpao. Thus $\widehat{\phi}_i$ converges uniformly in probability to ϕ_i .

In conclusion, the functions of (X_i, S_i) appearing in $\widehat{\underline{\psi}}_i$ and $\widehat{\overline{\psi}}_i$ converge uniformly to their pointwise limits. This implies that $(\widehat{\underline{\psi}}_i, \widehat{\overline{\psi}}_i) (\widehat{\underline{\psi}}_i, \widehat{\overline{\psi}}_i)'$ converges uniformly to $(\underline{\psi}_i, \overline{\psi}_i) (\underline{\psi}_i, \overline{\psi}_i)'$. As in Theorem 3, we obtain using the LLN that $\widehat{\Sigma} \xrightarrow{P} \Sigma$.

A.4 Proposition 4

First assume that $\beta_{0k} = 0$. Then $\Delta = 0$ and

$$P\left(\Delta \in \text{CI}_{1-\alpha}^1\right) \geq P(\varphi_\alpha = 0) \rightarrow 1 - \alpha,$$

where the latter follows since φ_α has asymptotic level α . Now, assume $\beta_{0k} \neq 0$. Then $\varphi_\alpha \xrightarrow{P} 1$, so that $\text{CI}_{1-\alpha}^1$ takes the first form wpa. Suppose first that $\underline{\Delta} < \bar{\Delta}$. By consistency of the bounds, consistency of $\widehat{\Sigma}$ and $\min(\Sigma_{11}, \Sigma_{12}) > 0$, we have

$$\frac{n^{1/2} \left(\widehat{\bar{\Delta}} - \widehat{\underline{\Delta}} \right)}{\max \left(\widehat{\Sigma}_{11}^{1/2}, \widehat{\Sigma}_{12}^{1/2} \right)} \xrightarrow{P} \infty.$$

Then, by Lemma 5.10 of van der Vaart (2000), $c_\alpha \rightarrow \Phi^{-1}(1 - \alpha)$. The result follows as in Lemma 2 of Imbens and Manski (2004). Next, assume $\underline{\Delta} = \bar{\Delta}$. Then, because $\widehat{\bar{\Delta}} \geq \widehat{\underline{\Delta}}$ a.s., $\bar{Z} - \underline{Z}$ must be degenerate, implying in turn $\bar{Z} = \underline{Z}$ a.s. Hence,

$$\frac{n^{1/2} \left(\widehat{\bar{\Delta}} - \widehat{\underline{\Delta}} \right)}{\max \left(\widehat{\Sigma}_{11}^{1/2}, \widehat{\Sigma}_{12}^{1/2} \right)} = o_P(1).$$

By, again, Lemma 5.10 of van der Vaart (2000), $c_\alpha \rightarrow \Phi^{-1}(1 - \alpha/2)$. The result follows using standard arguments for this point identified case.

A.5 Lemma 2

First, let $W_i = (X_i, S_i)$ and

$$g(W_i, \beta) = \beta_k \sum_{t=0}^{S_i} \frac{a_t(X_i, \beta) \binom{T-t}{S_i-t} \exp(S_i X_{iT}' \beta)}{C_{S_i}(X_i, \beta)},$$

so that $\tilde{\Delta} = E[g(W_1, \beta_0)]$ and $\widehat{\Delta} = \sum_{i=1}^n g(W_i, \widehat{\beta})/n$. By choosing appropriate \bar{M} , $\underline{\sigma}$ and A , $P \in \mathcal{P}$. Then, by Lemma 8, we have

$$\sqrt{n} \left(\widehat{\beta} - \beta_0 \right) = \frac{1}{n^{1/2}} \sum_{i=1}^n \phi_i + o_P(1). \quad (51)$$

Since $\beta \mapsto g(w, \beta)$ is differentiable for all w , by the mean value theorem, there exists $\bar{\beta}_i = t_i \widehat{\beta} + (1 - t_i) \beta_0$, with $t_i \in [0, 1]$, such that $g(W_i, \widehat{\beta}) - g(W_i, \beta_0) = \partial g / \partial \beta (W_i, \bar{\beta}_i) (\widehat{\beta} - \beta_0)$. Let

$$\widehat{G} = \frac{1}{n} \sum_{i=1}^n \frac{\partial g}{\partial \beta} (W_i, \bar{\beta}_i).$$

Then, by what precedes,

$$\sqrt{n} (\hat{\Delta} - \tilde{\Delta}) = \frac{1}{n^{1/2}} \sum_{i=1}^n [\hat{G}\phi_i + g(W_i, \beta_0) - \tilde{\Delta}] + o_P(1).$$

Because $\partial g/\partial\beta$ is continuous, $\text{Supp}(W)$ is compact and $\hat{\beta}$ is consistent, we have (see, e.g., Lemma 2.4 in Newey and McFadden, 1994)

$$\hat{G} \xrightarrow{P} G := E \left[\frac{\partial g}{\partial \beta}(W_i, \beta_0) \right].$$

Thus,

$$\sqrt{n} (\hat{\Delta} - \tilde{\Delta}) = \frac{1}{n^{1/2}} \sum_{i=1}^n [G\phi_i + g(W_i, \beta_0) - \tilde{\Delta}] + o_P(1).$$

The first result follows by the central limit theorem and a few algebra. The second result follows using the same reasoning as to prove $\hat{G} \xrightarrow{P} G$.

A.6 Theorem 2

1. First assume that $\bar{b} = 0$. Then $\tilde{\Delta} = \Delta$ and $\hat{b} \geq 0$. Since $b \mapsto q_\alpha(b)$ is increasing on \mathbb{R}^+ , we obtain

$$\begin{aligned} P \left(n^{1/2} \left| \frac{\hat{\Delta} - \Delta}{\hat{\sigma}} \right| \leq q_\alpha \left(n^{1/2} \frac{\hat{b}}{\hat{\sigma}} \right) \right) &\geq P \left(n^{1/2} \left| \frac{\hat{\Delta} - \tilde{\Delta}}{\hat{\sigma}} \right| \leq q_\alpha(0) \right) \\ &\rightarrow 1 - \alpha, \end{aligned}$$

where the convergence follows by Lemma 2.

Next, let us assume that $\bar{b} > 0$, and thus $R < \bar{R}$. Let $b := \tilde{\Delta} - \Delta$. Then, remark that $b < \bar{b}$. Let us define the event

$$E_n := \left\{ \hat{\sigma} q_\alpha \left(n^{1/2} \frac{\hat{b}}{\hat{\sigma}} \right) \geq \sigma q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right) \right\}.$$

Note that $q_\alpha(x) = x + z_{1-\alpha} + o(1)$ as $x \rightarrow \infty$, where we recall that $z_{1-\alpha}$ the quantile of order $1 - \alpha$ of a standard normal distribution. Fix $\eta \in (0, z_{1-\alpha})$. Then, for x large enough,

$$x + z_{1-\alpha} - \eta \leq q_\alpha(x) \leq x + z_{1-\alpha} + \eta.$$

Now, $\hat{\sigma}/\sigma \xrightarrow{P} 1$ by Lemma 2 and $\hat{b} \xrightarrow{P} \bar{b} > b$ by Lemmas 8 and 9. Thus, with probability approaching one, $\hat{\sigma}/\sigma > 1 - \eta/(z_{1-\alpha} - \eta)$ and $\hat{b} > b + 3n^{-1/2}\sigma\eta$. If so,

$$\hat{\sigma} q_\alpha \left(n^{1/2} \frac{\hat{b}}{\hat{\sigma}} \right) \geq \hat{\sigma} \left(n^{1/2} \frac{\hat{b}}{\hat{\sigma}} + z_{1-\alpha} - \eta \right)$$

$$\begin{aligned}
&\geq \sigma \left[n^{1/2} \frac{b + 3n^{-1/2}\sigma\eta}{\sigma} + \left(1 - \frac{\eta}{z_{1-\alpha} - \eta} \right) (z_{1-\alpha} - \eta) \right] \\
&\geq \sigma \left(n^{1/2} \frac{b}{\sigma} + z_{1-\alpha} + \eta \right) \\
&\geq \sigma q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right).
\end{aligned}$$

As a result, $P(E_n) \rightarrow 1$. Then, using $P(A \cap B) \geq P(A) + P(B) - 1$,

$$\begin{aligned}
P \left(n^{1/2} |\widehat{\Delta} - \Delta| \leq \widehat{\sigma} q_\alpha \left(n^{1/2} \frac{\widehat{b}}{\widehat{\sigma}} \right) \right) &\geq P \left(n^{1/2} |\widehat{\Delta} - \widetilde{\Delta} + b| \leq \sigma q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right) \right) \\
&\quad + P(E_n) - 1 \\
&\geq P \left(\left| \widetilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right) \right) \\
&\quad + P(E_n) - 1,
\end{aligned}$$

where $\widetilde{Z}_n := n^{1/2}(\widehat{\Delta} - \Delta)/\sigma$. Hence, by what precedes,

$$\liminf_{n \rightarrow \infty} P \left(n^{1/2} |\widehat{\Delta} - \Delta| \leq \widehat{\sigma} q_\alpha \left(n^{1/2} \frac{\widehat{b}}{\widehat{\sigma}} \right) \right) \geq \liminf_{n \rightarrow \infty} P \left(\left| \widetilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right) \right).$$

Now, let F_n denote the cdf of \widetilde{Z}_n and let $Z \sim \mathcal{N}(0, 1)$. We have:

$$\begin{aligned}
&\left| P \left(\left| \widetilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right) \right) - (1 - \alpha) \right| \\
&= \left| P \left(\left| \widetilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right) \right) - P \left(\left| Z + n^{1/2} \frac{b}{\sigma} \right| \leq q_\alpha \left(n^{1/2} \frac{b}{\sigma} \right) \right) \right| \\
&\leq \sup_{x \in \mathbb{R}} \left| P \left(\left| \widetilde{Z}_n + x \right| \leq q_\alpha(x) \right) - P \left(\left| Z + x \right| \leq q_\alpha(x) \right) \right| \\
&= \sup_{x \in \mathbb{R}} \left| F_n(x + q_\alpha(x)) - \Phi(x + q_\alpha(x)) - F_n(x - q_\alpha(x)) + \Phi(x - q_\alpha(x)) \right| \\
&\leq 2 \sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)|.
\end{aligned}$$

Finally, Lemma 2 implies that for all x , $F_n(x) \rightarrow \Phi(x)$ with Φ continuous. By Lemma 2.11 in van der Vaart (2000), the convergence is uniform. The result follows.

2. To show the result, it suffices to show that

$$\liminf_n P_n \left(\Delta \in \text{CI}_{1-\alpha}^3 \right) \geq 1 - \alpha, \tag{52}$$

for any sequence of probability distributions $(P_n)_{n \geq 1}$ in \mathcal{P} . Note that to simplify notation, we do not index parameters by P_n (nor by n). We proceed in three steps. We first show that

$$\liminf_n P_n \left(|\widehat{\beta}_k| + z_{1-\gamma} n^{-1/2} \widehat{\tau}_k \geq |\beta_{0k}| \right) \geq 1 - \gamma. \tag{53}$$

Then, we prove that for any η small enough,

$$\liminf_n P_n \left(\hat{\sigma} q_\delta \left(\frac{n^{1/2} \hat{b}_\gamma}{\hat{\sigma}} \right) \geq \sigma q_\delta \left(\frac{n^{1/2} b}{\sigma} \right) - \eta \right) \geq 1 - \gamma. \quad (54)$$

We finally establish (52) in the third step.

Step 1: (53) holds.

Define $Z_n = n^{1/2}(\hat{\beta}_k - \beta_{0k})/\tau_k$. If $A' \cap B \subset A \cap B$, then $P_n(A) \geq P_n(A') - P_n(A' \cap B^c) + P_n(A \cap B^c) \geq P_n(A') - P_n(B^c)$. Thus for any $\eta > 0$:

$$\begin{aligned} P_n \left(|\hat{\beta}_k| + z_{1-\gamma} n^{-1/2} \hat{\tau}_k \geq |\beta_{0k}| \right) &= P_n \left(|Z_n + n^{1/2} \beta_{0k}/\tau_k| \geq |n^{1/2} \beta_{0k}/\tau_k| - z_{1-\gamma} \frac{\hat{\tau}_k}{\tau_k} \right) \\ &\geq P_n \left(|Z_n + n^{1/2} \beta_{0k}/\tau_k| \geq |n^{1/2} \beta_{0k}/\tau_k| - z_{1-\gamma} (1 - \eta) \right) \\ &\quad - P_n \left(\frac{\hat{\tau}_k}{\tau_k} < 1 - \eta \right) \\ &\geq \left[\inf_{|x| \geq z_{1-\gamma} (1 - \eta)} P_n (|Z_n + x| \geq |x| - z_{1-\gamma} (1 - \eta)) \right] - o(1) \\ &\geq \min [F_n (z_{1-\gamma} (1 - \eta)), 1 - F_n (-z_{1-\gamma} (1 - \eta))] - o(1) \\ &\rightarrow \min [\Phi (z_{1-\gamma} (1 - \eta)), 1 - \Phi (-z_{1-\gamma} (1 - \eta))] = \Phi (z_{1-\gamma} (1 - \eta)) \end{aligned}$$

by Lemma 8 and uniform convergence of the estimator of the variance of $\hat{\beta}$. Because η is arbitrarily small and Φ is continuous everywhere, we conclude that

$$\liminf_n P_n \left(|\hat{\beta}_k| + z_{1-\gamma} n^{-1/2} \hat{\tau}_k \geq |\beta_{0k}| \right) \geq 1 - \gamma.$$

Step 2: (54) holds.

We will use below the following results on q_α . First, for all $x \geq 0$ (a restriction that we can make wlog),

$$x + z_{1-\alpha} \leq q_\alpha(x) \leq x + z_{1-\alpha/2}. \quad (55)$$

To see this, note that the first inequality comes from

$$\Phi(q_\alpha(x) - x) - \Phi(-x - q_\alpha(x)) = 1 - \alpha \quad (56)$$

and $\Phi(-x - q_\alpha(x)) \geq 0$. The second inequality comes from $-q_\alpha(x) - x \leq x - q_\alpha(x)$ and thus, from (56) again, $2\Phi(q_\alpha(x) - x) - 1 \leq 1 - \alpha$. Second, by differentiating (56) with respect to x , we obtain that $x \mapsto q_\alpha(x) - x$ is decreasing, from $z_{1-\alpha/2}$ at $x = 0$ to $z_{1-\alpha}$ at $x \rightarrow \infty$.

Now, fix $C > z_{1-\delta/2}/\zeta$ and let us first suppose that $n^{1/2}b/\sigma > C$. Fix $\eta \in (0, 1)$. We have

$$q_\delta \left(\frac{n^{1/2} b}{\sigma} \right) \leq \frac{n^{1/2} b}{\sigma} + z_{1-\delta/2}$$

$$\leq n^{1/2} \frac{|\beta_{0k}|R(1+\zeta)}{\sigma} + (1-\eta)z_{1-\delta}, \quad (57)$$

where the first inequality uses the second inequality in (55) and the second inequality follows from $z_{1-\delta/2} - (1-\eta)z_{1-\delta} \leq C\zeta \leq n^{1/2}b\zeta/\sigma$. From (55), we also have:

$$\begin{aligned} \hat{\sigma}q_\delta \left(n^{1/2} \frac{\widehat{b}_\gamma}{\widehat{\sigma}} \right) &\geq \hat{\sigma} \left(n^{1/2} \frac{\widehat{b}_\gamma}{\widehat{\sigma}} + z_{1-\delta} \right) \\ &= \sigma \left(n^{1/2} \frac{(|\widehat{\beta}_k| + n^{-1/2}\widehat{\tau}_k z_{1-\gamma})\widehat{R}}{\sigma} + \frac{\widehat{\sigma}}{\sigma} z_{1-\delta} \right). \end{aligned} \quad (58)$$

Moreover,

$$\begin{aligned} &\liminf_{n:n^{1/2}b/\sigma > C} P_n \left[\sigma \left(n^{1/2} \frac{(|\widehat{\beta}_k| + n^{-1/2}\widehat{\tau}_k z_{1-\gamma})\widehat{R}}{\sigma} + \frac{\widehat{\sigma}}{\sigma} z_{1-\delta} \right) \geq \sigma \left(n^{1/2} \frac{|\beta_{0k}|R(1+\zeta)}{\sigma} + (1-\eta)z_{1-\delta} \right) \right] \\ &\geq \liminf_{n:n^{1/2}b/\sigma > C} P_n \left(\left\{ |\widehat{\beta}_k| + z_{1-\gamma}n^{-1/2}\widehat{\tau}_k \geq |\beta_{0k}| \right\} \cap \left\{ \widehat{R} > R(1+\zeta) \right\} \cap \left\{ \frac{\widehat{\sigma}}{\sigma} \geq 1-\eta \right\} \right) \\ &\geq 1-\gamma, \end{aligned}$$

where the second inequality follows from (53), $\lim_n P_n(\widehat{R} > R(1+\zeta)) = 1$ (in view of $\bar{R} > R(1+\zeta)$ and uniform convergence of \widehat{R} shown in Lemma 9), $\lim_n P_n(\widehat{\sigma}/\sigma \geq 1-\eta) = 1$ (in view of $\sigma \geq \underline{\sigma} > 0$ and uniform convergence of $\widehat{\sigma}$ shown in Lemma 8) and $P_n(A \cap B \cap C) \geq P_n(A) + P_n(B) + P_n(C) - 2$. Combined with (57)-(58), this yields

$$\liminf_{n:n^{1/2}b/\sigma > C} P_n \left[\hat{\sigma}q_\delta \left(n^{1/2} \frac{\widehat{b}_\gamma}{\widehat{\sigma}} \right) \geq \sigma q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) \right] \geq 1-\gamma. \quad (59)$$

Next, assume that $n^{1/2}b/\sigma \leq C$. Because $x \mapsto q_\delta(x)$ is increasing:

$$\begin{aligned} P_n \left(q_\delta \left(\frac{n^{1/2}\widehat{b}_\gamma}{\widehat{\sigma}} \right) \geq q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) \right) &= P_n \left(n^{1/2}\widehat{b}_\gamma \geq n^{1/2}b \right) \\ &= P_n \left((|\widehat{\beta}_k| + z_{1-\gamma}n^{-1/2}\widehat{\tau}_k)\widehat{R} \geq |\beta_{0k}|R \right) \\ &\geq P_n \left(\left\{ (|\widehat{\beta}_k| + z_{1-\gamma}n^{-1/2}\widehat{\tau}_k) \geq |\beta_{0k}| \right\} \cap \left\{ \widehat{R} \geq R \right\} \right) \\ &\geq P_n \left(|\widehat{\beta}_k| + z_{1-\gamma}n^{-1/2}\widehat{\tau}_k \geq |\beta_{0k}| \right) - P_n \left(R > \widehat{R} \right) \end{aligned}$$

Since $\lim_n P_n \left(R > \widehat{R} \right) = 0$, (53) ensures that

$$\liminf_{n:n^{1/2}b/\sigma \leq C} P_n \left(q_\delta \left(\frac{n^{1/2}\widehat{b}_\gamma}{\widehat{\sigma}} \right) \geq q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) \right) \geq 1-\gamma. \quad (60)$$

Moreover, using the fact that $x \mapsto q_\delta(x) - x$ is decreasing and $x \mapsto q_\delta(x)$ is increasing, we get

$$\begin{aligned} q_\delta \left(\frac{n^{1/2}b}{\hat{\sigma}} \right) &\geq q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) - \max \left(0, 1 - \frac{\sigma}{\hat{\sigma}} \right) n^{1/2} \frac{b}{\sigma} \\ &\geq q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) - \max \left(0, 1 - \frac{\sigma}{\hat{\sigma}} \right) C. \end{aligned}$$

Thus, for any $\eta > 0$ small enough, we have with probability approaching one, using again $n^{1/2}b/\sigma \leq C$ and the fact that $\sigma < \bar{\sigma}$ for some $\bar{\sigma} < \infty$ by smoothness of the functions involved in the definition of ψ and compactness of Θ and of the support of X ,

$$\begin{aligned} \hat{\sigma} q_\delta \left(\frac{n^{1/2}b}{\hat{\sigma}} \right) &\geq \frac{\hat{\sigma}}{\sigma} \sigma q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) - \max \left(0, \frac{\hat{\sigma}}{\sigma} - 1 \right) \sigma C \\ &\geq \sigma q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) - \eta. \end{aligned}$$

From (60), we then have, for any $\eta > 0$,

$$\liminf_{n: n^{1/2}b/\sigma \leq C} P_n \left(\hat{\sigma} q_\delta \left(\frac{n^{1/2}\hat{b}_\gamma}{\hat{\sigma}} \right) \geq \sigma q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) - \eta \right) \geq 1 - \gamma \quad (61)$$

Combining (59) and (61), we finally obtain (54).

Step 3: conclusion.

Let $E_{n,\eta} = \left\{ \hat{\sigma} q_\delta \left(\frac{n^{1/2}\hat{b}_\gamma}{\hat{\sigma}} \right) \geq \sigma q_\delta \left(\frac{n^{1/2}b}{\sigma} \right) - \eta \right\}$, following the same line as in the proof of simple convergence, we have:

$$\begin{aligned} P_n \left(\Delta \in \text{CI}_{1-\alpha}^3 \right) &= P_n \left(n^{1/2} |\hat{\Delta} - \Delta| \leq \hat{\sigma} q_\delta \left(n^{1/2} \frac{\hat{b}_\gamma}{\hat{\sigma}} \right) \right) \\ &\geq P_n \left(n^{1/2} |\hat{\Delta} - \tilde{\Delta} + b| \leq \sigma q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) \\ &\quad + P_n(E_{n,\eta}) - 1 \\ &\geq P_n \left(\left| \tilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) \\ &\quad + P_n(E_{n,\eta}) - 1, \end{aligned}$$

where $\tilde{Z}_n := n^{1/2}(\hat{\Delta} - \tilde{\Delta})/\sigma$. Hence, by what precedes,

$$P_n \left(\Delta \in \text{CI}_{1-\alpha}^3 \right) \geq \liminf_{n \rightarrow \infty} P_n \left(\left| \tilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) - \gamma.$$

Now, let F_n denote the cdf of \tilde{Z}_n under P_n and let $Z \sim \mathcal{N}(0, 1)$. We have,

$$\begin{aligned}
& \left| P_n \left(\left| \tilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) - (1 - \delta) \right| \\
& \leq \left| P_n \left(\left| \tilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) - P_n \left(\left| Z + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) \right| \\
& \quad + \left| P_n \left(\left| Z + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) - P_n \left(\left| Z + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) \right) \right| \\
& \leq \sup_{x \in \mathbb{R}} \left| P_n \left(\left| \tilde{Z}_n + x \right| \leq q_\delta(x) - \eta \right) - P_n \left(n^{1/2} |Z + x| \leq q_\delta(x) - \eta \right) \right| + \sqrt{2/\pi} \eta \\
& = \sup_{x \in \mathbb{R}} |F_n(x + q_\delta(x) - \eta) - \Phi(x + q_\delta(x) - \eta) - F_n(x - q_\delta(x) + \eta) + \Phi(x - q_\delta(x) + \eta)| + \sqrt{2/\pi} \eta \\
& \leq 2 \sup_{x \in \mathbb{R}} |F_n(x) - \Phi(x)| + \sqrt{2/\pi} \eta.
\end{aligned}$$

Finally, Lemma 8 implies that for all x , $F_n(x) \rightarrow \Phi(x)$ with Φ continuous. By Lemma 2.11 in van der Vaart (2000), the convergence is uniform. Since η was arbitrary we obtain

$$\lim_{n \rightarrow \infty} \left| P_n \left(\left| \tilde{Z}_n + n^{1/2} \frac{b}{\sigma} \right| \leq q_\delta \left(n^{1/2} \frac{b}{\sigma} \right) - \eta \right) - (1 - \delta) \right| = 0.$$

The result follows because $\alpha = \delta + \gamma$.

B Technical lemmas

Lemma 7 *Suppose that Assumptions 1-3 and 5, 6 hold. Then, the five conditions in Part 1, Step 3 of the proof of Theorem 1 hold.*

Proof: We use the same notation as that introduced in the proof of Theorem 1.

Condition 1: By Assumption 3 and uniform consistency of \tilde{m} (see Step 1 in the proof of Theorem 3), $\tilde{m}(x) = m(\hat{\gamma}, x, \hat{\beta})$ lies in $\mathcal{M}_I^{\epsilon/2}$ for all $x \in \text{Supp}(X)$ wpao, where, for any $\eta > 0$

$$\mathcal{M}_I^\eta := \text{cl}\{m \in \mathcal{M}_I; \mathcal{B}(m, \eta) \subset \text{Int } \mathcal{M}_I\}.$$

It is known that q_T and \bar{q}_T are infinitely differentiable on $\mathcal{M}_I^{\epsilon/2}$. The function $(\gamma, x, \beta) \mapsto m(\gamma, x, \beta)$ depends on γ only through its value when evaluated at x , $\gamma(x)$, and is infinitely differentiable with respect to the vector $\gamma(x)$ and with respect to β . Moreover, the set \mathcal{V}_0 is constructed using the known β_0 and thus does not depend on $\hat{\beta}$. The function \underline{h} is therefore infinitely differentiable in the vector $\gamma(x)$ and in β . It is in particular Fréchet differentiable in γ and continuously differentiable in β and we have

$$|\underline{h}(X_i, S_i, \gamma_1, \beta_1) - \underline{h}(X_i, S_i, \gamma_2, \beta_2)|$$

$$\begin{aligned} &\leq \sup_{\beta \in \Theta} \|\partial_\beta \underline{h}(X_i, S_i, \gamma_2, \beta)\| \|\beta_1 - \beta_2\| + \sup_{\gamma(X_i), \gamma \in \mathcal{G}} \|\partial_\gamma \underline{h}(X_i, S_i, \gamma, \beta_2)\| \|\gamma_1(X_i) - \gamma_2(X_i)\| \\ &\leq b(X_i, S_i) (\|\beta_1 - \beta_2\| + \|\gamma_1 - \gamma_2\|_{\mathcal{G}}), \end{aligned}$$

where the suprema of the derivatives exist since Θ and $\{\gamma(X), \gamma \in \mathcal{G}\}$ are compact sets by Assumptions 5.2 and 6.2a. Note additionally that by similar smoothness arguments and because indicator functions are bounded, $E(b(X_i, S_i)^r) < \infty$ for any $r \geq 2$ as X and S have bounded support. Moreover using the definitions of Chen et al. (2003) and the results they cite from van der Vaart and Wellner (1996), the covering number of \mathcal{G} exists and is integrable if $\alpha > \dim(X)/2 = pT/2$. Thus by Theorem 3 of Chen et al. (2003), Condition 1 holds.

Condition 2: The difference

$$\begin{aligned} &|\underline{h}(X_i, S_i, \gamma, \beta) - \underline{h}(X_i, S_i, \gamma_0, \beta_0)| \\ &- D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)'[\gamma(X_i) - \gamma_0(X_i)] - D_\beta \underline{h}(X_i, S_i, \gamma_0, \beta_0)'[\beta - \beta_0], \end{aligned}$$

is equal to the second-order partial derivatives of \underline{h} evaluated at some point $\tilde{\gamma}(X_i)$ and $\tilde{\beta}$, and applied to $\gamma(X_i) - \gamma_0(X_i)$ and $\beta - \beta_0$. By the same argument as in Condition 1, the second-order derivatives of h can be bounded uniformly over β and $\gamma(X)$ and these bounds have finite expectation over (X_i, S_i) . The residual can thus be bounded by a constant multiplied by $(\|\gamma - \gamma_0\|_\infty^2 + |\beta - \beta_0|^2)$.

Condition 3: This condition holds by Lemma 8 below, with $\mathcal{P}' = \{P\}$ and

$$\phi(X_i, Y_i) = E \left[\frac{\partial^2 \ell_c}{\partial \beta^2}(Y_i | X_i; \beta_0) \right]^{-1} \frac{\partial \ell_c}{\partial \beta}(Y_i | X_i; \beta_0).$$

Condition 4: We apply Theorem 6 of Masry (1996) on the convergence rate of local polynomial estimators. This theorem requires the conditional density $f_{X|Z}$ to exist and be bounded, which holds here as $Z = (\mathbf{1}\{S=0\}, \dots, \mathbf{1}\{S=T\})'$ and X has compact support and bounded density. By Assumptions 5-6, the other conditions of the theorem hold. Thus, by Masry (1996)

$$\sup_{x \in \mathcal{D}} |\hat{\gamma}_j(x) - \gamma_{0j}(x)| = O \left(\left(\frac{\ln n}{nh_n^{pT}} \right)^{1/2} + h_n^{\ell+1} \right) \text{ almost surely.} \quad (62)$$

By Assumption 6.4b, $(\ln n / (nh_n^{pT}))^{1/2} + h_n^{\ell+1} = o(n^{-1/3})$ thus $\|\hat{\gamma} - \gamma_0\|_\infty = O_P(n^{-1/4})$. Theorem 6 of Masry (1996) also states that almost surely

$$\text{for } |a| \leq \ell, \sup_{x \in \mathcal{D}} \left| \frac{\partial^a \hat{\gamma}_j(x)}{\partial x^a} - \frac{\partial^a \gamma_{0j}(x)}{\partial x^a} \right| = O \left(\left(\frac{\ln n}{nh_n^{pT+2|a|}} \right)^{1/2} + h_n^{\ell+1-|a|} \right). \quad (63)$$

Define $\tilde{\epsilon}_n := \left[\ln n / (nh_n^{pT+2|\alpha|}) \right]^{1/2} + h_n^{\ell+1-|\alpha|}$. Then by $\alpha \leq \ell$, $\|\hat{\gamma} - \gamma_0\|_{\mathcal{G}} = O_P(\tilde{\epsilon}_n)$ and by $\alpha \leq \ell$, $\alpha \leq pT$ and Assumption 6.4b, $\tilde{\epsilon}_n \rightarrow 0$. Moreover, note that $\hat{\gamma}$ is continuous by construction and since $\gamma \in \mathcal{C}_{\mathcal{G}}^\alpha$, Equations (62) and (63) imply that $\hat{\gamma} \in \mathcal{G}$ w.p.a.1.

Condition 5: First, we apply Corollary 1 of Kong et al. (2010). To this end, we check their Assumptions A1-A7. To avoid confusion with the notations of Kong et al. (2010), let $p' = pT$. For $u, e, \theta \in \mathbb{R}^3$, let $\rho(u, \theta) = \frac{1}{2}(u-\theta)^2$ and $\varphi(e) = -e$, we have $\rho(u, \theta) = \rho(u, 0) + \int_0^\theta \varphi(u-t)dt$ and $E(\varphi(\epsilon_j)|X) = 0$ for $\epsilon_j = \mathbf{1}\{S = j\} - \gamma_{0j}(X)$.

First, because ϵ_j has a bounded support and a bounded density, A1 and A2 in Kong et al. (2010) hold for any value of the parameter ν_1 as defined in Kong et al. (2010).

Assumption 6.3 ensures that for any $\alpha = (\alpha_{j,t}) \in \mathbb{N}^{p'}$ such that $\sum_{j,t} \alpha_{j,t} \leq 2\ell+1$, $u \mapsto u^\alpha K(u)$ is Lipschitz on any compact set (as a product of Lipschitz functions) and on $\mathbb{R}^{p'} \setminus \text{Supp}(K)$ (as the null function). If $u \in \text{Supp}(K)$ and $v \in \mathbb{R}^{p'} \setminus \text{Supp}(K)$ there exists $w \in \{\mu u + (1-\mu)v : \mu \in [0; 1]\} \cap (2 \cdot \text{Supp}(K)) \cap (\mathbb{R}^{p'} \setminus \text{Supp}(K))$. Because $2 \cdot \text{Supp}(K)$ is a compact containing $\text{Supp}(K)$, we have:

$$\begin{aligned} |u^\alpha K(u) - v^\alpha K(v)| &\leq |u^\alpha K(u) - w^\alpha K(w)| + |w^\alpha K(w) - v^\alpha K(v)| \\ &\leq C(|u - w| + |w - v|) = C|u - v|, \end{aligned}$$

ensuring that $u \mapsto u^\alpha K(u)$ is Lipschitz on $\mathbb{R}^{p'}$. So, A3 holds.

Assumptions 5.1 and 6.1 (resp. 6.2) imply that A4 (resp. A5) holds.

To check A6, we fix the values $\lambda_1 = 3/4$, $\lambda_2 = 1/2$, $p = \ell$ and take any $\nu_2 > 12$, borrowing here the notation of Kong et al. (2010). Then, tedious algebra shows that under Assumption 6.4b, the three conditions on the bandwidth h_n in A6 hold.

Finally, the Bayes formula and Assumptions 1 and 6.1 ensure that $X|S$ admits a bounded density with respect to the Lebesgue measure. By independence across $i = 1, \dots, n$, A7 holds.

Hence, by Corollary 1 in Kong et al. (2010), we have with probability 1 and uniformly in $x \in \mathcal{K}$, a compact subset of \mathbb{R}^{pT} ,

$$\hat{\gamma}_j(x) - \gamma_{0j}(x) = T_j(x) + O\left[\left(\frac{\log n}{nh_n^{pT}}\right)^{3/4}\right] + o(h_n^{\ell+1}), \quad (64)$$

where

$$T_j(x) := \alpha(x)h_n^{\ell+1} + \frac{1}{n}e'S_{h_n}(x)^{-1} \sum_{i=1}^n K\left(\frac{X_i - x}{h_n}\right) \epsilon_{ij}w(X_i - x).$$

In this expression, e is the first element of the canonical basis of the corresponding vector space, the dimension of which depends on ℓ and pT (it is the number of polynomials of X of degree less than or equal to ℓ). The function $\alpha(x)$ is a bounded function of x , $w(x)$ is the vector $(1, x, \dots, x^k, \dots)'$ for all $|k| \leq \ell$ ordered by increasing degree and $S_{h_n}(x)$ is the matrix $E[w((X-x)/h_n)w((X-x)/h_n)'K((X-x)/h_n)]$.

Under Assumption 6.4b, we have $(\ln n/(nh_n^{pT}))^{3/4} = o(n^{-1/2})$ and $h_n^{\ell+1} = o(n^{-1/2})$. Thus (64) implies

$$\begin{aligned} & E^* (D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)' [\hat{\gamma}(X) - \gamma_0(X)]) \\ &= \int_{x \in \mathbb{R}^{pT}} [D_\gamma \underline{h}(X_i, S_i, \gamma_0, \beta_0)]' T_j(x) f_X(x) dx + o_P(n^{-1/2}) \\ &= \frac{1}{n} \sum_{i=1}^n \sum_{j \leq T+1} \epsilon_{ij} \int_{x \in \mathbb{R}^{pT}} e' S_{h_n}(u)^{-1} w(X_i - x) \lambda_{h,j}(x) K\left(\frac{X_i - x}{h_n}\right) f_X(x) dx \\ &\quad + o_P(n^{-1/2}), \end{aligned}$$

where $\lambda_{h,j}(x)$ is the derivative of $\underline{h}(x, s, \gamma_0, \beta_0)$ with respect to the j^{th} component of $\gamma(x)$, written $D_{\gamma,j} \underline{h}(x, s, \gamma_0, \beta_0)$. Note that this derivative is not a function of s , as

$$\begin{aligned} \lambda_{h,j}(x) &= \beta_k D_{\gamma,j} c_0(\gamma_0, x, \beta_0) \lambda_{T+1}(x, \beta_0) \left[\underline{q}_T(\gamma_0, x, \beta_0) \mathbf{1}\{\lambda_{T+1}(x, \beta_0) > 0\} \right. \\ &\quad \left. + \bar{q}_T(\gamma_0, x, \beta_0) \mathbf{1}\{\lambda_{T+1}(x, \beta_0) < 0\} \right] \\ &\quad + \beta_k c_0(\gamma_0, x, \beta_0) \lambda_{T+1}(x, \beta_0) \left[D_{\gamma,j} \underline{q}_T(\gamma_0, x, \beta_0) \mathbf{1}\{\lambda_{T+1}(x, \beta_0) > 0\} \right. \\ &\quad \left. + D_{\gamma,j} \bar{q}_T(\gamma_0, x, \beta_0) \mathbf{1}\{\lambda_{T+1}(x, \beta_0) < 0\} \right]. \end{aligned} \quad (65)$$

Also, $\lambda_{h,j}(x)$ is a continuous function of x . Let $I_{i,j}$ denote the integral in the display above. After a change of variable, $I_{i,j}$ is equal to

$$I_{i,j} = h_n^{pT} \int_u e' S_{h_n}(X_i - h_n u)^{-1} w(h_n u) \lambda_{h,j}(X_i - h_n u) K(u) f_X(X_i - h_n u) du.$$

Assumptions A3-A6 of Kong et al. (2010) hold. Thus, by their Lemma 8,

$$\sup_{x \in \mathcal{D}} |S_{h_n}(x)/(h_n^{pT}) - f_X(x) S_\ell| = O(\nu_n),$$

with $\nu_n := h_n + [nh_n^{pT}/\ln n]^{-1/2}$. Then, we have

$$I_{i,j} = \int_u e' S_\ell^{-1} w(h_n u) \lambda_{h,j}(X_i - h_n u) K(u) du + g_{j,n}(X_i),$$

where $g_{j,n}$ is a deterministic function and because \mathcal{K} is compact, $\sup_{\mathcal{K}} |g_{j,n}| = O(\nu_n)$. While $\lambda_{h,j}$ is not differentiable, it is directionally differentiable and we can write $\lambda_{h,j}(X_i - h_n u) =$

$\lambda_{h,j}(X_i) - h_n u' \nabla \lambda_{h,j}(\tilde{X}, u)$, for some \tilde{X} , where $\nabla \lambda_{h,j}(\tilde{X}, u)$ is uniformly bounded over \tilde{X} and u . Including this new residual in the definition of $g_{j,n}$ and ν_n and noting that $w(h_n u)' e = 1$,

$$\begin{aligned} I_{i,j} &= \int_u e' S_\ell^{-1} w(h_n u) w(h_n u)' e \lambda_{h,j}(X_i) K(u) du + g_{j,n}(X_i) \\ &= e' S_\ell^{-1} \int_u H_n w(u) w(h_n u)' K(u) du e \lambda_{h,j}(X_i) + g_{j,n}(X_i) \\ &= e' S_\ell^{-1} H_n S_\ell H_n e \lambda_{h,j}(X_i) + g_{j,n}(X_i), \end{aligned}$$

where again g_n is deterministic and such that $\sup_{\mathcal{K}} |g_{j,n}| = O(\nu_n)$, and H_n is a diagonal matrix with diagonal entries $h_n^{|r|}$ for $|r| \leq \ell$. Their entries are ordered in the same order as for the polynomial terms in $w(x)$. One can show that

$$H_n S_\ell H_n = S_\ell + O(h_n)$$

where the $O(h_n)$ is an entry-wise bound. This gives $I_{i,j} = \lambda_{h,j}(X_i) + g_{j,n}(X_i)$, changing again the definition of $g_{j,n}$ to a function with the same properties. By Assumption 6.4b, $\nu_n \rightarrow 0$. Then, by Chebyshev's inequality and since $E(\epsilon_{ij}|X_i) = 0$, we have $\frac{1}{\sqrt{n}} \sum_{i=1}^n \epsilon_{ij} g_{j,n}(X_i) = o_P(1)$. Thus,

$$\begin{aligned} \sqrt{n} E^* (D_\gamma h(X_i, S_i, \gamma_0, \beta_0)' [\hat{\gamma}(X_i) - \gamma_0(X_i)]) &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j \leq T+1} \epsilon_{ij} [\lambda_{h,j}(X_i) + g_{j,n}(X_i)] \\ &= \frac{1}{\sqrt{n}} \sum_{i=1}^n \sum_{j \leq T+1} \epsilon_{ij} \lambda_{h,j}(X_i) + o_P(1). \end{aligned}$$

Hence, Condition 5 follows.

Lemma 8 *Let \mathcal{P}' be defined as \mathcal{P} (see (16)) but without the constraint on R_P and $\underline{\sigma} \geq 0$. Suppose that Assumption 5 holds. Then:*

$$\limsup_{n \rightarrow \infty} \sup_{P \in \mathcal{P}'} P \left(\left\| n^{1/2} (\hat{\beta} - \beta_0) - \frac{1}{n^{1/2}} \sum_{i=1}^n \phi_i \right\| > \eta \right) = 0. \quad (66)$$

Moreover, for $\tau = E(\phi_i \phi_i')$ and $\hat{\tau}$ its plug-in estimator, $\hat{\tau} \xrightarrow{P} \tau$ holds uniformly over \mathcal{P}' and if $\underline{\sigma} > 0$, Lemma 2 holds uniformly over \mathcal{P}' .

Proof: To show these results, it suffices to show that they hold along any sequence of probability distribution $(P_n)_{n \geq 1}$ in \mathcal{P}' . We use the same notation as in the other proofs but index parameters, variables and the expectation operator by n to underline their dependence on P_n when deemed necessary. Relatedly, we use $o_{P_n}(1)$ as a shortcut for a sequence of random variable ε_n satisfying $P_n(\|\varepsilon_n\| > \eta) \rightarrow 0$ for all $\eta > 0$.

To prove the first point, let us first prove that $\widehat{\beta} - \beta_{0n} = o_{P_n}(1)$. To that end, consider the class of functions $\mathcal{L} := \{(y, x) \mapsto \ell_c(y|x; \beta); \beta \in \Theta\}$. We apply a version of Glivenko-Cantelli theorem on \mathcal{L} that is uniform over P . The functions $(y, x, \beta) \mapsto \ell_c(y|x; \beta)$ are C^1 on $\{0, 1\}^T \times \text{Supp}(X) \times \Theta$, which is a compact set. The class \mathcal{L} thus satisfies the Lipschitz requirement of Theorem 2.7.11 of van der Vaart and Wellner (1996). Then, by that theorem and the fact that Θ is compact,

$$N(\epsilon \|F\|_{Q,1}, \mathcal{L}, L_1(Q)) \leq N_{[\cdot]}(\epsilon \|F\|_{Q,1}, \mathcal{L}, L_1(Q)) \leq N(\epsilon/2, \Theta, \|\cdot\|) < \infty,$$

where $N_{[\cdot]}$ denotes bracketing numbers, N denotes covering numbers and F is the envelope function defined in the same theorem. Hence,

$$\sup_Q \log N(\epsilon \|F\|_{Q,1}, \mathcal{L}, L_1(Q)) < \infty.$$

In view of the comment after its proof, we can then apply Theorem 2.8.1 of van der Vaart and Wellner (1996). As a result,

$$\sup_{\beta \in \Theta} \left| \frac{1}{n} \sum_{i=1}^n \ell_c(Y_i | X_i; \beta) - E_n[\ell_c(Y | X; \beta)] \right| = o_{P_n}(1). \quad (67)$$

We establish below a uniform version of the well-separation condition by proving that for all $\eta > 0$, there exists $\nu > 0$ such that for all $n \geq 1$,

$$\sup_{\beta: \|\beta - \beta_{0n}\| > \eta} M_n(\beta) < M_n(\beta_{0n}) - \nu, \quad (68)$$

where $M_n(\beta) = E_n[\ell_c(Y | X; \beta)]$. By suitably modifying the proof of Theorem 5.7 in van der Vaart (2000) to the sequence (P_n) , the result follows.

Now, we prove that for any $\eta > 0$, there exists $\nu > 0$ such that (68) holds. For any β such that $\|\beta - \beta_{0n}\| > \eta$, let

$$\beta' = \frac{\eta}{\|\beta - \beta_{0n}\|} \beta + \left(1 - \frac{\eta}{\|\beta - \beta_{0n}\|}\right) \beta_{0n}.$$

Then $\|\beta' - \beta_{0n}\| = \eta$. Moreover, by concavity of M_n ,

$$M_n(\beta') \geq \frac{\eta}{\|\beta - \beta_{0n}\|} M_n(\beta) + \left(1 - \frac{\eta}{\|\beta - \beta_{0n}\|}\right) M_n(\beta_{0n}) \geq M_n(\beta).$$

Thus,

$$\sup_{\beta: \|\beta - \beta_{0n}\| > \eta} M_n(\beta) \leq \sup_{\beta \in S_{n,\eta}} M_n(\beta),$$

where $S_{n,\eta} = \{\beta : \|\beta - \beta_{0n}\| = \eta\}$. Next, for any $\beta \in S_{n,\eta}$ by a Taylor expansion of M_n at β_{0n} ,

$$M_n(\beta) = M_n(\beta_{0n}) - \frac{1}{2}(\beta - \beta_{0n})' \mathcal{I}_{n,0}(\beta - \beta_{0n}) + \frac{\partial^3 M_n}{\partial \beta \partial \beta'}(\tilde{\beta})[\beta - \beta_{0n}],$$

where $\tilde{\beta} = t\beta + (1-t)\beta_{0n}$ for some $t \in (0, 1)$ and $\frac{\partial^3 M_n}{\partial \beta \partial \beta'}(\tilde{\beta})[\beta - \beta_{0n}]$ is the third order differential of M_n at $\tilde{\beta}$ evaluated at $\beta - \beta_{0n}$. We know that $\mathcal{I}_{n,0} \gg A$, write $\underline{\rho}$ the smallest eigenvalue of A . By Assumption 5, there exists $B > 0$ such that $\left| \frac{\partial^3 M_n}{\partial \beta \partial \beta'}(\tilde{\beta})[\beta - \beta_{0n}] \right| \leq B\eta^3$, which gives

$$M_n(\beta) \leq M_n(\beta_{0n}) + \eta^2(B\eta - \frac{1}{2}\underline{\rho}) \leq M_n(\beta_{0n}) - \varepsilon\eta^2$$

if $\eta \leq (\frac{1}{2}\underline{\rho} - \varepsilon)/B$ for some $\varepsilon > 0$. Taking η small enough is without loss of generality, thus (68) follows.

Next, we prove (66). By a Taylor expansion, there exists $t_n \in (0, 1)$ such that

$$\frac{1}{n} \sum_{i=1}^n \frac{\partial \ell_c}{\partial \beta}(Y_i | X_i; \beta_{0n}) + \left[\frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \tilde{\beta}_n) \right] (\hat{\beta} - \beta_{0n}) = 0,$$

where $\tilde{\beta}_n = \hat{\beta} + (1 - t_n)\beta_{0n}$. Thus, by definition of $\phi_{n,i}$,

$$\mathcal{I}_{n,0}^{-1} \left[\frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \tilde{\beta}_n) \right] \sqrt{n} (\hat{\beta} - \beta_{0n}) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \phi_{n,i}. \quad (69)$$

Now, by the triangle inequality and the fact that the third derivatives of ℓ_c are uniformly bounded, there exists $C > 0$ such that

$$\begin{aligned} \left\| \frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \tilde{\beta}_n) - \mathcal{I}_{n,0} \right\| &\leq \left\| \frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \tilde{\beta}_n) - \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \beta_{0n}) \right\| \\ &\quad + \left\| \frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \beta_{0n}) - \mathcal{I}_{n,0} \right\| \\ &\leq C \|\hat{\beta} - \beta_{0n}\| + \left\| \frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \beta_{0n}) - \mathcal{I}_{n,0} \right\|. \end{aligned}$$

By what precedes, the first term is an $o_{P_n(1)}$. Moreover, for all i and n , each element of the matrix $\partial^2 \ell_c / \partial \beta \partial \beta'(Y_i | X_i; \beta_{0n})$ is bounded almost surely. Thus, the uniform integrability condition of Gut (1992) holds for this variable. Then, by his weak LLN, the second term of the right-hand side above is also an $o_{P_n(1)}$. Thus, because $\mathcal{I}_{n,0}^{-1} \ll \underline{A}^{-1}$ (since $P_n \in \mathcal{P}'$), we have

$$\mathcal{I}_{n,0}^{-1} \left[\frac{1}{n} \sum_{i=1}^n \frac{\partial^2 \ell_c}{\partial \beta \partial \beta'}(Y_i | X_i; \tilde{\beta}_n) \right] = \text{Id} + o_{P_n}(1).$$

Next, for all i and n , we have

$$E_n[\phi_{n,i}] = 0, \quad V_n(\phi_{n,i}) = \mathcal{I}_{n,0}^{-1} \ll \underline{A}^{-1}. \quad (70)$$

Hence, by Chebyshev's inequality, the right-hand side of (69) is bounded in probability uniformly over n . Thus, this is also the case of $\sqrt{n}(\hat{\beta} - \beta_{0n})$. Hence,

$$\sqrt{n}(\hat{\beta} - \beta_0) = \frac{1}{\sqrt{n}} \sum_{i=1}^n \phi_{n,i} + o_{P_n}(1).$$

In other words, (66) holds.

We now show that $\hat{\tau}$ converges uniformly over \mathcal{P}' to τ . First, we have

$$\begin{aligned} \hat{\phi}_i &= \left[\frac{1}{n} \sum_{j=1}^n \frac{\partial^2 \ell_c}{\partial \beta^2}(Y_j | X_j; \hat{\beta}) \right]^{-1} \frac{\partial \ell_c}{\partial \beta}(Y_i | X_i; \hat{\beta}) \\ &= \left\{ \left[\frac{1}{n} \sum_{j=1}^n \frac{\partial^2 \ell_c}{\partial \beta^2}(Y_j | X_j; \hat{\beta}) \right]^{-1} - \mathcal{I}_0^{-1} \right\} \frac{\partial \ell_c}{\partial \beta}(Y_i | X_i; \hat{\beta}) + \mathcal{I}_0^{-1} \frac{\partial \ell_c}{\partial \beta}(Y_i | X_i; \hat{\beta}). \end{aligned}$$

Denote with $\hat{\phi}_{i1}$ and $\hat{\phi}_{i2}$ the first and second terms respectively on the right hand side of the second equality above. By the same argument as below Equation (69), using sequences of probability distributions and replacing $\tilde{\beta}_n$ with $\hat{\beta}$, one can show that $\frac{1}{n} \sum_{j=1}^n \partial^2 \ell_c / \partial \beta^2 (Y_j | X_j; \hat{\beta})$ converges uniformly to \mathcal{I}_0 . And since $\mathcal{I}^{-1} \ll \underline{A}^{-1}$, $\left[\frac{1}{n} \sum_{j=1}^n \partial^2 \ell_c / \partial \beta^2 (Y_j | X_j; \hat{\beta}) \right]^{-1}$ converges uniformly to \mathcal{I}_0^{-1} . Moreover, since $C_s(x, \beta)$ is bounded away from 0 uniformly over (s, x, β) , $\partial \ell_c / \partial \beta(y|x; \beta)$ is a continuous function of (y, x, β) and since the support of (Y_i, X_i, β) is a compact set, $|\partial \ell_c / \partial \beta(Y_i | X_i; \hat{\beta})| \leq C$ with probability going to 1 uniformly. This implies that in $\hat{\phi}_i \hat{\phi}'_i = \hat{\phi}_{i1} \hat{\phi}'_{i1} + \hat{\phi}_{i1} \hat{\phi}'_{i2} + \hat{\phi}_{i2} \hat{\phi}'_{i1} + \hat{\phi}_{i2} \hat{\phi}'_{i2}$, the sample average of all terms including $\hat{\phi}_{i1}$ converges uniformly to 0. As for the term $\hat{\phi}_{i2} \hat{\phi}'_{i2}$, writing $\phi_{i2} = \mathcal{I}_0^{-1} \partial \ell_c / \partial \beta(Y_i | X_i; \beta)$ then $\|\hat{\phi}_{i2} - \phi_{i2}\| \leq C|\hat{\beta} - \beta_0|$ with probability uniformly going to 1. Since ϕ_{i2} is uniformly bounded, we obtain $\sum_{i=1}^n \hat{\phi}_{i2} \hat{\phi}'_{i2} / n \xrightarrow{P} \sum_{i=1}^n \phi_{i2} \phi'_{i2} / n$ uniformly over \mathcal{P}' .

We now show that Lemma 2 holds uniformly over \mathcal{P}' . Let us start with the asymptotic normality. Reasoning as in the proof of Lemma 2 and using the first point above, we get

$$\sqrt{n}(\hat{\Delta} - \tilde{\Delta}) = \frac{1}{n^{1/2}} \sum_{i=1}^n \left[\hat{G} \phi_{n,i} + g(W_i, \beta_{0n}) - \tilde{\Delta} \right] + o_{P_n}(1).$$

Note that g is C^2 on the compact set $\text{Supp}(W) \times \Theta$. Moreover, $\bar{\beta}_i$ as defined in Lemma 2 satisfies $\|\bar{\beta}_i - \beta_{0n}\| \leq \|\hat{\beta} - \beta_{0n}\|$. Hence, there exists $M > 0$ such that

$$\|\hat{G} - G_n\| \leq M \|\hat{\beta} - \beta_{0n}\| + \left\| \frac{1}{n} \sum_{i=1}^n \frac{\partial g}{\partial \beta}(W_i, \beta_{0n}) - G_n \right\|. \quad (71)$$

By the first part of the proof, $\hat{\beta} - \beta_{0n} = o_{P_n}(1)$. Next, because $\partial g / \partial \beta(\cdot, \beta_{0n})$ is bounded on $\text{Supp}(W)$, the uniform integrability condition of Gut (1992) also holds for this variable.

Then, by his weak LLN,, the second term of (71) is an $o_{P_n}(1)$. Thus, $\|\widehat{G} - G_n\| = o_{P_n}(1)$. As a result,

$$\sqrt{n} \frac{\widehat{\Delta} - \widetilde{\Delta}}{\sigma_n} = \frac{1}{n^{1/2}} \sum_{i=1}^n \frac{G_n \phi_{n,i} + g(W_i, \beta_{0n}) - E_n[g(W_i, \beta_{0n})]}{\sigma_n} + o_{P_n}(1).$$

Now, by the triangle and Cauchy-Schwarz inequalities, we have

$$|G_n \phi_{n,i} + g(W_i, \beta_{0n}) - E_n[g(W_i, \beta_{0n})]| \leq \|G_n\| \|\phi_{n,i}\| + |g(W_i, \beta_{0n}) - E_n[g(W_i, \beta_{0n})]|. \quad (72)$$

$\|G_n\|$ is bounded uniformly over n . The variables $|g(W_i, \beta_{0n}) - E_n[g(W_i, \beta_{0n})]|$ are also bounded. Next, $\phi_{n,i} = \mathcal{I}_{n0}^{-1} V_{n,i}$ where $V_{n,i}$ is a bounded vector (with $\|V_{n,i}\| \leq C$, say). Moreover, because $P_n \in \mathcal{P}'$,

$$\|\mathcal{I}_{n0}^{-1} V_{n,i}\| \leq \|\underline{A}^{-1} V_{n,i}\| \leq \underline{\rho}^{-1} C,$$

where $\underline{\rho} > 0$ denotes the smallest eigenvalue of \underline{A} . Then, using (72) and $\sigma_n \geq \underline{\sigma}$, the variables $(G_n \phi_{n,i} + g(W_i, \beta_{0n})/\sigma_n)$ are bounded by a constant independent of n . Thus, they satisfy the Lindeberg condition. Then, by the central limit theorem for triangular arrays,

$$\sqrt{n} \frac{\widehat{\Delta} - \widetilde{\Delta}}{\sigma_n} \xrightarrow{d} \mathcal{N}(0, 1).$$

We now show that $\widehat{\sigma}$ converges to σ uniformly over \mathcal{P}' . First note that using the notation of the proof of Lemma 2, we have

$$\widehat{\psi}_i = \left[\frac{1}{n} \sum_{j=1}^n g(W_j, \widehat{\beta}) \right] \widehat{\phi}_{ik} + \widehat{\beta}_k g(W_i, \widehat{\beta}) + \left[\frac{1}{n} \sum_{j=1}^n \frac{\partial g}{\partial \beta}(W_j, \widehat{\beta}) \right]' \widehat{\phi}_i.$$

Since $\partial g/\partial \beta$ is continuous in β , $\frac{1}{n} \sum_{j=1}^n g(W_j, \widehat{\beta})$ converges uniformly to $E(g(W_j, \beta_{0n}))$. Similarly since $\partial^2 g/\partial \beta^2$ is continuous, $\frac{1}{n} \sum_{j=1}^n \partial g/\partial \beta(W_j, \widehat{\beta})$ converges uniformly to $E(\partial g/\partial \beta(W_j, \beta_{0n}))$. Thus since $\widehat{\phi}_{ik}$, $g(W_i, \widehat{\beta})$ and $\widehat{\phi}_i$ are all bounded with probability uniformly going to 1, to show that $\widehat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n \widehat{\psi}_i^2$ converges uniformly to σ^2 it suffices to show that $\frac{1}{n} \sum_{i=1}^n \widetilde{\psi}_i^2$ converges uniformly to σ^2 , where

$$\widetilde{\psi}_i = E[g(W_j, \beta_{0n})] \widehat{\phi}_{ik} + \beta_k g(W_i, \widehat{\beta}) + E \left[\frac{\partial g}{\partial \beta}(W_j, \beta_{0n}) \right]' \widehat{\phi}_i.$$

Each of the terms in this sum are bounded with probability uniformly going to 1. They are converging uniformly for all i to $E[g(W_j, \beta_{0n})] \phi_{ik}$, $\beta_k g(W_i, \beta_{0n})$ and $E \left[\frac{\partial g}{\partial \beta}(W_j, \beta_{0n}) \right]' \phi_i$ respectively, by the proof of uniform convergence of $\widehat{\tau}$ for $\widehat{\phi}$, and continuity of $\partial g/\partial \beta$. Thus one can conclude that $\frac{1}{n} \sum_{i=1}^n \widetilde{\psi}_i^2$ converges uniformly to σ^2 .

Lemma 9 *Suppose that Assumption 5 holds. Then \widehat{R} converges to \overline{R} and if $\zeta > 0$ in the definition of \mathcal{P} , this convergence holds uniformly over \mathcal{P} .*

Proof: Define

$$f(W_i, \beta) := \binom{T}{S_i} \frac{\lambda_T(X_i, \beta) \exp(S_i X'_{iT} \beta)}{2 \times 4^T \times C_{S_i}(X_i, \beta)}.$$

Then, $\widehat{R} = \sum_{i=1}^n |f(W_i, \widehat{\beta})|/n$. The function $w \mapsto f(w, \cdot)$ is C^1 and $(w, \beta) \mapsto \partial f(w, \beta)/\partial \beta$ is continuous over the compact set $\text{Supp}(W) \times \Theta'$ where Θ' is a compact set such that $\Theta \subsetneq \Theta'$. Hence, there exists $M > 0$ such that $\beta \mapsto f(w, \beta)$ is Lipschitz with coefficient M for all $\beta \in \Theta'$ and $w \in \text{Supp}(W)$. The same property then holds for $\beta \mapsto |f(w, \beta)|$. Since $\widehat{\beta} \in \Theta'$ with probability uniformly going to 1 and $|f(W_i, \beta)|$ is bounded almost surely when $\beta \in \Theta'$, one can use the argument below Equation (69) to show that \widehat{R} converges to \overline{R} and this convergence holds uniformly over \mathcal{P} .

C Further details on the simulations

First, to estimate $\gamma_{0t}(x)$, we use a local linear estimator with a common bandwidth h_t for the T components of X . To choose h_t , we aim at reaching a certain ratio between the (integrated) bias and standard deviation of the estimator. Specifically, let $B_t(x, h)$ and $\sigma_t^2(x, h)$ denote respectively the asymptotic bias and variance of $\widehat{\gamma}_t(x)$ with a bandwidth equal to h . Then (see, e.g. Ruppert and Wand, 1994),

$$B_t(x, h) = h^2 \left(\int u^2 K(u) du \right) \sum_{j=1}^{pT} \frac{\partial^2 \gamma_{0t}}{\partial x_j^2}(x),$$

$$\sigma_t^2(x, h) = \frac{1}{nh^T} \frac{(\int K(u)^2 du)^T \gamma_{0t}(x)(1 - \gamma_{0t}(x))}{f_X(x)}.$$

Then, define $B_t^2(h) := E[B_t^2(X, h)]$ and $\sigma_t^2(h) := E[\sigma_t^2(X, h)]$. Assuming first that $B_t^2(h)$ and $\sigma_t^2(h)$ are known, we would choose h_t so that $\sigma_t^2(h_t) = R_n \times B_t^2(h_t)$, where $R_n > 0$ fixes to the degree of undersmoothing. For instance, $R_n = 1$ corresponds to the optimal bandwidth in terms of asymptotic mean integrated squared error. We use $R_n = 5(n/500)^2$ in our simulations. Now, $B_t^2(h)$ and $\sigma_t^2(h)$ are actually unknown. We estimate both assuming that α is constant. Then, we can estimate this constant by MLE (plugging the CMLE $\widehat{\beta}$ in the log-likelihood) and then estimate $\gamma_{0t}(x)$ by plug-in, using (2).

Finally, to obtain \widehat{m} , we must choose a threshold c_n . We actually slightly modify $\widehat{I}(x)$, by

letting

$$\hat{I}(x) := \max \left\{ t \in \{1, \dots, T\} : \underline{H}_t(\tilde{m}_{\rightarrow t}(x)) \geq \underline{c}_{nt}(x) \text{ and } \overline{H}_t(\tilde{m}_{\rightarrow t}(x)) \geq \overline{c}_{nt}(x) \right\},$$

where $\underline{c}_{nt}(x) := \hat{\sigma}_t[2 \ln \ln(n)]^{1/2}$, $\overline{c}_{nt}(x) := \widehat{\sigma}_t(x)[2 \ln \ln(n)]^{1/2}$ and $\hat{\sigma}_t^2(x)$ (resp. $\widehat{\sigma}_t^2(x)$) is an estimator of the asymptotic variance of $\underline{H}_t(m_{\rightarrow t}(x))$ (resp. $\overline{H}_t(m_{\rightarrow t}(x))$).